Appetizer

Scallop

Starter Puzzles

Chapter 0: A Miscellany of Puzzles

Problem 1 is an introduction for the puzzles in Chapter 1, Problem 2 for Chapter 2, and so on. There is no introductory problem for Chapters 10 or 11 because each is itself a miscellany of puzzles.

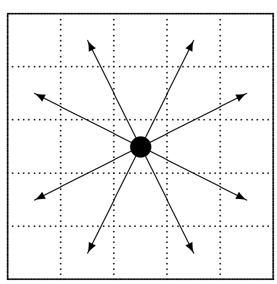
Somewhere under the rainbow, there is a mystic seafood restaurant in the land of the Rising Sun. Both the manager and the chef are very fond of recreational mathematics.

Problem 1.
Available at the take-out counter of the restaurant are 3×3 sushi plates. Each of the 9 pieces is either salmon or tuna. Two pieces are said to be neighbors if they occupy squares sharing a common side. Two configurations are equivalent if they may be obtained from each other by rotation or reflection.

(a) In Plate A, each tuna piece has exactly one salmon piece among its neighbors, and no salmon piece has exactly one other salmon piece among its neighbors. Find all possible non-equivalent configurations of Plate A.

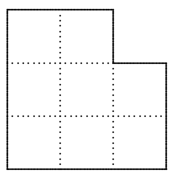
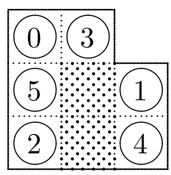
(b) In Plate B, each tuna piece has exactly two other tuna pieces among its neighbors, and no salmon piece has exactly two tuna pieces among its neighbors. Find all possible non-equivalent configurations of Plate B.

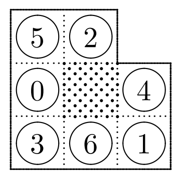
Problem 2.
The chef of the restaurant is an aficionado of chess. He is mainly fascinated by the move of the knight, as shown in the diagram below.



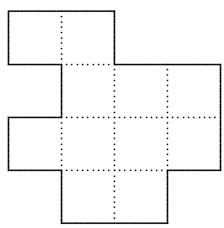
The lunch menu offers a sushi special. For a fixed price, the customer places a sushi piece on any square of the plate in the diagram below on the left. The next piece may be placed on any square a knight's move away, and so on, until no further placement is possible.

There is an additional requirement. The empty space on the plate may not be separated into two or more regions by the occupied squares at any time. Two squares sharing only a vertex are considered to be separated. The diagram below in the middle shows that the customer can get as many as six pieces. It may appear that the customer can get seven pieces, as shown in the diagram below on the right. However, a violation of the rule occurs when the piece numbered 2 is placed.



The diagram below shows the plate for the sushi special in the dinner menu. What is the maximum number of pieces the customer can get?

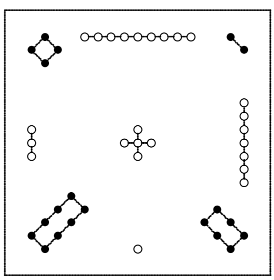


Remark.
This problem is based on a puzzle from the wonderful book *Arithmetical, Geometrical and Combinatorial Puzzles from Japan* by **Tadao Kitazawa**. It was published by the American Mathematical Society in 2021. The statement of the problem appears on page 171 and the solution appears on page 176.

Problem 3.
The restaurant wishes to hire three new servers. Instead of advertising, the manager goes on a recruitment tour.

She comes across the Island of Fish-Eaters. Each islander eats only one kind of fish. The majority eat trout, and answer questions truthfully. The minority eat featherback, and answer questions falsely. Although truth can be deduced from the responses of a Featherback-Eater just as well as a Trout-Eater, provided that we know which is which, the manager prefers hiring only Trout-Eaters. She interviewed 20 candidates in the morning and 20 more in the afternoon. The candidates in the afternoon are asked how many Featherback-Eaters are among those interviewed in the morning. They all know the correct answer, but respond in their own ways. The first one says there is at least one of them, the second one says there are at least two of them, and so on. The last one says that all of them are Featherback-Eaters. How many Featherback-Eaters are among the 40 candidates?

