Preface

"To Think Deeply about Simple Things" was a motto of Arnold Ross, founder of the Ross Mathematics Program. Ross's philosophy has provided inspiration for many content-based professional development programs, each with its own personality and design, including the Summer School Teacher Program (SSTP) at the Park City Mathematics Institute (PCMI) and the PROMYS for Teachers Program (PfT) at Boston University.

More information about the SSTP can be found at `mathforum.org/pcmi/hstp`. For PfT, see `www.promys.org/pft/`.

PCMI, sponsored by the Institute for Advanced Study, is a three-week summer program for those involved in mathematics: research mathematicians, graduate students, undergraduate faculty, undergraduate students, and precollege teachers.

The SSTP has been an integral part of PCMI from the beginning. Since 2001, the mathematics course has been designed for precollege teachers by a collaborative of teachers, educators, and mathematicians from PfT. The SSTP course meets for 15 two-hour sessions during which the teachers investigate an aspect of mathematics loosely related to the overall mathematical theme of that summer's PCMI.

This book is based on the program from 2007, *Probability through Algebra*. The goals of the course are to introduce participants to the algebraic properties of expected value and variance through analysis of games, to the use of generating functions and formal algebra as combinatorial tools, and to some applications of these ideas to questions in probabilistic number theory.

Another goal, one that runs across all the summers, is to take up mathematics that helps teachers put the topics in their curricula into the broader landscape of mathematics as a scientific discipline.

The Design of the Course

The central feature of the SSTP course is a set of intricately sequenced questions that engage participants in doing mathematics in ways that exemplify the *Common Core* Standards for Mathematical Practice. What's important here is the structure of the program, because the program is not a "course" in the traditional sense. The materials provide participants with the opportunity for authentic mathematical discovery—participants build mathematical structures by investigating patterns, use reasoning to test and formalize their ideas, offer and negotiate mathematical definitions, and apply their theories and mathematical machinery to solve problems. Through this experience, participants develop habits of mind for thinking about and doing mathematics, deepening their mathematical intuition, sense-making, and reasoning skills.

Each fall and spring, a team drawn from the PfT community meets regularly to create two or three major themes for the upcoming SSTP course, and for each theme, it creates a "soup" of potential problems and investigations that might be used at PCMI. Once in Park City, the instructors create daily problem sets, revised each night to reflect what happened in the day's session. After the course, the problem sets are revised once more, solutions and hints are written, and the course is prepared for publication.

The problem sets are separated into three sections: Important Stuff, Neat Stuff, and Tough Stuff. The problems in Important Stuff contain the fundamental concepts that should enable everyone to move forward. The problems in Neat Stuff and Tough Stuff are there for those who are curious or looking for a challenge.

The distinguishing features of the program have stayed constant over the years:

Teachers as professionals These materials are designed and implemented by practicing teachers in collaboration with mathematicians and mathematics educators. Experienced teachers mentor teachers new to the program by acting as "table leaders." The connections between the program and the teaching profession are real, because teachers are involved at every level.

Serious mathematics connected to secondary teaching Each experience is designed to connect to the mathematics teachers use in their professional lives. This "applied" mathematics is sometimes around underpinnings that will end up in the hands of students. But it may also take up mathematics that helps teachers put the topics in their curricula into the broader mathematical landscape.

In one summer at the SSTP, teachers studied how formal algebra with polynomials can be used to bring coherence to combinatorial problems. In another summer, teachers learned to apply the arithmetic of algebraic numbers to the problem of designing tasks for their students that "come out nice."

Experience before formality Participants experience first-hand the effectiveness of struggling with new ideas and connections *before* they are brought to closure. The role of the instructor is to pull together the participants at several points to collate conjectures,

> logical arguments, and extensions. With help from those around them, teachers refine and prove their own conjectures, sometimes over the course of several days. This style of learning, emblematic of the intent of *Common Core's* Standards for Mathematical Practice, has had an immense effect on how teachers approach their own classes and how they view the discipline they teach.

Teachers work in tables of 5–6 participants, accompanied by a table leader who typically responds to questions with well-posed additional questions. Tablemates work on carefully crafted problem sets, slowly abstracting general principles from calculations and experiments.

