

CHAPTER 3

Growing Up

Young Hassler's formative years, with stories

3.1 Hassler's first bit of research

Hass learned to multiply numbers when he was about eight or nine. At one point the teacher gave a hint about memorizing the multiplication table: Whenever you multiply by 9, the answer's digits always add up to 9. That is, in any of 9, 18, 27, 36, ..., 81, the digits sum to 9. Hassler thought this was amazing.

He began to experiment with 9 times larger numbers such as 9 times 10, 11, 12, 13, ..., and found that summing the digits until you get a single digit still always ended up giving 9. He also tried really big numbers, like 9 times 8213, and the phenomenon held even for those. He thought about this for a few days, and finally convinced himself that it was always true, no matter what big integer you multiplied 9 by. In what can aptly be called "curiosity-driven research," he wondered what would happen if he used 8 instead of 9. When he tested the idea by multiplying 1, 2, 3, ... by 8, he discovered that instead of always summing to 9, the sum decreased. That is, 8×1 gives 8, $8 \times 2 = 16$ gives 7, $8 \times 3 = 24$ gives 6, and so on. He obtained an endlessly-repeating cycle: 8, 7, 6, 5, 4, 3, 2, 1, 9, 8, When multiplying by 7, he got another sequence that went down by 2: 7, 5, 3, 1, 8, 6, 4, 2, 9, 7, Symmetrically, if he multiplied by 10, which is 1 more than 9, the sequence went up by 1: 1, 2, 3, Multiplying by 11 (2 more than 9) gave a sequence that increased by 2. Young Hassler was excited by finding these regularities, and he always regarded this as "my first little bit of mathematical research." Over a half-century later he looked back on the experience as distinctly formative, and that somehow he was like a little toy train car that—perhaps accidentally—got placed on tracks and started moving forward. He felt that schools should do more to set youngsters on little tracks so they could move forward, too, and experience some of those same emotions.



Figure 3.1. Hass with his sister Lisa, shortly after his “first little bit of mathematical research.” Lisa attended Vassar and, like her mother, became an artist.

3.2 Capturing his imagination

During Hassler’s early teens, the magazine *Popular Science* played a large role in his development as a budding scientist. He eagerly awaited each month’s issue and would devour its contents. The magazine became for him an important source of ideas, inspiration, and general science information. Each issue ran around 150 pages; it had well-written articles and a good selection of drawings and photographs. The magazine’s articles on general science, technology, and practical engineering were meant to inform and inspire. Its formula may not have worked for all, but it certainly did for Hassler. Its articles left life-long impressions—they introduced new horizons, different ways of viewing the world, and different ways of thinking.

Some of the articles asked, “What if?” What if we had telescopic eyes that could magnify images a thousandfold or more? Others quantified and compared:

How much horsepower does Jack Frost need to spread frost throughout New York City in one night? Another article entitled “How Big Can They Build Them?” talked about building airplanes larger than theory says is possible and exposed an error in applying theory. Other articles connected the unlikely, like a searchlight so powerful that it melts lead. Many of the mental templates used in the articles would informally resurface in his own conversation decades later.

Here are some short condensations of the sort of articles that he read.

Popular Science June 1920

If the Eye Were a Telescope



The magnificent cluster of stars in Hercules, which appears merely as a speck of light, but which to the eye of telescopic power would resemble a bursting rocket. Suns of many colors are in this swarm.

If the Eye Were a Telescope (June, 1920). If the night is dark and very clear, we are able see some 3,000 stars, ranging from bright jewels to just barely discernable flecks of light. Some of those flecks hold great surprises, one example being the Hercules Cluster. If we had telescopic eyes that could magnify a

thousand times, we would discover that this almost vanishingly small bit of light is actually a huge globe made up of a vast swarm of stars—many suns like ours, and countless others that are hundreds of times larger. No two suns would have exactly the same temperature, so no two would have exactly the same color. With telescopic eyes, we could also see Saturn with its beautiful rings. If we constantly turn up the telescopic power so our field of vision sees ever tinier portions of the planet, our eyes receive fewer and fewer photons, and what we see eventually fades away to nothingness! And our moon? Seeing it rising would be an unforgettable experience: A gigantic disk majestically rises, and in a few minutes a full moon would nearly fill the sky. It would be hard to believe that such an enormous body could long remain in the sky and not fall toward us. Craters that seemed so small and insignificant to our naked eye become huge and menacing. We could also clearly see the red planet Mars, and during some months of the year we could witness its glistening white icecaps slowly melting away as weeks pass, only to reappear months later.



Rings of dust and meteorites surround the globe Saturn. To see them without a telescope would be one of the greatest sights permitted the eye of man.

If Jack Frost Could Be Put to Work (January, 1921). Jack Frost's job is to make ice. If we look at what he does in New York City alone, his gigantic ice-plant there uses energy 300,000 times faster than man's combined energy use throughout the entire world! The key to making this comparison is using a *unit of energy*, one example being the British Thermal Unit, or BTU. It takes 144 BTUs to melt one pound of ice at 32 degrees. That energy could come from the sun, from a fire, from surrounding water or air. No matter, it's always 144 BTUs that are needed. The reverse is true, too. A perfectly efficient refrigerator needs 144 BTUs of energy to turn one pound of water at 32 degrees into ice. Jack Frost, on the other hand, works on a far grander scale. In 1918, there were four days in which the amount of rain on New York City totaled 1 inch—quite a bit of water because the area of New York is the same as a square 17.5 miles on each side. Think of the water as being one inch deep in a huge pan. Now Jack Frost can easily freeze this pan of water during just a single night. The energy his ice-plant uses for this is staggering—64 trillion BTUs! In making the ice, where does all the energy removed from the pan of water go? Although you might not realize it, that removed energy heats up the surrounding atmosphere. That is, the air warms up somewhat as the water freezes. Jack does his job overnight, so that tells us the *rate* of energy use. If a hundred-pound weight is lifted five and a half feet in one second, that's one horsepower, or 1 HP—a very respectable feat for a well-conditioned man. An automobile with a 90 HP engine is therefore quite strong. An actual calculation shows that Jack Frost, doing an overnight rush job on his pan of water, works at the rate of nearly a *trillion* horsepower—more exactly, 916,403,891,235 HP. That would be equivalent to more than 10 billion 90 HP automobiles using all their capacity to freeze water. All the automobiles in the world don't add up to even a tiny fraction of 10 billion!

How Big Can They Build Them? (October, 1920) During the war [WWI], our government desperately required larger aeroplanes, and aeronautical engineers were instructed to design larger craft, no matter what limitations theory put on size. This at once pitted two huge forces against each other: theory versus practice. Theory says that if you double the size of an aeroplane, increasing every dimension by a factor of two, then the surface area—basic to wing surface and lifting ability—goes up as the square, that is, by $2^2 = 4$, but mass, just like volume, goes up as the *cube*, this often being called the “Law of the Cube.”¹ That means doubling the size increases the weight by a factor of $2^3 = 8$. Such a rapid increase in weight puts a serious restriction on aeroplane size, because lifting ability can't keep up with the faster increase in weight. But despite this law, today giant machines of the type the government envisioned now routinely “wet their wings in the misty clouds.” What happened? Where did theory go wrong? The answer is that in applying the Law of the Cube, the theoreticians made an unwarranted assumption. The Law of the Cube applies when the larger model is

an exact replica of the original. That means the shape is exactly the same, and all the materials used are the same, too. But in the practical world, we don't have to make these assumptions! We can use lighter materials and change the shape of the aeroplane to make it more streamlined. That's what the engineers did. They succeeded, with the result that they became more reliable, could carry heavier loads, and fly longer distances. We learned that with theory, we need to keep the underlying assumptions in mind, and that in the real world those assumptions may not always be necessary.