Problem 4.
On the wall of the dining room of the restaurant is a copy of the diagram below. On the left is the *Luoshu* from an ancient Chinese scroll. On the right, the strings of beads have been converted to Roman numerals. Later, the restaurant decides to replace the Roman numerals by Arabic numbers. Unfortunately, a mistake is made in that the II is converted into an 11 instead of a 2. Rearrange the numbers into a new magic square (see definition after the diagram).



IV	IX	II
III	V	VII
VIII	I	VI

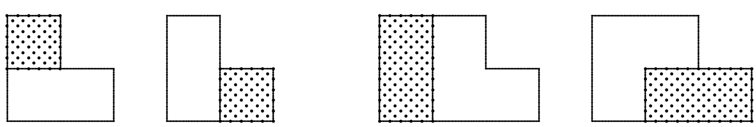
An $n \times n$ table is called a *magic square* of *order* n with *magic constant* k if it satisfies the following three conditions:
(1) The sum of the numbers in each row is k.
(2) The sum of the numbers in each column is k.
(3) The sum of the numbers in each diagonal of length n is k.
Thus the *Luoshu* is a magic square of order 3 with magic constant 15. Moreover, the numbers are consecutive, starting from 1.

Problem 5.
In the waiting area of the restaurant, there are sheets each containing a puzzle. Here is a sample. The diagram below represents a multiplication in which all but two of the digits have been replaced by $\star$s. Reconstruct the computation.

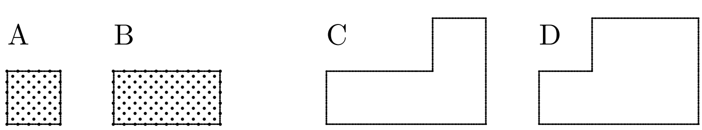
$$\begin{array}{cccc} & & \star & \star \\ \times & & 2 & \star \\ \hline & & \star & \star \\ & \star & \star & \\ \hline \star & \star & 1 & \star \end{array}$$

Problem 6.
The restaurant is a major sponsor of the Aquatic Boxers' Club. From time to time, two octopuses from the ABC are invited to the restaurant to put on a boxing match in an aquarium, for the entertainment of the customers. The ABC sends along eight gloves in each of eight colors mixed in a bag. For the octopuses, there is no distinction between left or right gloves. What is the minimum number of gloves that must be drawn at random from the bag to guarantee that each octopus can wear eight gloves of different colors?

Problem 7.
The chef comes up with the idea of a Wagashi Combo. It consists of a thin yokan slice and a thin mochi slice, put together to form the same figure in two different ways. The pieces may be rotated or reflected but may not overlap. The first attempt, as shown in the diagram below on the left, is not a success, as the two ways become the same after rotation or reflection. A successful attempt is shown in the diagram below on the right.



The actual offerings consist of the following slices.



Prepare a Wagashi Combo using the following combination of slices:

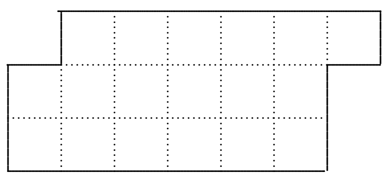
(a) A and D;

(b) B and C;

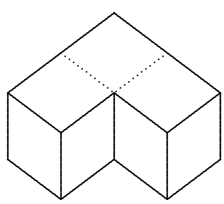
(c) B and D.

A Wagashi Combo normally has a single solution. For these starting exercises, multiple solutions are allowed.

Problem 8.
As a promotion, the restaurant offers the yokan slice in the diagram below. To get it for free, the customer must cut it into two parts which are identical, after rotation or reflection if necessary. Moreover, the customer must do this in three different ways. How can the customer take advantage of this Yokan Special?



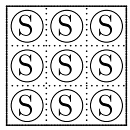
Problem 9.
The chef prepares nine thick mochi pieces consisting of three unit cubes joined face to face, as shown in the diagram below. Is it possible to pack them into a $3 \times 3 \times 3$ box?

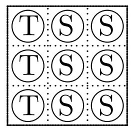


Solutions

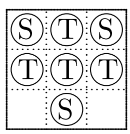
Problem 1.

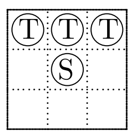
(a) We may have no tuna pieces at all, as shown in the diagram below on the left. Suppose there is at least one. We claim that the plate must be as shown in the diagram below on the right.





