Preface

From a scientific standpoint, the 18th century was the Age of Euler. During its early years, Newton and Leibniz were late in their careers, and the Bernoulli brothers, Johann and Jakob, were just passing their peaks. The end of the century featured the likes of Gauss, Lagrange and Laplace. Euler, though, dominated the middle of the century. There were other giants—d'Alembert, Clairaut, Maupertuis, Maclaurin—but Euler alone more than matched their combined productivity. Clifford Truesdell [T] has estimated that Euler made more than a third of the century's discoveries in mathematics and science.

Euler was the preeminent mathematician and scientist during the European Enlightenment, the intellectual revolution of the 18th century that shaped our modern ideas of liberty and human rights. The great thinkers of the Enlightenment—Rousseau, Voltaire, Diderot, Jefferson, Locke and the rest—all wanted to believe that their social and political ideas were rooted in scientific principles, and that, in turn, those scientific principles were based in the firm rigor of mathematics. They worked hard to understand science and mathematics, or at least to gain reputations for understanding.

World leaders, kings and empresses collected scientists in much the way they collected works of art and overseas colonies, partly for power and economic gain, but also partly for show. Euler was a great prize in this game of the powerful. He began his career in the new academy of Peter the Great, though Peter died shortly before the Academy opened. Then Euler moved to Berlin to help reform the academy of Frederick the Great, and finally came back to St. Petersburg during the reign of Catherine the Great. Part of greatness is recognizing and recruiting good help.

Ironically, in many ways Euler was out of step with the Enlightenment. Many of the most charismatic characters of the period were enthusiastic atheists, witty salonists and fawning courtiers. Euler, on the other hand, was religiously faithful. He served on his church council. He was not witty and charming in crowds. Legend has it that he was asked at a party why he spoke so seldom. He supposedly answered something like "Ma'am, where I used to live, people who speak out are hanged." The incident probably never happened, but, as such stories often do, it probably captures the truth that Euler was seldom the life of a large party. He is reported to have been quite charming in small parties, though.

Most readers know some of Euler's discoveries in mathematics and science, especially his pioneering work in number theory. Readers are also likely to have heard that he pioneered the use of $f(x)$ notation for functions and standardized the symbols e and π. We live in an age of specialization, and many of us know Euler's contributions to our own specialty, be it graph theory, fluid flow, Diophantine equations, calculus of variations, mechanics, the physics of music or any of a dozen other fields where the foundational results are due to Euler. But few of us know much of Euler's discoveries outside our own specialty, so we don't appreciate the true depth and breadth of his contributions.

This book is intended as a mathematical biography of Euler. It is an article-by-article description of the fifty or so mathematical articles Euler completed through 1741, the year he left the Imperial Academy in St. Petersburg to join the employ of Frederick the Great in Berlin. I will dwell on the details of his papers far more than on the details of his life or his times, though I will point out connections between the work and the rest of the world when I can.

I also attempt to recapture some of the mathematical and scientific spirit of the 18th century. As the years pass, Euler grows as a scientist and at the same time world changes around him. I will try to show how his early papers were very much in the spirit of the 17th century in ways described in the text, and at the same time they were clearly the work of a young and sometimes naïve scholar of as-yet unproven abilities and potential.

After a good deal of thought, I decided to present the articles in the order in which scholars believe they were written. The authoritative source in this chronology is the Eneström index [E]. That index assigned a number to each Euler work, in the order they were published, but the appendix also listed them in the order they were written. These orders are only approximately the same. For example, Euler's first published article, E-1, was also apparently the first one he wrote. It was written in 1725 and published in 1726. On the other hand, the last article we consider in this book, E-790, "On the usefulness of higher mathematics," was written in 1741, but for reasons we can only guess, it was not published until 1847, over a hundred years later. Other less dramatic permutations in the chronology arise from simple things like publication delay, or for more dramatic reasons, such as late in his career when Euler withheld a result in the calculus of variations because he saw that Lagrange's method was superior.

Euler's mathematical work between 1725 and 1741 has been divided into segments. Most are just a year long, but some cover two or three years. Each segment is introduced with a brief description of Euler's other activities in that time interval, some of the events in his life, and some highlights of then-current events. I hope that these "Interludes" will help illuminate Euler's work in a wider context.

Euler had modest, though by no means impoverished, origins. His father Paul was pastor of a parish near Basel, Switzerland, but had studied with the great Jakob Bernoulli at the University in Basel. When Leonhard went to the university, his teacher was Jakob Bernoulli's younger brother Johann. Euler was an apt student, and Bernoulli's tutelage turned his career ambitions to science. By 1725, when Euler was 18, Bernoulli had him writing papers for *Acta Eruditorum*, one of the most widely read scholarly journals of the era.