Why Does Iron Rust? (June, 1920) Of all the metals in the world, iron is the most used and the most useful. In fact without it, civilization as we know it would not be possible. But if not protected, rust can reduce any piece of iron to a useless powder, ruining the largest steel bridge or the mightiest ship. But exactly how does rust occur? What is actually needed to form rust? Iron is a pure element, while rust is a chemical combination of iron and oxygen. There can be varying amounts of oxygen in rust, making different varieties of it, and there can also be other elements involved such as hydrogen or carbon. To begin the rusting process, we need both oxygen and water. Now dry air contains oxygen but no water, so iron won't rust in dry air. A piece of iron could lie in a dry desert for years and not get rusty! At the opposite extreme, pure water with no oxygen dissolved in it will not rust iron, either. Ordinary water from a lake or stream contains oxygen, and fish depend on it to live. But if you drop a piece of iron into a capped jar full of water that has been distilled to remove all oxygen, and if you leave no air pocket at the top, then that iron would remain rust-free indefinitely. One way to prevent rust is to coat the iron with a good barrier against water and oxygen, such as a high quality paint. Another way that has proved successful is to build a very large tank with many layers of thin, perforated iron and slowly pass water through these layers. These layers are there exactly to get rusty, and are replaceable. Making rust requires oxygen, so dissolved oxygen in the entering water gets used up as the water passes through the layers of iron, rusting it on the way. The rusting process takes time, but with a very big tank, water can easily stay in the tank for a day or two, and when it finally does leave the tank, it has very little oxygen left in it. So oxygen in the entering water rusts the replaceable layers of perforated iron instead of the iron pipes in your home!

So Powerful Is This Searchlight that It Melts Lead (June, 1920). How does a searchlight work? When current passes through a conductor like a piece of wire, it heats the wire somewhat, and if the resistance is greater, it heats it up more. The tungsten filament in the newer light bulbs has a great deal of resistance, and passing house current through the filament heats it so much that it becomes white hot, giving off light. When you turn on a searchlight, the "filament" starts off as two touching carbon rods that vaporize some carbon where

they touch. The rods are gradually separated up to about four inches, the arc between their ends still conducting current and giving off a tremendous amount of light—much more than a tungsten filament. As the carbon vaporizes the gap between the rods increases, so the rods must be continually slowly moved together to keep the separation at only four inches. The intense light rays emanate from the arc in all directions, and these are reflected off the searchlight's parabolic mirror to create a straight beam of light. This beam is so intense that objects even miles away can be illuminated or "searched for." Early models generated nearly six million candlepower, equivalent to about six million candles all burning at once, but an inventor, Mr. Elmer Sperry, has improved the design by more precisely aligning the two rods, and in addition making them rotate along their long axis so the carbon is burned off more evenly. He now has a searchlight generating over a *billion* candlepower. If you place a bar of lead in the beam's path 12 feet away from the arc, the beam will melt the lead!

35 miles. In May of 1921, 14-year-old Hassler traveled to Switzerland with his mother and younger sister Lisa for a two-year stay. His brother Roger was already there, and the three other siblings, all older, were by that time living independently. Hassler kept a diary to record events, feelings, and impressions during the exciting trip. He noted facts, numbers, comparisons, and conversions. He was tremendously fascinated with the awesome steamship *Orbita* that carried them across the Atlantic, and his diary provides a look at how his mind worked. He notes that the ship was "over 550 feet long and nearly 70 feet wide, weighed over 15,500 gross tons, and its engines generated some 18,000 horsepower. The engine room was 30 feet below the water, and there was a huge staircase going from a glass-topped ceiling down to the bottom of the engine room." In a letter to his aunts dated May 29, 1921, he repeats this information and includes a detailed drawing of the cross-section of the staircase and engine room, reporting that he made the trip all the way to the bottom, where it was unbelievably hot. Back up on deck, he calculates that it would take a line of horses 35 miles long to generate the same power that the ship's engines do. He learns that the ship uses 120 tons of fresh water each day, and converts that, saying it would take a person 250 years to drink that much water. He is taking large numbers and converting them into dramatic visual terms.

3.3 Life in Switzerland

Hassler's mother Josepha believed it was important for her children to have exposure to the wider world and see other ways of life. As for Hassler's education, the idea was that he would learn French the first year, German the second, and continue his piano lessons both years. Hass did in fact become fluent in both languages and once commented that at one time he spoke French better than

English. But the Hassler you see in the passport photo in Figure 3.2 had another, far more intense goal. He dreamed of climbing the beautiful Swiss Alps with his brother Roger, and he was already absorbing as much information as he could about them from as many sources as he could find.



Figure 3.2. Passport photo with Josepha and sister Lisa, when Hass was 14. Hass was headed to Switzerland for a two-year stay, and at the time of this photograph was eagerly looking forward to climbing in the Alps.

He had carefully studied other climbers' routes, the kinds of difficulties each route posed, and various reports of various climbers' experiences. Hassler's mother, always supportive, made an important contact: Gaston Clerc in Switzerland was an experienced mountain climbing coach who thoroughly "knew the ropes." It was arranged for him to give both Hassler and Roger climbing lessons. He also gave Hass French lessons the first year. The contact was especially fortuitous because Clerc's wife was an accomplished pianist and give Hassler piano lessons during his two years in Switzerland.

3.4 Some math in Switzerland

From early on, Hass was attracted to numbers, quantity, and measuring things. Letters from his teenage years were filled with references to the depth of snowfall, the time it took to hike a certain route, and the altitude of various mountains. He



Figure 3.3. Here's Hass, above his coach Gaston Clerc. This candid photo was taken shortly after Hass arrived in Switzerland.

kept a dairy and faithfully noted how long he slept each night. Numbers and measurement were very central to his nature. It is no surprise, then, that when his sister Caroline gave Hass a slide rule as a going-away gift before he left for Switzerland, the rule seemed absolutely magical to him. How could its sliders with numbers marked on them actually multiply numbers? The instructions that came with the slide rule were clear about how to use it, but not at all clear about why it worked. The instructions did mention logarithms, and he knew the most basic thing about them: If x and y are two numbers, then

$$\log xy = \log x + \log y.$$

He eventually concluded that the x and y must be the actual numbers printed on the sliders, and that $\log x$ and $\log y$ must be physical distances. Because the total length of two straight sticks abutting each other is the sum of the sticks' lengths, that must correspond to the "+" in $\log x + \log y$. An example of his basic logic is depicted in Figure 3.4: To multiply 2 times 3, put the left edge of the top slider

over the “2” on the bottom scale. As the picture suggests, the physical distance from “1” to “2” is $\log 2$. In the same way, the physical distance from “1” to “3” is $\log 3$, and by moving the slider so as to effectively abut two sticks, their total length is $\log 2 + \log 3$. According to the basic log formula, this length $\log 2 + \log 3$ is also $\log(2 \cdot 3)$, or $\log 6$, and any printed number x is at distance $\log x$ from the beginning. So it’s the printed “6”—the answer—that’s at that distance $\log 6$.