The goal of the PCMI teacher program is to provide teachers with opportunities to:

- deepen their understanding of mathematics,
- reflect on the practice of teaching, and
- serve as mathematical resources for their colleagues.

All of this has worked. Exit reviews of the summer programs and in-depth interviews by external evaluators of teachers from varying backgrounds and school systems have provided evidence that the program has helped hundreds of teachers become more effective in—and more satisfied with—their professional lives. According to one evaluation report, many participants have been influenced by PCMI to revise their roles as teachers, acting more as facilitators, rebalancing how much time they allow students to talk versus talking themselves, and giving students more responsibility for their own learning. They consciously change the ways they question students and answer questions, and make reasoning and sense making a core goal of their instruction.

Teachers meet after each mathematical session to discuss the work of teaching. One summer focused on how to make classrooms a place where questioning is central to learning. In another summer, participants considered how to manage discourse that made students the center of the discussion. Activities are designed around artifacts of practice, such as student work, classroom videos, assessment, or lesson design.

While these materials were developed for use at the SSTP, they have been piloted in other settings and have been successfully adapted for use as a capstone course for preservice mathematics teachers and as an elective in a mathematics or mathematics education program where students did one problem set per week. While the course does not replace a standard probability course, it prepares students to take such a course, with the problem sets building towards a mathematical conclusion. The materials could be used as a course in a graduate program or summer institute for teachers, or in a variety of different professional development configurations. Intermediate conclusions typically occur every four or five problem sets, so a shortened course might consist of the first four or five problem sets or the eighth or ninth problem sets. In settings where credit is offered, some instructors have assigned projects or reports on specified readings related to the work. While the mathematics may seem appropriate only for teachers with a strong mathematical background, it was designed to provide all teachers with a rich mathematical experience, regardless of their backgrounds.

Navigating the Problem Sets

To allow access for multiple levels of experience, any topic in algebra or beyond is approached as though it is newly encountered. For example, the work on quadratics is designed to lead to an understanding of the quadratic formula, not requiring it, and generalizations come from repeated iterations of examples, not from an initial exposure. Many learners find their first answers by testing options, and when an unusual one comes along, they use the patterns from their previous work to figure out what might happen in this new case. In some cases, the proof uses all of the key ideas from the openers (boxed problems). The problems are not built to support a lecture but rather deliberately constructed for students to pursue a general solution over time, with the goal of enabling the learners to build their own understanding from the problems as they work through them.

Note that because the problems are carefully crafted to let the mathematics unfold through the experience of actually doing the work, facilitators should do all of the problems, at least in Important Stuff, themselves before engaging students so as to anticipate students' thought processes and to encourage them during explorations and discoveries.

A few things are important about how the materials are structured:

- The boxed openers are meant to be answered just like the rest of the problems. Consider them "Problem 0"; they are boxed because they are intended to be more important than others. A general note: keep looking forward in the problem sets. The solutions do not go immediately for proof, but rather the course tends to let ideas sit for a while. For example, a proof of the Set 3 opener might be found in a problem in Set 4.
- There are problem categories: *Important Stuff*, *Neat Stuff*, *Tough Stuff*, and maybe other stuff sometimes. All the mathematics that is central to the program can be found and developed in the Important Stuff. That's why it's Important Stuff.

- As mentioned earlier, the materials provide experience before formality, where the participant uses examples to build intuition. Definitions and theorems appear as capstones, not foundations.
- The problems should lead to the appropriate mathematics rather than requiring it. The same goes for technology: the problems should lead to the appropriate use of technology rather than requiring it.
- The problems have multiple points of entry where everyone, regardless of their level of confidence or experience, can begin.
- The materials have a low threshold and a high ceiling—they allow everyone to experience success and are also designed so that participants will feel challenged regardless of skill or experience level.
- The problems explicitly link different content areas and encourage participants to seek multiple representations and solutions.
- A problem presented from one perspective (algebraically, say) may be repeated in another (geometrically, for example).
- The problems often foreshadow key ideas that are not introduced formally until later problem sets.
- The problems repeat and connect throughout all of the sets. A goal is to have the participants look for connections rather than being surprised when they notice relationships.