In April, 1727, just after his 20th birthday, Euler left Basel to join his friends Nicolas II and Daniel Bernoulli at the St. Petersburg Academy. Euler remained in St. Petersburg until 1741, when political unrest, particularly directed at the German community in Russia, made him welcome an opportunity to join the Academy of Frederick II in Berlin.

Initially, Frederick II and Euler got along very well, but over the next 25 years their relationship gradually deteriorated. In 1766, in the wake of the Seven Years War, Euler persuaded Frederick to let him accept an invitation to return to St. Petersburg, where he was welcomed as a hero.

By the time Euler died in 1783, he had published about 550 books and papers. Another 250 or 300 papers were published posthumously, appearing regularly until about 1820, and then sporadically until 1862, 79 years after his death.

Some remarks are appropriate about the quantity and variety of Euler's publications. They are all cataloged in the Eneström index [E]. Eneström lists 866 publications, and this is usually cited as the "official" number. Closer examination, though, reveals that this is a little too high. For example, about 20 entries late in the list, after E-800, are collections of letters, assembled after Euler's death and probably never intended for publication. Some others are excerpts from the public notebook, the so-called "Day Book" in which Academy members sketched ideas. Still others are clearly "fragments," papers and books started but never finished. Some articles appear twice on the list, in two different languages. Taking all this into account, we have several numbers we could cite as Euler's total number of publications. We could use Eneström's last index number, 866. The number 856 is also often cited, the number of the last article on the list. The entries 857 to 866 are all letters. Here, though, we will prefer the vague but truthful "over 800" published books and papers.

Having settled on the value "over 800," some readers may enjoy asking how this output compares with others. In particular, "Who published more, Euler or Erdős?" It depends on how we make the rules. Erdős authored or co-authored well over 1000 books and papers. Euler wrote about 800, so, in the first analysis, Erdős published more. Moreover, Erdős papers are still being published. Euler hasn't published anything new for over 140 years. If, on the other hand, for some reason we want to penalize Erdős for co-publishing, then Euler comes out ahead. Euler and Erdős combined, though, are more than outdone by Christian Wolff. Today Wolff is mostly forgotten, but in his own time (1679–1754) he was a very influential scientist, philosopher and mathematician. Wolff was the last defender of Leibniz's philosophy of monads. His collected works run to over 300 volumes. Should Wolff be disqualified just because his work is mostly forgotten and was not entirely mathematics? Such questions make pleasant conversation, but seldom lead to valuable conclusions.

Some remarks are appropriate on the philosophy of history behind this book. I focus primarily on the mathematical details of Euler's work up to the time he left St. Petersburg for Berlin in 1741. As such, the work is prima facie what is called internalist.

Moreover, it is chronological, describing Euler's mathematical work in the order he wrote it, which, as pointed out earlier, is only approximately the order in which it was published. The decision to proceed chronologically, rather than thematically, was a difficult one. Eventually, though, it became clear that the many threads of Euler's work cross so often, forming more a web than a bundle of threads, that they would not retain their shape if anyone tried to untangle them. The chronological exposition also lets the reader better appreciate the broad number of projects Euler pursued at any given time.

At least one person who has seen this manuscript has called the work "crypto-internalist." Though I am not exactly sure what the word means, and suspect it was coined just for the occasion, I choose to take it as a compliment. I think and hope it means that the book describes the mathematical details themselves, but that it also makes connections among the various papers, even if they are not obviously part of the same thread. Moreover, it relates the mathematical details to the events in Euler's life, the rest of the scientific community and the world at large.

These considerations on the way this book was written leave the reader a good deal of flexibility in how to read the book. Those with plenty of time and broad interests might want to read it straight through and enjoy its many plots and subplots. Those with particular interests, say number theory or the calculus of variations, can read just those threads. The annotated table of contents should help such readers. Alternatively, there is a collection of some topically related lists of papers at the end of this Preface. Those with broader interests but less time will find that some articles are marked with one, two or three stars. These are the ones that I thought were most important or more interesting. Such judgments are, of course, subjective and vary with individual tastes.

There is a particular pleasure reading Euler in the original language, usually Latin. Euler's writing is as clear and carefully crafted as his mathematics. He seems to make a deliberate effort to keep the language itself relatively simple, knowing, perhaps, that Latin was nobody's first language. I hope that some readers will use this book as a guide to choose what Euler they would like to read in the original.