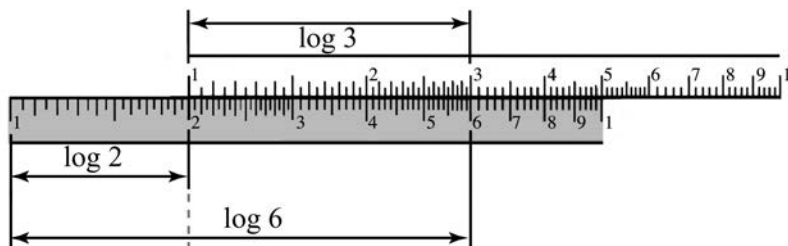


Figure 3.4. Multiplying 2 times 3 using a slide rule.

log 2

Once in Switzerland, his curiosity about the slide rule led to further thought, and his understanding of it began to grow. He realized that the results are never perfectly accurate, because you can see only so closely, and besides that, the scale lines have thickness to them—the magic rule has limitations! Could *he*, Hassler, do better than the slide rule? He decided that accurately finding $\log 2$ would be an excellent test of his own powers. According to the slide rule, it should be a little over 0.3, but how much over? Was there some way to find the value exactly, getting a better answer than his slide rule could? He cooked up a plan of attack. If you multiply 2 by itself a bunch of times to get 2^n , then the logarithm of that is $n \log 2$, while if you multiply 10 by itself a bunch of times to get 10^m , that has a very simple logarithm, just m . His strategy was to keep doubling 2 until you get some 10^m . At that point, $n \log 2 = m$ means that $\log 2 = m/n$. But as he doubled 2 over and over again, he realized that the last digit cycles on forever as 2, 4, 8, 6, 2, 4, 8, 6, ... On the other hand, any 10^m ends with all zeros, so his plan hit a wall. He certainly didn’t know this observation essentially proves that $\log 2$ is irrational! Seeing that he couldn’t obtain an exact answer, he backtracked to getting a good approximation. He got somewhat close when n is 10, 20, 30, 40, but after filling page after page with powers of 2 and always missing his goal of beating the slide rule, he finally gave up. He really tangled with this problem and gained a lot of respect for it. Increasingly things were happening in math that seemed mysterious to him.

He frequently wrote to Caroline about such matters. She was his favorite sounding board as his imagination roamed around numbers and other mathematical ideas. In one letter to her he mused: “What are the logarithms of negative numbers? Do they have any relation to imaginary numbers? If they haven’t, I should think they would be pretty useful anyway.” Later, he began to wonder why certain graphs have “excluded regions.” Even the parts to the left and right of a circle’s plot are excluded, aren’t they? Is it because of imaginary numbers, again?² He was meeting more and more mysteries, and his subconscious was becoming prepared for answers.

Graphs and a bug. It delighted Hass to see that abstract combinations of symbols as in polynomials could be pictured geometrically. He would make up a polynomial and use his slide rule to compute points on its graph, then connect the points to get the full graph. When he stumbled upon $y = 1/x$, something weird happened—the graph (a hyperbola) broke up into two parts. Could he make up things to plot that broke up into *three* parts? Or even more parts? In a long letter to Caroline written over a few days, he chronicled his progress in this, and his understanding was clearly growing. A little later he stumbled upon a different idea: Replace y by y^2 . Now, suddenly, the graph is symmetric about the x -axis. In exploring this idea, he happened to plot $y^2 = x^3 + x^2$, and to his surprise, he found it looked like a bug’s head together with two antennae coming out! He’d been fascinated by bugs and crawly things for some time, and now an equation could replicate some of that. He was on a roll, and wanted some way to give the bug a body. He eventually realized that if you plot $f(x, y) = 0$ and $g(x, y) = 0$, then the union of the two plots is given by the plot of $fg = 0$. So by appropriately multiplying equations, he was able to add an ellipse-shaped body to the head, as shown in Figure 3.5. Using this same idea, he could add small circles or ellipses to represent dots of the kind you see on a ladybug.

3.5 Mostly a handbook

Hass returned to the States in 1923 and had just one more year before college. His hankering for measuring things now extended to finding lengths of various curves. By this time he knew that calculus could help in finding lengths, areas, and volumes. He owned a calculus book, but never had the slightest inclination to sit down and read it or do exercises in it. For him, the book was a resource, a handbook that he could use to help solve problems that interested him. One such problem he thought about was this: “If an ant crawls along one arch of the sine curve $y = \sin \pi x$, how far has it traveled?” After setting up the correct integral, he found that nowhere in the table of integrals at the back of the book was there anything that could solve the problem for him. He ended up with an approximation, using his slide rule to get the lengths of small pieces of the curve

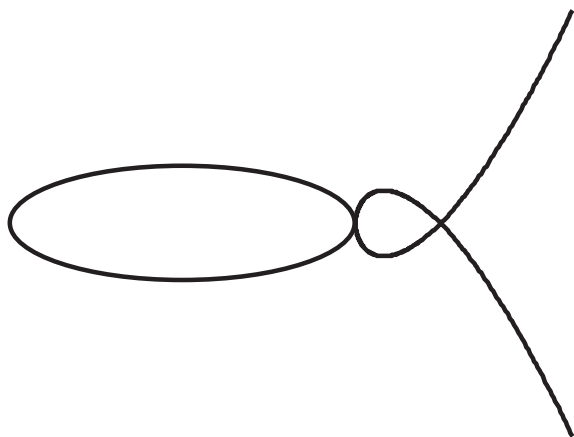


Figure 3.5. Through experimentation, Hass found that if he chose polynomials $f = y^2 - (x^3 + x^2)$ and $g = (x+3)^2 + (4y)^2 - 4$, then $fg = 0$ gave him this “bug,” made up of an alpha-shaped curve for its head and antennae, together with an ellipse for the body. The bug’s equation is therefore the product of f and g , $[y^2 - (x^3 + x^2)][(x + 3)^2 + (4y)^2 - 4] = 0$.

and adding them up. Experiences of this sort made him quite a bit wiser than students who would read about arc length, do the book’s problems, get the right answers, take a test and receive an “A”. He once said, “They come away thinking they can find arc lengths! Gee, the book carefully avoids problems where things don’t work out, so the students see only those rare examples where things can be solved nicely. They can go the rest of their life having a totally unrealistic view.”



Figure 3.6. Hass shortly before entering Yale.

CHAPTER 12

Whitney's Extension Theorems

When Whitney began publishing mathematics, it was already well known that any function continuous on a closed subset of real n -space can be extended to a continuous function on the entire space. Whitney wondered about differentiable analogs. What if “continuous” were replaced by “differentiable” or “infinitely differentiable”? His answers played a significant role in his becoming a widely known mathematician.

12.1 Searching for that piece of gold

Despite his love for and progress on the four-color problem, Hass never intended to spend his entire career on it. In him was a need to do something he thought was “real mathematics” in the sense of more traditional analysis—something that would involve differentiable functions. After all, he had spent years in physics working with polynomials, trigonometric and exponential functions, and so on, and they were his friends. During this period around 1933–4 he was spending more and more time thinking about differentiable functions in one or several variables and less time on the four-color problem. He began to gain a deeper understanding of real functions and their derivatives and started to keep an eye out for some problem involving differentiable functions whose resolution would be important. It had to be elementary, simply stated, require little extra background to make progress, and involve differentiable functions in some way.

His quarry proved surprisingly elusive: His search was a bit analogous to a beachcomber with a metal detector, except that he found the beach to be much larger than he expected, and it took months before he finally stumbled upon what he wanted: an eight-page article written by William Whyburn ([W]). This article, “Non-isolated critical points of functions,” appeared in the *Bulletin of the AMS* 35 (1929), pp. 701–708 ([W]) and guarantees, for a closed set S in n -space, nonconstant functions having zero partial derivatives at each point of S . The methods used in the paper gave Hass the idea that he might be able to extend the Tietze

extension theorem for continuous functions to certain differentiable ones. In real n -space \mathbb{R}^n , the Tietze theorem says, in part, that a function continuous on a closed set in \mathbb{R}^n has a continuous extension to all of \mathbb{R}^n . The paper inspired Hassler to ask more general extension questions. Here are two of them:

- If in some sense a function is r -times differentiable on any closed set S of \mathbb{R}^n , can it be extended to a function that is r -times differentiable on all of \mathbb{R}^n ?
- If the function is infinitely differentiable on S , is there an infinitely differentiable extension to \mathbb{R}^n ?