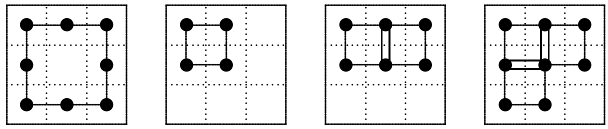
Suppose there is a tuna piece on the central square. Since it has exactly one salmon piece among its neighbors, we may assume that the salmon piece is to its south, as shown in the diagram below on the left. Then neither of the northwest and the northeast corner squares can be tuna pieces. However, this means that the tuna piece in the middle of the north edge has two salmon neighbors.



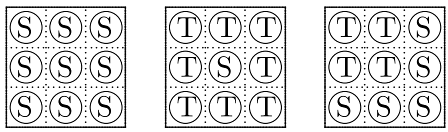


It follows that we must have a salmon piece on the central square. If all of its neighbors are also salmon pieces, then there cannot be any tuna pieces. Hence we may assume that there is a tuna piece on the middle of the north edge. Then all three pieces on the north edge are tuna, as shown in the diagram above on the right. We can then deduce that the remaining pieces are indeed salmon.

(b) Since each tuna piece has exactly two tuna neighbors, the tuna pieces together form disjoint unions of cycles of even lengths (possibly empty). There are four such cycles which fit in a 3×3 plate. However, the last two cannot be used as they are short-circuited along the double lines.

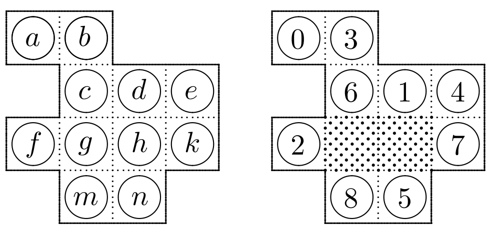


There may be no tuna pieces, as shown in the diagram below on the left. If there are tuna pieces, we have seen that they must form a cycle as shown in the diagram below in the middle or on the right.



Problem 2.

The customer can get as many as nine pieces of sushi. In the diagram below on the left, suppose h is the first piece placed. Then that is all the customer would get since only b is a knight's move away, and placing b violates the rule. If h is the last piece placed, then a must be placed before b. We cannot go from d to a and then to g, or in the reverse order, as otherwise the rule is violated. Hence a must be the first piece placed, followed by d. If we then go to m, we cannot proceed to k. Hence we must go from d to f, n, e and b, but now we cannot proceed to h. It follows that h cannot be placed at all. An analogous but more involved analysis shows that g cannot be placed either. The diagram below on the right shows how the customer may get all the remaining pieces.



Problem 3.

If all the candidates interviewed in the morning are Trout-Eaters, then all the candidates interviewed in the afternoon are Featherback-Eaters, and there are 20 of them. Suppose exactly k of the morning candidates are Featherback-Eaters, where $1 \le k \le 20$. Then the first k afternoon candidates are Trout-Eaters while the remaining $20-k$ are Featherback-Eaters. Again, the number of Featherback-Eaters is $k + (20 - k) = 20$.

Problem 4.

The sum of the nine numbers is $1+11+3+4+5+6+7+8+9 = 54$. Hence the magic constant is $54 \div 3 = 18$. We list all sums of these numbers taken three at a time and equal to 18: 1+6+11, 3+4+11, 1+8+9, 3+6+9, 4+5+9, 3+7+8, 4+6+8 and 5+6+7. The number 6 appears in four sums, each of the numbers 3, 4, 8 and 9 appears in three sums, and each of the numbers 1, 5, 7 and 11 appears in two sums. The new magic square is shown in the diagram below.

9	1	8
5	6	7
4	11	3

Problem 5.

To explain the approach, we first replace the $\star$s by letters, as shown in the diagram below.

$$\begin{array}{cccc} & & A & B \\ \times & & 2 & C \\ \hline & & D & E \\ & F & G & \\ \hline H & I & 1 & J \end{array}$$

Let us focus on the addition of the third and the fourth rows to yield the product in the fifth row We have $H = 1$ from a carry-over. Another carry-over results in $1 + F = 10 + I$. Hence $F = 9$ and $I = 0$. A third carry-over occurs at $D + G = 11$. Finally, note that $E = J$.