The reader deserves to be warned of some conventions followed in this book. First, I often refer to Euler's individual works by their numbers in the Eneström index. Certainly "E-20" is a good deal more concise than "De summatione innumerabilium progressionum." Eneström numbers are chronological in order of publication. We sometimes refer to an article by its original title, usually in Latin, and sometimes by an English translation of that title. If this becomes confusing, citations at the end of each chapter should help.

Moreover, it has become standard practice to use two or more dates to cite Euler's works. For example, Euler wrote E-20 in 1731, and it appeared in volume 5 of the journal of the St. Petersburg Academy. That volume contains papers presented to the Academy for the years 1731 and 1732, and was published in 1738. Moreover, E-20 appears in Series I volume 14 of Euler's *Opera Omnia*, and facsimiles of this, and almost all of Euler's works are available online at EulerArchive.com. The bibliographic citation in the footnote at the beginning of the chapter on E-20 contains much of this information. It says

Comm. Acad. Sci. Imp. Petropol. 5 (1730/31) 1738, 91–105; *Opera Omnia* I.14, 25–41

The abbreviation "*Comm. Acad. Sci. Imp. Petropol.* 5" tells us that this article appeared in volume 5 *Commentarii academiae scientiarum Peteropolitanei*, the journal of the St. Petersburg Academy. The years 1730/31 and 1738 are the dates described above, and "*Opera Omnia* I.14, 25–41" tell us where to find the article in the *Opera Omnia.*

In Euler's time, Russia still used the Julian calendar. His correspondents in the rest of Europe mostly used the newer Gregorian calendar, so when it was November 12, 1739, it was already November 23 in Berlin. Eighteenth century mail services were much better than most people would expect, so occasionally it was possible for a letter to seem to be answered before it had been written! We make note of these calendar problems whenever they arise.

As much as possible I try to use the same notation that Euler did. Euler wrote "$l\,x$" where we would write "$\ln x$." Later in his career, he would write $\int n$ where we would write $\sum n$, and he usually wrote xx rather than x^2, but never xxx for the cube. He had no subscript notation. (There are many times it would have been very helpful.) I will explain his notation for definite integrals when the time comes.

There are two main exceptions to this faithfulness to Euler's notation. First, occasionally Euler uses the old German Fraktur alphabet. I find this font very difficult to read, and the printers I was using didn't have the font installed, so I made substitutions. Second, Euler's radical signs don't usually have bars on them. They look like $\surd$ instead of $\sqrt{}$. It simply became too tedious to untangle his radical signs over and over again, and then to try to explain what the correct reading of the expression was, so I finally gave up and switched to the modern notation.

The scholarship in this book is based almost entirely on original sources. This lets us avoid repeating the errors of earlier scholars. One is reminded of one reader's criticism of Descartes: "He wiped away the errors of the Ancients and replaced them with his own."

Moreover, there are inevitably a great number of loose ends and interesting threads that could be tied up, perhaps should be, but aren't. I am aware of some of these and blissfully ignorant of many others. It would be interesting, for example, to learn more about Henry Pemberton and the origins of the problems of reciprocal trajectories. Euler studied these in several papers beginning with E-3. One wonders why Pemberton would pose such problems. Also, I suspect there is something more in E-44 and E-45 than I see.

I have been extraordinarily fortunate to be able to read much of Euler from the original 18th century volumes in the library collections of Yale University. Most readers will not be so lucky, but they have at least two recourses. First, in 1907 the Swiss Academy of Sciences began to publish the complete works of Euler under the title *Opera Omnia*. The first volume, *Algebra*, appeared in 1911, and the series is not yet complete. All of Euler's published works in mathematics and physics have been republished in this series, but most of his correspondence and all of his notebooks are still in preparation. The scholarship that has gone into the preparation of the *Opera Omnia* includes much of the 20th century's best historical and mathematical work on Euler. I refer frequently to the *Opera Omnia*.

Second, more and more of Euler's original work is becoming available on line. I regularly use the resources at EulerArchive.org, which has over 95% of Euler's publications scanned and on line. Additional Euler is available at the Berlin Academy of Sciences, the National Library in France and at the Cornell Digital Library, and certainly more is being added all the time.

There are surely errors in this work and places I did not understand what Euler was doing. For this, I apologize and take full responsibility. I hope that they do not obscure the picture of Euler, his early work, and its role in the 18th and 21st centuries that I am trying to paint.

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Ed Sandifer
Newtown, CT
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