Answers would be significant no matter how they turned out, and in what can now be called classic Whitney style he considered lots of examples, drew many pictures, considered various ways of creating extensions, and through persistence finally got local extensions to match up properly. It was a breakthrough for him, and a breakthrough for mathematics. He admitted “It wasn’t that easy,” but in 1934 his 27-page article appeared in the *AMS Transactions* ([44]). Two more articles in the *Annals* ([45], [46]) continued in this vein. 1934 was a good year for the 27-year-old.

It was Whitney’s style to think, argue, and explore geometrically, but when writing up his results he typically translated the math into an algebraic form with little reference to his original, intuitive pictures. This was in part because he submitted articles to journals where it was essential that his arguments be highly rigorous. It was in the algebraic setting where he succeeded in making algebraic arguments having an appropriate level of rigor; in fact, in all the 27 pages of the *Transactions* paper, there is not a single figure. Although he adopted this method for most of his papers, in private conversations he was quite different, revealing his vividly geometric mind. In this chapter we explore the problem in the way that he typically would have, using concrete examples to progressively lead to a solution. Along the way we will meet some of the essential difficulties he faced and how he overcame them, and will give the reader an appreciation of his early blockbuster result.

12.2 First questions and examples

Let’s begin with a very simple example of extending a continuous function—say, a function continuous on the closed subset $S = [0, 1]$ of the line \mathbb{R}^1 and taking on real values. Here, “continuous” means we can draw its graph without the pencil ever leaving the paper. To extend it leftward, place the pencil point on the leftmost point of the graph (which exists because $[0, 1]$ is closed) and just drag further and further left without ever lifting the pencil. In fact, drawing a nonvertical half line would do just fine. Similarly for extending it rightward. If the closed set S consists of two intervals separated by a gap, then one could span the graph’s gap

with a line segment connecting the two graph endpoints. It's easy to see how to extend this idea to other closed subsets of the real line.

What if the function on S is differentiable there, and we want to extend it to a function on \mathbb{R}^1 differentiable at each point of \mathbb{R}^1 ? To start, let's consider just a first-derivative problem. On a single interval such as $[0, 1]$, we agree that "differentiable at an endpoint" means the appropriate one-sided derivative exists there.¹ To extend the function f leftward from 0, we may choose a half line whose slope agrees with the right-sided derivative of f there. Likewise, to extend the function rightward from 1, we may choose a half line whose slope agrees with the left-sided derivative there. But if S consists of more than one piece—say, the two closed intervals $[-1, 0]$ and $[1, 2]$ —the problem suddenly becomes less trivial. Figure 12.1 shows such an example. We now need to make the left and right slopes at $x = 0$ match up, and ditto at $x = 1$. It's clear that if the extended function is to be differentiable on the whole horizontal axis, line segments are not up to the job.

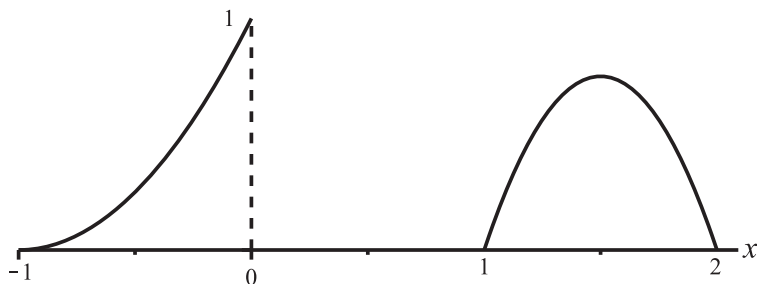


Figure 12.1. How can you differentially extend this function on $[-1, 0] \cup [1, 2]$ to all of \mathbb{R} ?

In Figure 12.1, let's say that above the interval $[-1, 0]$ is a section of the parabola $y = (x + 1)^2$, which has slope $+2$ at $x = 0$, and that above $[1, 2]$ is part of another parabola, $y = -3x^2 + 9x - 6$, with slope $+3$ at $x = 1$. One way to solve this problem is by finding a polynomial $p(x)$ for which

$$\begin{aligned} p(0) &= 1; & p'(0) &= 2; \\ p(1) &= 0; & p'(1) &= 3. \end{aligned}$$

The general cubic $p(x) = ax^3 + bx^2 + cx + d$ is a polynomial with four coefficients, and each of these coefficients can be assigned arbitrary values. In this case, $p(0) = 1$ becomes $d = 1$ and $p'(0) = 2$ becomes $c = 2$. Substituting $x = 1$ then leads to simultaneous equations

$$a + b = -3, \quad 3a + 2b = 1,$$

having solutions $a = 7$ and $b = -10$. Thus, $p(x) = 7x^3 - 10x^2 + 2x + 1$ defined on $[0, 1]$ connects the two parabolic pieces, with its values and slopes agreeing at the glue joints. To the left of $x = 0$ and the right of $x = 1$, we can extend by simply using half lines of the correct slopes. The result appears in Figure 12.2, with the heavily drawn additions giving a function on \mathbb{R} differentiable at each point there, and extending the original function on $S = [-1, 0] \cup [1, 2]$.

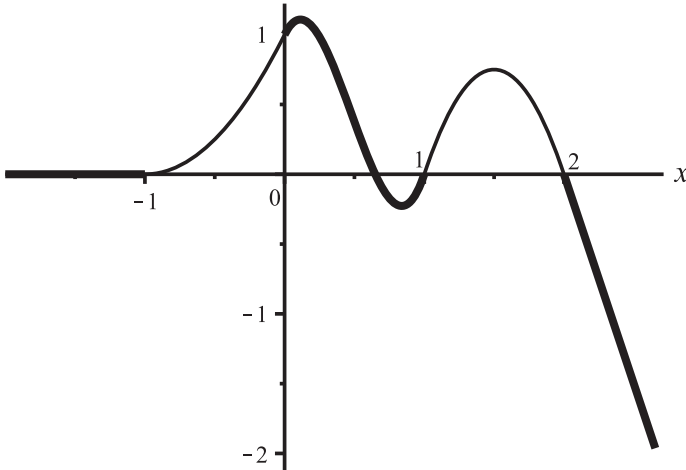


Figure 12.2. The heavily drawn parts extend the function in Figure 12.1 to a function differentiable on all of \mathbb{R} .

This general approach can be used for higher derivatives, as well as for any finite number of closed intervals. If the two parabolic pieces in Figure 12.1 were replaced by arbitrary ten-times differentiable functions—that is, members of \mathcal{C}^{10} —then for perfect matching there would be 11 conditions at $x = 0$ reflecting values of the 0th through the 10th derivatives there, and another 11 at $x = 1$. These lead to a system of 22 linear equations in 22 variables, those variables being the coefficients of a general polynomial of degree 21. And if there were additional gaps to bridge, there would correspondingly be additional 22-variable systems, one for each gap. All this adds up to considerable work, but in theory it's possible to make such extensions.

So far we've been content to match up only finitely many derivatives. But more generally, there's the question of extending pieces of analytic functions like trigonometric or exponential ones. These entail matching up not a finite number of derivatives, but *infinitely* many, and the method outlined above fails. At this point a really huge problem rears its head: A simple example shows we can't generally bridge gaps between analytic pieces with other analytic pieces because, for example, if you remove part of a cosine curve, there is essentially only one

way to extend it analytically—put the removed part back where it was originally. In Figure 12.3, a cosine curve has been cut at $x = 0$ and the right part pushed rightward by one unit. The heavily drawn part depicts a function that does bridge the gap, with all the infinitely many derivatives matching up at the glue joints. This bridging function must actually be defined at the glue joints so that we can take derivatives there. It turns out that the function whose graph is depicted in Figure 12.3 isn't analytic at either of the glue points. A promise: Before the end of this chapter, we'll tell you what the bridging function is, and how we got it.