We now switch to the multiplications. Since $2 \times A < 9$, $A = 4$ and $C = 1$ or 2. Suppose $C = 1$. Then $D = A = 4$ and $G = 2 \times B$ is even. This contradicts $D + G = 11$. Hence $C = 2$ and $D = F = 9$. It follows that $J = E = G = 2$ and $B = 6$. The multiplication is $46 \times 22 = 1012$.

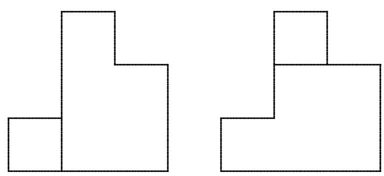
$$\begin{array}{cccc} & & 4 & 6 \\ \times & & 2 & 2 \\ \hline & & 9 & 2 \\ & 9 & 2 & \\ \hline 1 & 0 & 1 & 2 \end{array}$$

Problem 6.

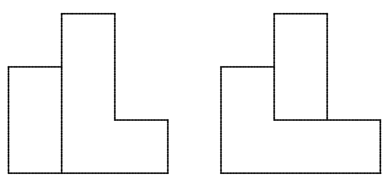
With 57 gloves, we may have all 8 gloves of seven of the colors and 1 glove of the last color. With 58 gloves, the smallest number of gloves of any color is at least 2. Thus the minimum number of gloves that must be drawn at random is 58.

Problem 7.

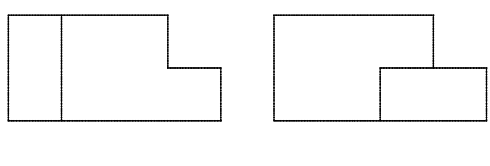
(a) See the diagram below.



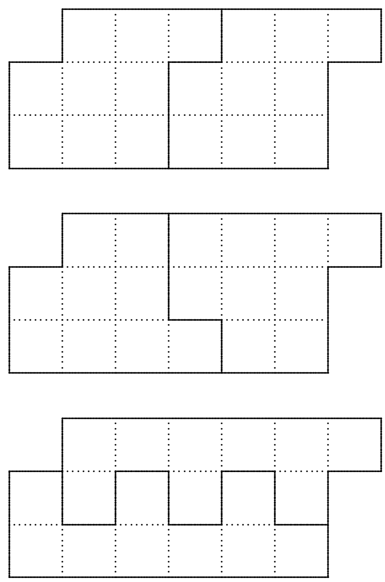
(b) See the diagram below.



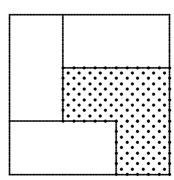
(c) See the diagram below.



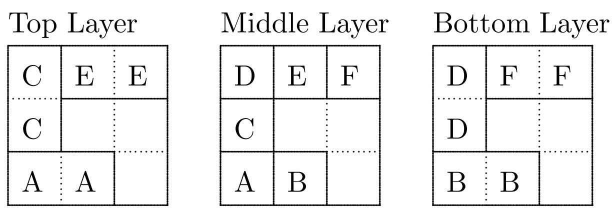
Problem 8.
The three solutions are shown in the diagrams below.



Problem 9.
Such a packing is possible. Two pieces of mochi can form a $1 \times 2 \times 3$ block. Use six pieces to build three of them. The diagram below shows the base of a $3 \times 3 \times 3$ cube divided into three dominoes and a fourth region. The three $1 \times 2 \times 3$ blocks stand on the dominoes while the remaining three pieces of mochi pile on top of one another on the fourth region.



The diagram below shows a different presentation of the construction. Cubes with the same label belong to the same piece. Unlabeled cubes belong to pieces which lie within single layers of the box.



Soup

Oyster

Other Puzzles

Chapter 10: The International Seafood Festival

The restaurant has accepted an invitation to take part in an International Seafood Festival in Hawaii. Apart from the hosting American seafood restaurant, a Chinese seafood restaurant is also coming.

Problem 1.
In preparation for foriegn travel, the manager gets hold of a bundle of four translation programs.

Program	Manual	Function
A	In Japanese	English into Japanese
B	In English	Chinese into English
C	In Chinese	Japanese into English
D	In Chinese	English into Chinese

She does not know any English or Chinese, so that Program A is the only one she knows how to use. Can she use the programs to translate Manuals B, C and D into Japanese?