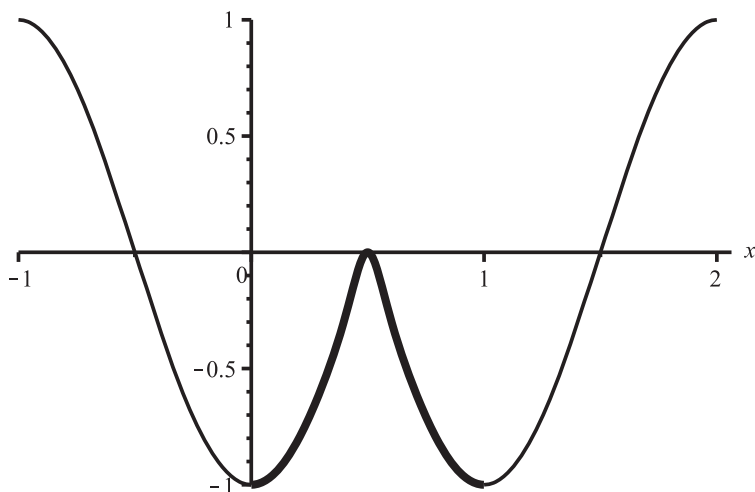


Figure 12.3. The heavily drawn curve represents one possible smooth bridging between the parts of a split cosine curve. Connecting the two lowest points of the split cosine curve with a horizontal line fails to give a smooth extension.

Except in very special cases, bridging analytic pieces requires functions that are infinitely differentiable, yet not analytic. These are the C^∞ functions. Figure 12.4 depicts that polynomials are more special than analytic functions, which are in turn more special than C^∞ functions. The class of analytic functions \mathcal{A} is a subset of C^∞ . For a function f to be analytic, the Taylor series² of an analytic function f formed from the derivatives evaluated at a point must be f itself, not some other function. The class of polynomials shown in the figure as \mathcal{P} is a subset of \mathcal{A} . Polynomials are not only analytic, but all sufficiently high derivatives must be 0. C^∞ functions are also called *smooth*, so all functions in Figure 12.4 are smooth.

A fair question: Is the set C^∞ definitely larger than \mathcal{A} ? That is, is there at least one infinitely differentiable function f whose Taylor series defines a function g

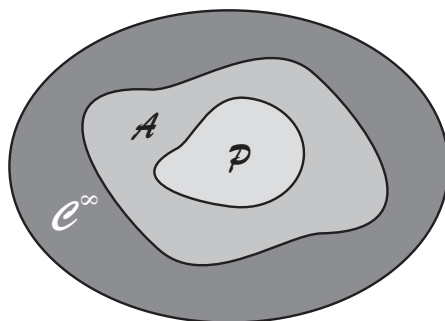


Figure 12.4. Three levels of generality for functions.

not equal to f ? Yes. We are about to meet a famous function that is analytic at each point of the x -axis except $x = 0$. It is C^∞ there—all of its derivatives exist (they're all 0 at $x = 0$). But with all derivatives being zero, its Taylor series is identically zero. The series doesn't represent this famous function because the function increases as $|x|$ grows and even has $y = 1$ as an asymptote. The function is

$$y = e^{-1/x^2} \text{ if } x \neq 0; \quad y = 0 \text{ if } x = 0.$$

Figure 12.5 shows two views of this famous function.

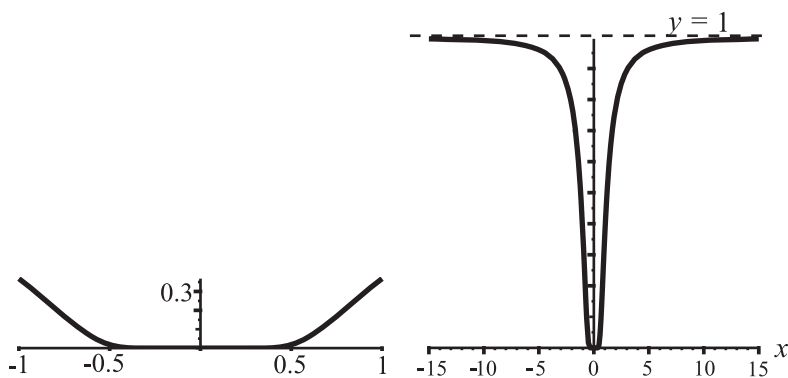


Figure 12.5. Zoomed-in and zoomed-out views of an infinitely differentiable but nonanalytic function. At $x = 0$, the function's derivatives are all zero. Therefore its Taylor series about $x = 0$ fails to represent the function because the Taylor series is $0 + 0x + 0x^2 + \dots$ —the identically zero function.

Not only have we found one function in C^∞ not in \mathcal{A} —that is, a smooth non-analytic function—we've in fact found infinitely many of them, since $y = e^{-1/x^2}$

can be multiplied by any nonzero real number to get a different function in \mathcal{C}^∞ that's not in \mathcal{A} . You can get even more of them by simply adding any analytic function you please! Thus, nonanalytic \mathcal{C}^∞ functions represent a rich source of bridging functions, and we'll soon see that there are so many of them that we can bridge gaps between polynomial, analytic, or \mathcal{C}^∞ functions.

As one more example, let $f(x)$ be e^x to the right of $x = 1$, and let $g(x)$ be $\sin(2\pi x)$ to the left of $x = 0$. The heavily drawn curve in Figure 12.6 depicts the graph of one \mathcal{C}^∞ -function bridging the gap between f and g . Another promise: Later in this chapter we'll get the graph's actual function.

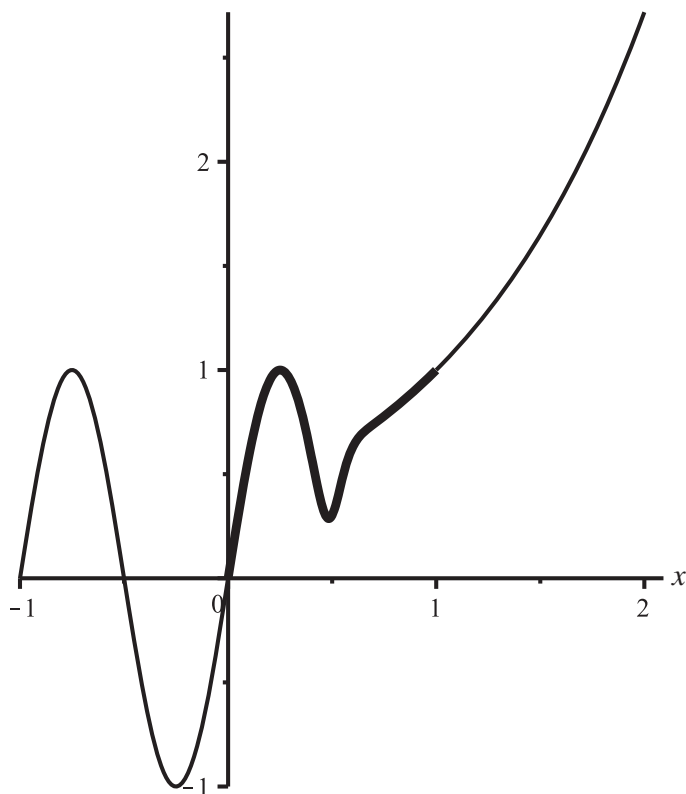


Figure 12.6. A smooth extension of a sine curve to the left of $x = 0$ and an exponential curve to the right of $x = 1$.

12.3 Smooth versus just *looking* smooth

We can turn the graph in Figure 12.2 into one looking even smoother—just replace its straight-line extensions by the natural parabolic ones, to get the graph in

Figure 12.7. This entire graph looks pleasingly smooth. However in mathematics, smoothness is not just skin deep—it goes all the way to the bone. That is, a function can look nice and smooth to the eye, but have nonsmoothness secretly hiding in it like skeletons in a closet. You may need to open many doors to reach the closet, and opening a door mathematically corresponds to differentiating. So a smooth function is one that, no matter how many times it's differentiated, still looks smooth. That is, it is a member of C^∞ . So what about the function depicted in Figure 12.7? If it is in fact smooth, then all right and left derivatives at $x = 0$ must agree; ditto at $x = 1$. Let's compute: At $x = 0$, the second left derivative of $(x + 1)^2$ is 2, while the second right derivative of the bridging function $7x^3 - 10x^2 + 2x + 1$ is -20 . That means our nice-looking function is not smooth at all! Although it no longer matters, the situation at $x = 1$ isn't any better, since the second one-sided derivatives are 22 and -6 .

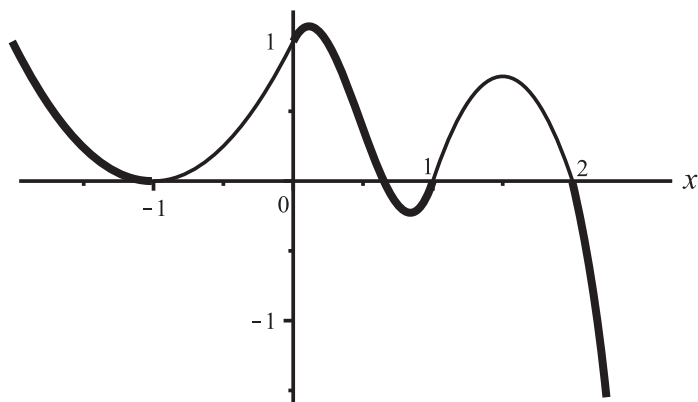


Figure 12.7. The graph in Figure 12.2 looks even smoother when the straight lines are replaced by natural parabolic extensions.

As for Whitney's first bulleted question on p. 144, our foray into extending one-variable polynomial pieces does teach us something: These polynomials are surely differentiable, and we can extend them to piecewise polynomial functions that are also differentiable on \mathbb{R} . However, polynomials are not only differentiable, but infinitely differentiable—smooth—and we've just seen that in the above example our polynomial bridging function doesn't give a function smooth on \mathbb{R} . It turns out that polynomial bridgings are typically not good enough to give smooth extensions of polynomially defined pieces. Even at the polynomial level, we need functions sharing the DNA of $y = e^{-1/x^2}$. Functions of this type appear throughout Whitney's big papers on extensions ([44], [45], [46]), and we'll

show how he used them to extend functions C^∞ on closed sets of a real space, to a function C^∞ on all of that real space.

12.4 Extending infinitely differentiable functions

We've seen how different polynomial graphs on disjoint intervals of the x -axis can be patched together, usually with polynomials of higher degree, to form a function on the whole x -axis that at least looks nice at the glue joints. We have also seen that the function

$$y = e^{-1/x^2} \text{ if } x \neq 0; \quad y = 0 \text{ if } x = 0$$

whose graph is depicted in Figure 12.5 can be used as a smooth function to patch together one-variable differentiable functions more general than polynomials. To see the role this function can play in extending more general smooth functions, let's look at one of the most simply stated and basic cases where this function is used:

Extend the function that is identically 0 to the left of $x = 0$, and identically 1 to the right of $x = 1$, to a function that is infinitely differentiable on the entire x -axis.

For this step function problem, the challenge is finding a function on the closed interval $[0, 1]$ so that all three pieces form the graph of a function infinitely differentiable at every point of the x -axis. One thing is clear: The C^∞ bridging function can't be analytic throughout $[0, 1]$. If it were, it would continue the function that is zero for all negative x , meaning the function would be identically 0 on the entire x -axis, while it would also continue the function constantly 1 to the right of $x = 1$. It can't do both!

Our exponential function displayed above goes about half way toward solving the problem, but the left picture in Figure 12.5 shows that it rises to only about 0.3 when $x = 1$ —too slowly for what we need. But by tinkering a bit in the denominator, we can make the function do what we want: Divide e^{-1/x^2} by $e^{-1/x^2} + e^{-1/(x-1)^2}$ to get

$$y = \frac{e^{-1/x^2}}{e^{-1/x^2} + e^{-1/(x-1)^2}}.$$

When x gets very close to 0, the denominator approaches something nonzero and finite, so the whole quotient approaches 0 essentially as if the denominator were the original value of 1. When x gets very close to 1, the denominator approaches $1/e$, and so does the numerator, so the whole quotient approaches 1 as x approaches 1. Figure 12.8 depicts this function's graph rising from $y = 0$ to $y = 1$ as x goes from 0 to 1.

All derivatives at $x = 0$ remain 0 and fortunately so do all derivatives at $x = 1$. Why is that? The graph appears to be symmetric about its middle point. If

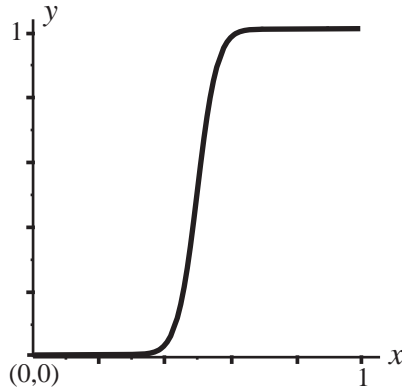


Figure 12.8. The ramp function $y = \frac{e^{-1/x^2}}{e^{-1/x^2} + e^{-1/(x-1)^2}}$ is \mathcal{C}^∞ and is an especially useful smooth function.

we can establish that it really is symmetric, then the identical shapes of the graph at each end will tell us that its derivatives at $x = 0$ and $x = 1$ are the same. Here's the proof of symmetry: Replace each x by $(x + \frac{1}{2})$, and y by $(y + \frac{1}{2})$; doing this translates the graph down by $\frac{1}{2}$ and left by $\frac{1}{2}$,³ to the graph of a function $y = f(x)$. Its graph is symmetric about the origin if and only if $f(x) = -f(-x)$, which is to say if and only if $f(x) + f(-x) = 0$. (Any such function is called *odd*.⁴) Making these replacements and simplifying in fact lead to $f(x) + f(-x) = 1 - \frac{1}{2} - \frac{1}{2} = 0$, establishing symmetry.

The function depicted in Figure 12.8 is not analytic at either $x=0$ or $x=1$, but is infinitely differentiable at every point on the x -axis. This solves our basic step problem of extending the function to a \mathcal{C}^∞ (smooth) function that is identically 0 to the left of $x = 0$ and identically 1 to the right of $x = 1$. As a nice bonus, this extension is actually analytic in the interior $(0, 1)$ of the gap. We denote this extended function on \mathbb{R} by $\phi_{0,1}(x)$.

Symmetrically, we get, almost for free, a function $\phi_{1,0}(x)$ on \mathbb{R} that decreases from $y = 1$ to $y = 0$ as x goes from 0 to 1:

$$\phi_{1,0}(x) = 1 - \phi_{0,1}(x).$$

Figure 12.9 depicts the parts of these graphs between $x = 0$ and $x = 1$; the graph of $\phi_{0,1}(x)$ is drawn solidly, the graph of $\phi_{1,0}(x)$ is drawn dashed. By the way $\phi_{1,0}(x)$ is defined, we see that

$$\phi_{0,1}(x) + \phi_{1,0}(x) \equiv 1.$$

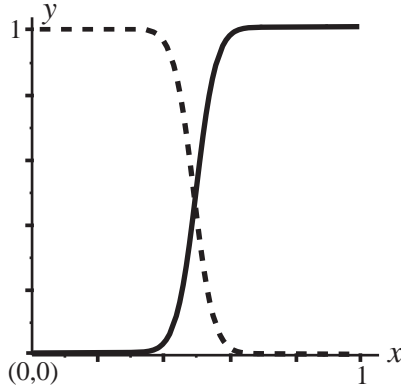


Figure 12.9. The graphs of the C^∞ ramp functions $\phi_{0,1}(x)$ and $\phi_{1,0}(x)$ sum to the function that is identically 1.