Problem 2.
As a gift to the host, the restaurant is bringing a manuscript containing some Japanese recipes. The pages are to be numbered consecutively from 1, using gold-plated stickers each containing a single digit and costing 500 yens. The budget is 1000 yens for each page in the manuscript. What is the maximum number of pages the manuscript may have if the actual cost matches the budget exactly?

Problem 3.
Upon arrival at Honolulu, the manager decides to hire some local help to serve as housekeepers during the festival. There are six applicants. Amanda wants a day off after working 2 days. Belinda wants a day off after working 4 days. Christina wants a day off after working 6 days. Denise wants a day off after working 8 days. Evelyn wants a day off after working 10 days. Felicia wants a day off after working 12 days. What is the minimum number of these applicants we must hire in order to have at least one housekeeper working each day?

Problem 4.

A straight stretch of the Waikiki Beach is divided into seven segments of equal lengths by the huts P, Q, R, S, T and U in that order. The hut T is used as the public toilet. Each of the three restaurants will use a different hut as its kiosk. It is assumed that a customer will eat lunch at the nearest kiosk. Naturally, each restaurant wishes to maximize the length of the beach it will control, assuming that the customers are uniformly distributed along the beach.

(a) The Chinese choose first. Which hut could they take?

(b) The Japanese choose next. Which hut could we take?

(c) The Americans choose last. Everything else being equal, they would like to cut down the length of the beach controlled by the Chinese restaurant. Which hut could they take?

Problem 5.

Before the plan worked out in Problem 4 is implemented, the setting is revised so that the stretch of beach now has eleven segments of equal lengths, divided by the huts P, Q, R, S, T, U, V, W, X and Y in that order. Moreover, hut T is no longer used as the public toilet, and is available if a restaurant chooses to put its kiosk there. What would the new plan be like?

Problem 6.

For lunch at the kiosk, the chef prepares 5 packs of crab, 4 packs of abalone, 2 packs of lobster and 3 packs of vegetables each week. On each day, 2 packs of food are served. They may be the same but not a double offering of vegetables. The same kind of food may not be served on consecutive days within the week, though the same kind of food may be served on Saturday of one week and on Sunday of the following week. The same combination of food may not be served more than once in the same week. Design a possible menu for the week from Sunday to Saturday.

Problem 7.

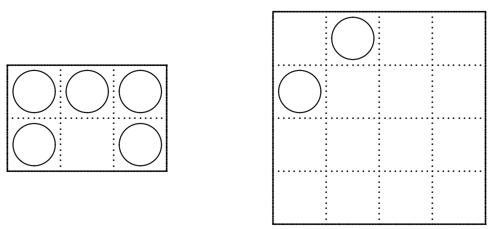
There is a gala dinner every evening. Only one of the three restaurants is scheduled to cook on each day. The American restaurant cooks on opening day. The Chinese restaurant will only cook on days right after the American restaurant cooks. For any two consecutive blocks of days of equal length, the schedule within each block must not be identical.

In particular, no restaurant cooks on two consecutive days. Design a schedule for the gala dinner so that it lasts the maximum number of days.

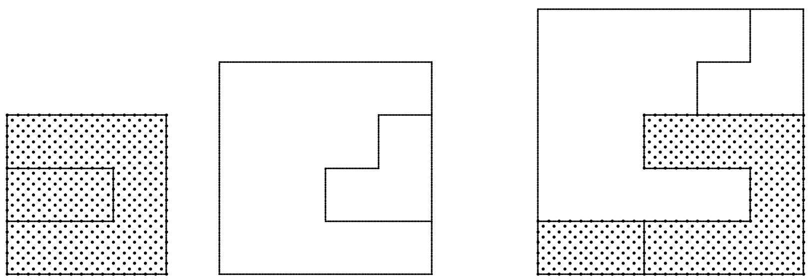
The chef estimates that the Japanese restaurant will be cooking for four days. So he prepares four special puzzles.

Problem 8.
Seven pieces of sashimi are arranged in a ring on a round tray. Each piece is either salmon or tuna. *Wasabi* is put on a salmon piece if and only if it is between two tuna pieces, and on a tuna piece if and only if it is between a salmon piece and a tuna piece. If there is wasabi on only one piece, what is the maximum number of tuna pieces on the tray?