12.5 Exponential bridging functions in action

Can we use these two C^∞ functions to solve other bridging problems? The answer is not only yes, but *emphatically* yes. As an example, let's dress up the above step problem by replacing the function that's identically 1 to the right of $x = 1$ by $y = e^x$ to the right of $x = 1$. Now the challenge is to find some function that smoothly bridges the function identically 0 to the left of $x = 0$ and our exponential function to the right of $x = 1$. The answer is

$$y = e^x \phi_{0,1}(x),$$

and the way it solves the problem is almost magical. Let's take a look:

When $x \leq 0$,

$$e^x \phi_{0,1}(x) = e^x \cdot 0 = 0.$$

When $x \geq 1$,

$$e^x \phi_{0,1}(x) = e^x \cdot 1 = e^x.$$

So $y = e^x \phi_{0,1}(x)$ correctly matches up the values at the glue joints $x = 0$ and $x = 1$. But what about matching up all *derivatives* at those two glue joints? Let's compute. For the first derivative, Leibniz's rule

$$(fg)' = f'g + fg'$$

gives

$$(e^x \phi_{0,1}(x))' = (e^x)' \phi_{0,1}(x) + e^x (\phi_{0,1}(x))'$$

which at $x = 0$ is

$$1 \cdot 0 + 1 \cdot 0 = 0,$$

One can replay the above arguments to see that the mate to this function— $\phi_{1,0}(x)$ —is equally impressive, and with equal ease solves the “mirror” problem. That is, if you have any function $g(x)$ like the ones just mentioned, then $g(x)\phi_{1,0}(x)$ agrees with $g(x)$ to the left of $x = 0$, and morphs in a C^∞ manner to the identically zero function to the right of $x = 1$.

Before meeting Whitney’s famous extension result at the beginning of the next section, we need to make an important reality check. Above, we saw the almost magical behavior of $y = e^x\phi_{0,1}(x)$. But when we multiplied e^x by $\phi_{0,1}(x)$, we tacitly assumed that e^x is defined in $[0, 1]$. Of course it is, but in general all we’re given to start off with is a function defined and everywhere differentiable in the closed set S —in particular, it need not be defined in $\mathbb{R} \setminus S$. In the above example, if instead of an exponential (or polynomial, or sine, or cosine) function, we chose a function $f(x)$ defined only to the right of $x = 1$, and this function is only C^∞ there, then at this stage of the game, there’s nothing to multiply $\phi_{0,1}(x)$ by within the gap $(0, 1)$. But because our C^∞ selection has all orders of right-sided derivatives at $x = 1$, we can use these numbers $f^{(m)}(1)$ to assemble the Taylor series centered at $x = 1$:

$$F(x) = \sum_{m=0}^{\infty} \left(\frac{f^{(m)}(1)}{m!} \right) (x-1)^m.$$

Under mild conditions, this is analytic and defines a nice function in the gap $(0, 1)$. It’s as if we’re standing at shore’s edge $x = 1$ looking toward the left. Without a Taylor series, there’s nothing to see in that gap. A Taylor series gives us something concrete between the two shores.

Analogously, suppose we have a function $g(x)$ defined only to the left of $x = 0$, and that this function is only C^∞ there. All orders of left-sided derivatives are defined at $x = 0$, so we can form the Taylor series centered at this other shore, $x = 0$:

$$G(x) = \sum_{m=0}^{\infty} \left(\frac{g^{(m)}(0)}{m!} \right) x^m,$$

and once again, under mild conditions, this is analytic in the gap $(0,1)$. We therefore have two possibly very different analytic functions in $(0, 1)$. We would like to gracefully morph one into the other.

12.6 Taming the clash of the titans

Here we go. Let $g(x)$ be any C^∞ function defined on $S_1 = (-\infty, 0]$ and suppose $f(x)$ is any C^∞ function defined on $S_2 = [1, +\infty)$. (S_1 and S_2 are each closed in \mathbb{R} .) Let $G(x)$ and $F(x)$ be their associated Taylor functions. Here’s the big result: The function

$$F(x)\phi_{0,1}(x) + G(x)\phi_{1,0}(x)$$

is analytic on $(0, 1)$ and extends f and g to the entire x -axis \mathbb{R} . As we move along the x -axis from 0 to 1, $\phi_{1,0}(x)$ picks up $G(x)$ with 100% strength at $x = 0$, and the multiplier $\phi_{1,0}(x)$ diminishes the strength of $G(x)$ to 0% by the time x reaches 1. During this same journey, $\phi_{0,1}(x)$ makes $F(x)$ start off with no strength and increases it to 100% at $x = 1$. So during the trip, the sum $F(x)\phi_{0,1}(x) + G(x)\phi_{1,0}(x)$ morphs from $G(x)$ to $F(x)$. All derivatives match appropriately at $x = 0$ and at $x = 1$ to form a function extending the original one on the closed set $S = S_1 \cup S_2$ to a function on the entire x -axis, which is furthermore analytic on $(0, 1)$.

12.7 Examples revisited

Let's revisit some previous examples. To start, we can now make good on our promises of finding concrete functions that accomplish the bridging.

Figure 12.3 on p. 147. What's the actual function bridging the gap in the figure? Thanks to our morphing functions $\phi_{0,1}(x)$ and $\phi_{1,0}(x)$, we now have an answer:

$$F(x)\phi_{0,1}(x) + G(x)\phi_{1,0}(x).$$

Specifically, this is

$$-\phi_{0,1}(x)\cos(\pi(x-1)) - \phi_{1,0}(x)\cos(\pi x).$$

This joins the two pieces to form a C^∞ function, analytic at every point except $x = 0$ and $x = 1$.

Figure 12.6 on p. 149. We can now give a specific function bridging the gap between the sine and exponential functions. In this example, $f(x)$ is e^x to the right of $x = 1$, and $g(x)$ is $\sin(2\pi x)$ to the left of $x = 0$. The heavily drawn curve in Figure 12.6 depicts the function

$$F(x)\phi_{0,1}(x) + G(x)\phi_{1,0}(x) = \phi_{0,1}(x)e^{(x-1)} + \phi_{1,0}(x)\sin(2\pi x)$$

in $[0, 1]$. The union of the three graphs is analytic at all points except $x = 0$ and $x = 1$. These two exceptions mean that the function on the entire x -axis is only C^∞ there.

Breaking the graph of the famous function at its lowest point. In analogy to breaking the cosine curve at its lowest point, we can do the same with

$$y = e^{-1/x^2} \text{ if } x \neq 0; \quad y = 0 \text{ if } x = 0.$$

Unlike the cosine curve, it isn't analytic at its lowest point (which in this case is the origin). All derivatives are 0 there, so after translating the right half to the right one unit, both $G(x)$ and $F(x)$ are identically zero in $[0, 1]$. This broken graph function on $S = S_1 \cup S_2 = (-\infty, 0] \cup [1, +\infty)$ gets extended to all \mathbb{R} simply by adding the horizontal line segment $[0, 1]$ to the graph. This extended function is analytic at every point except $x = 0$ and $x = 1$.

What if the pieces to be extended are polynomial? Using the morphers $\phi_{0,1}(x)$ and $\phi_{1,0}(x)$ make solving systems of linear equations unnecessary. For example, we can use them to get a transition which is merely analytic in $(0, 1)$ instead of a polynomial transition. Insisting that the transition be a polynomial means that all Taylor coefficients are zero after some point, while being analytic doesn't require that. Figure 12.11 compares the results of the two methods—our polynomial transition on the left, and the morphers transition on the right. Notice how the morphers transition function both rises and falls more than the polynomial transition. This makes sense since, for example, $\phi_{1,0}(x)$ hugs closely to 1 from $x = 0$ until around $x = 0.3$, so the parabolic piece on the left will continue to rise during this time nearly like the original $y = (x + 1)^2$. Analogously for the other parabolic piece, from around $x = 0.7$ to $x = 1$.