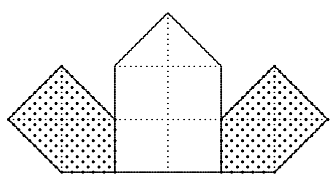
Problem 9.
Five of the sixteen pieces of sushi on a 4×4 tray have wasabi. They occupy five squares contained within a 2×3 rectangle, as shown in the diagram below on the left. The rectangle may be rotated. Neither piece in the two squares near the northwest corner marked in the diagram below on the right has wasabi. How many pieces must be tried in order to taste a piece with wasabi?



Problem 10.
A 3×3 yokan slice and a 4×4 mochi slice have been cut into a total of four pieces and reassembled into a 5×5 square. The pieces may not be rotated or reflected. The diagram below shows one solution. Find three others.



Problem 11.
The diagram below shows one mochi slice and two yokan slides on a plate. All three pieces have the same shape, and the mochi slice is twice as large as each yokan slice. What is the minimum number of copies of this plate that must be used in order to cover the mochi slices completely with the yokan slices?



Solutions

Problem 1.
First, the manager uses Program A to translate Manual B from English into Japanese. She now uses Program B to translate Manuals C and D from Chinese into English. Finally, she uses Program A to translate Manuals C and D from English into Japanese.

Problem 2.
From page 1 to page 9, we save 500 yens per page, for a total saving of 4500 yens. From page 10 to page 99, the we break even. From page 100 on, we takes a loss. To accumulate a lost of 4500 yens, the manuscript must have nine pages beyond page 99, so that the number of pages must be 108.

Problem 3.
Since every candidate wants days off, hiring only one will not suffice. Amanda wants a day off every 3 days. Belinda wants a day off every 5 days. Christina wants one day off every 7 days. Denise wants one day off every 9 days. Evelyn wants a day off every 11 days. Felicia wants a day off every 13 days. Of these six numbers, only 3 and 9 have a common divisor greater than 1. The manager should hire Amanda and Denise, and give Denise days off when Amanda is working.

Problem 4.
By symmetry, hut U is a better choice for the Chinese restaurant than hut P and hut S is a better choice for the Chinese restaurant than hut R. Hence there are three cases.
Case 1. The Chinese restaurant takes hut Q.
The Japanese restaurant's best choice is hut U. This is because the best the American restaurant can do is to control 2 segments, by taking either hut R or S. It will take hut R, limiting the length of the beach controlled by the Chinese restaurant to $2\frac{1}{2}$ segments. That leaves $2\frac{1}{2}$ segments for the Japanese restaurant.
Case 2. The Chinese restaurant takes hut S.
The Japanese restaurant's best choice is hut Q. This is because the best the American restaurant can do is to control 2 segments, by taking hut U, limiting the Chinese restaurant's control to 2 segments. That leaves 3 segments for the Japanese restaurant.

Case 3. The Chinese restaurant takes hut U.
The Japanese restaurant's best choice is hut Q. This is because the best the American restaurant can do is to control 2 segments, by taking either hut R or S. It will take hut S, limiting the length of the beach controlled by the Chinese restaurant to $2\frac{1}{2}$ segments. That leaves $2\frac{1}{2}$ segments for the Japanese restaurant.
All cases considered, the Chinese restaurant's best choice is to take hut Q or U, controlling $2\frac{1}{2}$ segments. The Japanese restaurant will take hut U or Q, respectively, also controlling $2\frac{1}{2}$ segments. This leaves 2 segments for the American restaurant, taking hut R and S respectively.

Problem 5.
By symmetry, we may assume that the Chinese restaurant takes one of huts U, V, W, X and Y. Hence there are five cases.
Case 1. The Chinese restaurant takes hut U.
The Japanese restaurant's best choice is hut T. This is because the American restaurant will take V instead of S, limiting the length of the beach controlled by the Chinese restaurant to 1 segment while controlling $4\frac{1}{2}$ segments itself. That leaves $5\frac{1}{2}$ segments for the Japanese restaurant.
Case 2. The Chinese restaurant takes hut V.
The Japanese restaurant's best choice is hut S. This is because the American restaurant will take W instead of R, limiting the length of the beach controlled by the Chinese restaurant to 2 segments while controlling $3\frac{1}{2}$ segments itself. That leaves $5\frac{1}{2}$ segments for the Japanese restaurant.
Case 3. The Chinese restaurant takes hut W.
The Japanese restaurant's best choice is hut R. This is because the American restaurant will take X instead of V, S or Q, limiting the length of the beach controlled by the Chinese restaurant to 3 segments while controlling $2\frac{1}{2}$ segments itself. That leaves $5\frac{1}{2}$ segments for the Japanese restaurant.
Case 4. The Chinese restaurant takes hut X.
The Japanese restaurant's best choice is hut R. The American restaurant will take W instead of S, limiting the length of the beach controlled by the Chinese restaurant to $2\frac{1}{2}$ segments while controlling 3 segments itself. That leaves $5\frac{1}{2}$ segments for the Japanese restaurant.

Case 5. The Chinese restaurant takes hut Y.
The Japanese restaurant's best choice is hut R. This is because the American restaurant will take X instead of S, limiting the length of the beach controlled by the Chinese restaurant to $1\frac{1}{2}$ segments while controlling $3\frac{1}{2}$ segments itself. That leaves 6 segments for the Japanese restaurant.
All cases considered, the Chinese restaurant's best choice is to take hut W, controlling 3 segments. The Japanese restaurant will take hut R, controlling $5\frac{1}{2}$ segments. This leaves $2\frac{1}{2}$ segments for the American restaurant, taking hut X.

Problem 6.
There are 7 days in a week and 5 packs of crab. We can serve a double portion of crab at most once, and may not serve crab on consecutive days within the week. So we must serve crab on Sunday, Tuesday, Thursday and Saturday, with a double portion on one of these days. We cannot serve abalone on Monday, Wednesday and Friday, with a double portion on one of these days, as otherwise vegetables will be serve twice with crab or twice with abalone. Hence abalone must be served in a double portion as well as with crab during the week. The following chart is a possible menu.

Day of the Week	Food Served
Sunday	Crab with vegetables
Monday	Double abalone
Tuesday	Crab with lobster
Wednesday	Abalone with vegetables
Thursday	Double crab
Friday	Lobster with vegetables
Saturday	Crab with abalone

Problem 7.
We denote the Chinese, Japanese and American restaurants by C, J and A respectively. The only forbidden pair for cooking on consecutive days, apart from (C,C), (J,J) and (A,A), is (J,C). The following chart traces each possible schedule of cooking until it can no longer continue. Schedule 1 lasts 12 days. It cannot be continued because we cannot have (JA)(JA). This is also the case with Schedules 3, 4 and 6.

Note that Schedules 2 and 5 cannot continue because of (AC)(AC) or (ACAJ)(ACAJ). Schedule 7 cannot continue because of (AJ)(AJ) or (AJAC)(AJAC). The maximum number of days we can have a gala dinner is 12.

Program Number	Day 1	2	3	4	5	6	7	8	9	10	11	12
1	A	C	J	A	C	A	J	A	C	J	A	J
2										A	–	–
3					J	–	–	–	–	–	–	–
4			A	J	A	C	J	A	J	–	–	–
5							A	–	–	–	–	–
6		J	A	C	J	A	J	–	–	–	–	–
7					A	J	A	–	–	–	–	–

Problem 8.

Let the pieces be P, Q, R, S, T, U and V in cyclic order, and let S be the lone piece which has wasabi. We consider two cases.

Case 1. S is a salmon piece.

Then R and T are both tuna pieces. Since neither of them has wasabi, Q and U must be salmon pieces. Neither of them has wasabi either. So P and V are also salmon pieces, and the number of tuna pieces is two.

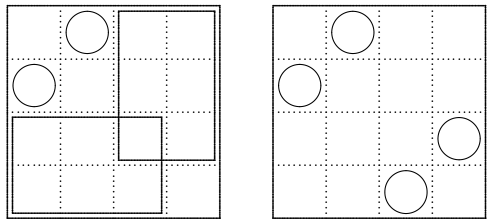
Case 2. S is a tuna piece.

Then one of R and T is a salmon piece and the other a tuna piece. By symmetry, we may assume that R is a salmon piece and T is a tuna piece. Since T does not have wasabi, U must be also be a tuna piece. In fact, all of V, P and Q must in turns be tuna pieces. However, this means that R should have wasabi, which is a contradiction. Hence this case is impossible.