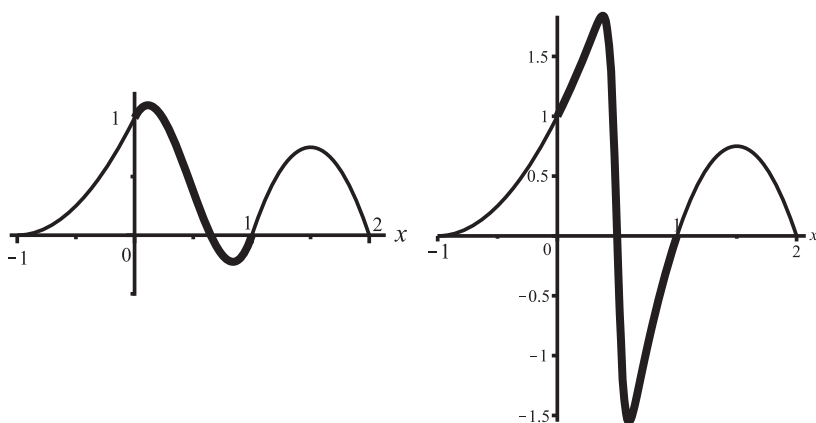


Figure 12.11. The sketch on the left depicts an extension by a polynomial. On the right, the extension is obtained using the morphing functions $\phi_{0,1}(x)$ and $\phi_{1,0}(x)$.

Generalizing from the unit interval. Although we have defined each morpher ϕ to change values over the interval $[0, 1]$, we can create a wide variety of more general examples in which $[0, 1]$ is replaced by $[a, b]$ ($a < b$). To do this, replace each $\phi(x)$ by the corresponding $\phi\left(\frac{x-a}{b-a}\right)$. This gives a pair of morphers we can use to bridge the gap $[a, b]$. Furthermore, even if we have a large number of disjoint closed intervals whose union forms a closed set S , we can apply a morpher pair to each gap in $\mathbb{R} \setminus S$ to extend a C^∞ function on $\mathbb{R} \setminus S$ to a function that is C^∞ on the entire x -axis.

12.8 Beyond the first dimension

Hass found it a real challenge to go beyond the first dimension. He drew picture after picture, but the problem seemed stubbornly intent on putting up a succession of frustrating barriers. His eventual success in 1933 was a real tour de force for the 26-year-old; the following is just enough to give the flavor of his approach.

All the basic ideas in dimension one encountered up to this point have higher-dimensional analogs: a closed set S in Euclidean space; \mathcal{C}^m or \mathcal{C}^∞ functions on S ; spaces between components of S ; Taylor series;⁵ morphing functions that resolve clashes between different Taylor series defined over the same region. But in higher dimensions the borders of the “oceans” separating the components of S require a lot more attention because even in dimension two, these borders are not just isolated points, but curves that can have a lot of wiggles. In dimension one, the gaps between components are just intervals, and morphing took place in the gaps. But what, and how, to morph in higher dimensions?

In going from morphing functions on an interval to morphing functions in higher dimensions, Hass found it most promising to replace “intervals” by “coordinate n -cubes.” To simplify exposition, let's assume we're working in the plane, so n -cubes are squares with horizontal and vertical edges. Now look at Figure 12.12. Here the components S_1 and S_2 are simple—just disks. If we wish to smoothly extend to the whole plane smooth functions F on S_1 and G on S_2 , Hass saw that the easiest way is to tile the ocean—the plane minus S —with nonoverlapping squares. Because the boundary of S will generally have curves and wiggles, such a tiling will require using ever smaller tiles—their sizes approaching 0—as we approach that boundary (the two circles bounding the disks in Figure 12.12).

Roughly, the overall philosophy Hass used in creating an extension is to first shrink each cube about its center so its linear dimension goes down by, say, half. Doing this creates room to morph. In dimension two, shrinking the tiles creates little roadways between them—room that allows appropriate morphers to do their thing. On each shrunken tile T_i we then create a function: If P_i is the nearest point in boundary(S) to a particular point Q_i in T_i , then take the function's value at P_i together with all its derivatives and create a Taylor series centered at Q_i in T_i . The devil is in the details, but Whitney very carefully chose things so that any morphed function is smooth. His last step was to create morphing functions that smoothly blend the function values on all the T_i to extend F and G to a function smooth on the entire plane. These functions, though quite complicated-looking, have the same basic DNA as our one-variable morphers in that they're multi-variable cousins to step functions like $y = \frac{e^{-1/x^2}}{e^{-1/x^2} + e^{-1/(x-1)^2}}$.

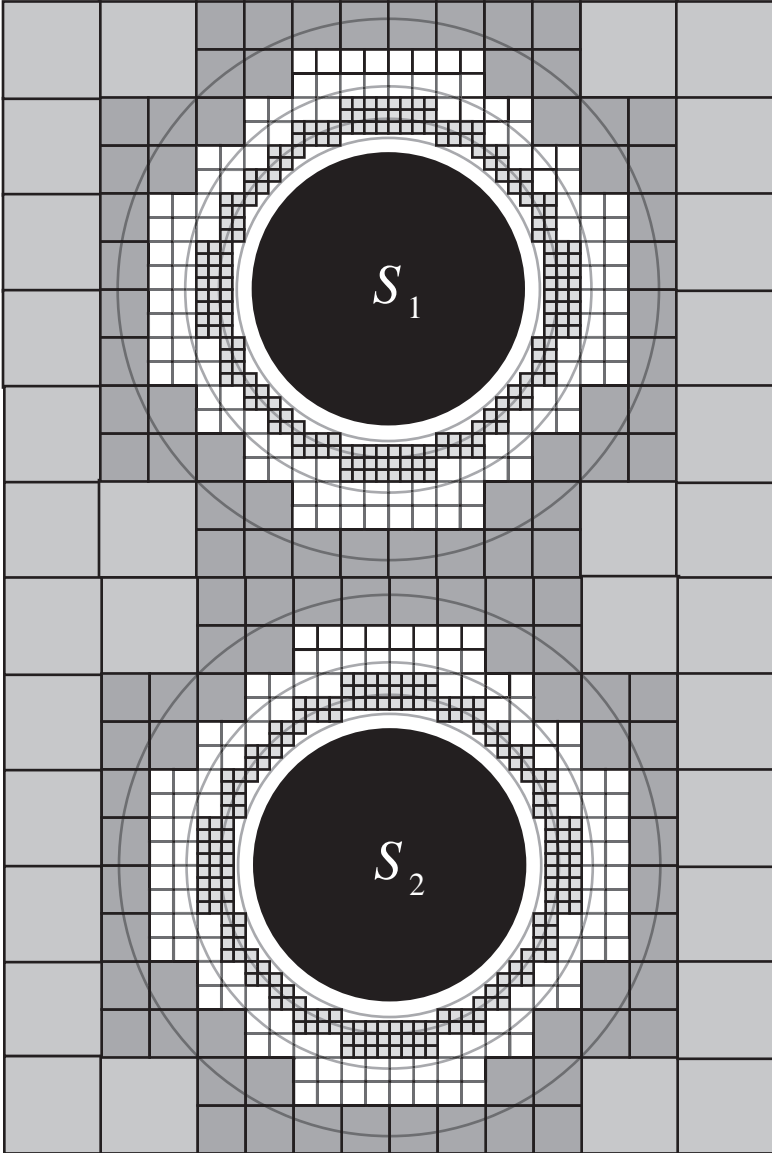


Figure 12.12. Every point of the “ocean” surrounding $S_1 \cup S_2$ is covered by 2-cubes.