In summary, the number of tuna pieces can only be two.

Problem 9.

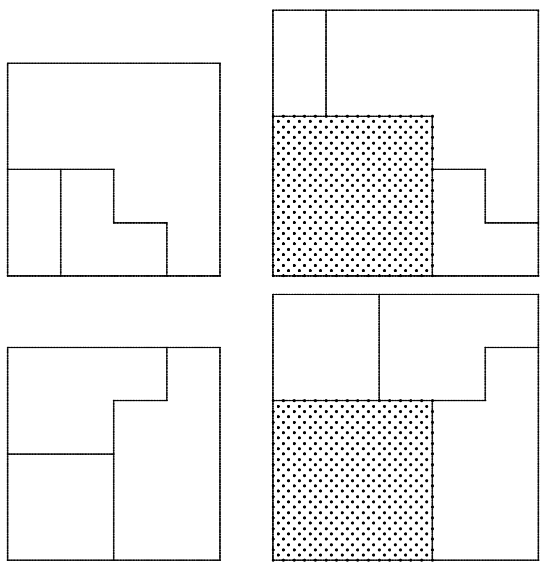
The diagram below on the left shows two 2×3 rectangles overlapping at a single square. All five pieces with wasabi may be inside either of them. If only one more piece is to be tried, it must be in the square which is the intersection of the two rectangles. However, the pieces with wasabi may be around this square but not in that square.

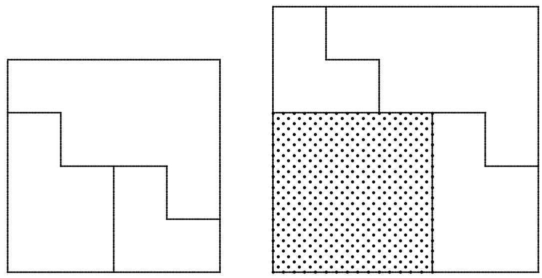


It follows that at least two more pieces must be tried. If the two pieces near the south east corner marked in the diagram above on the right, at least one of them must have wasabi, because any 2×3 rectangle must cover one of the squares containing a tried piece.

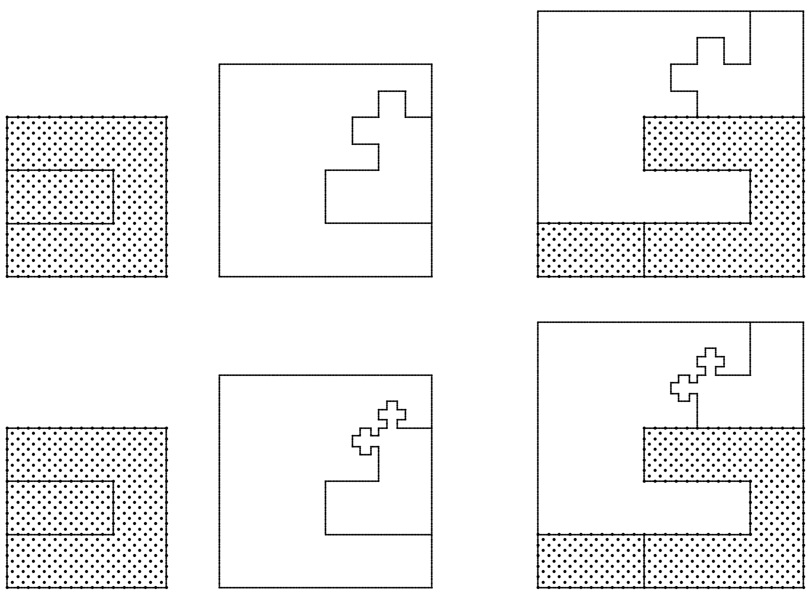
Problem 10.

In each of the three solutions in the diagram below, the yokan slice is not cut.





Remark.
If rotations and reflections are permitted, there are infinitely many solutions, obtained by modifying the solution given in the statement of the problem. The diagram below shows two of them.



Problem 11.

A minimum of four copies of the plate would be required. The mochi slices are arranged as shown in the diagram below on the left, and the covering yokan slices are arranged as shown in the diagram below on the right.