Problem Set 1

Opener

Get geometry software working on your computer. The actual opener will be done later.

There are several geometry software packages you can use, but if you are working with others, it is best to standardize on one. Some provided files have been built for Geometer's Sketchpad.

Important Stuff

1. Sameer tells you the perimeter and area of a rectangle. Is it possible to confidently determine the dimensions of the rectangle?

2. Here are some perimeters and areas. Find the dimensions of each rectangle.

a. perimeter 24, area 36
b. perimeter 24, area 35
c. perimeter 24, area 32
d. perimeter 24, area 27
e. perimeter 24, area 34
f. perimeter 24, area 37

3. Find all solutions to each equation.

a. $x^2 - 12x + 36 = 0$
b. $x^2 - 12x + 35 = 0$
c. $x^2 - 12x + 32 = 0$
d. $x^2 - 12x + 27 = 0$
e. $x^2 - 12x + 34 = 0$
f. $x^2 - 12x + 37 = 0$

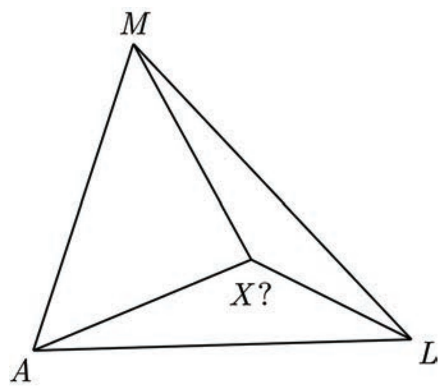
4. A rectangle has perimeter 36. What could its area be?

5. Chance tells you the surface area and volume of a rectangular box. Is it possible to confidently determine the dimensions of the box?

Opener

Three cities located at points M, A, L get together to build an airport. Where should the airport be placed to minimize the lengths of the new roads that need to be built?

In geometric language, the goal is to find point X so that $MX + AX + LX$ is minimized.



Build this sketch, then use geometry software to figure out where point X should be placed. Is there anything special about this point?

Neat Stuff

6. For each point, decide if it is the same distance from $(7, 1)$ and $(-2, 9)$.

a. $(7, 10)$ **b.** $(-1, 1)$ **c.** $(-17, -17)$ **d.** $(-2, 0)$

This problem has four parts, but it's *not* multiple choice!

7. Find some rectangles whose perimeter and area have the same numeric value. Then find some more.

8. Find some rectangular boxes whose surface area and volume have the same numeric value. Then find some more, if there are any.

9. Rina tells you the perimeter and area of a triangle. Is it possible to confidently determine the side lengths of the triangle?

10. On July 4, 2011, the date was written as 7/4/11, and $7 + 4 = 11$.

a. After July 4, 2011, how many more times in the 21st Century is there a day like this?

b. How many times *next* century will there be a day like this?

c. How can your second answer help you check the first?

The next one after 7/4/11 was 8/3/11.

Tough Stuff

11. A triangle has perimeter 24. Find its maximum possible area, and explain how you know that this *must* be it.

12. Given positive integer n, the unit fraction $\frac{1}{n}$ can be written as the sum of two other unit fractions:

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$$

Find a rule for the number of ways to write $\frac{1}{n}$ as the sum of two unit fractions.

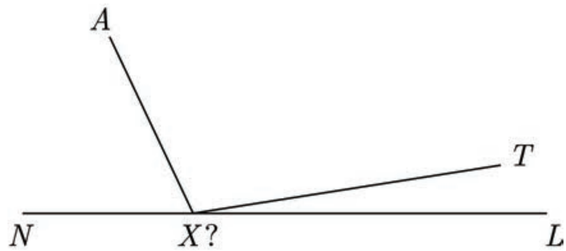
Like the blood type, a and b must be positive. Unlike the blood type, they must be integers.

13. Find a rule for the number of ways to write $\frac{1}{n}$ as the sum of *three* unit fractions.

Problem Set 2

Opener

Art (at point A) has to take a high-stakes test at point T. He is extremely thirsty, and needs to take a big gulp from nearby river NL.



Where should he run toward (point X) to minimize the total distance $AX + XT$?

If you figure out where point X should be, start thinking about how you could *construct* point X more directly, how you could *prove* that point X must be the right one, or how to change the problem in interesting ways.

Important Stuff

1. **a.** What's $(11+\sqrt{2})+(11-\sqrt{2})$?
b. What's $(11+\sqrt{2})(11-\sqrt{2})$?
c. What's $(11+x)(11-x)$?

2. **a.** A rectangle has perimeter 44 and area 117. What are its length and width?
b. A rectangle has perimeter 44 and area 119. What are its length and width?
c. A rectangle has perimeter 44 and area 100. What are its length and width?
d. A rectangle has perimeter 44 and area 130. What are its length and width?

3. Find all solutions to each quadratic equation.

a. $w^2 - 22w + 121 = 0$ **b.** $w^2 - 22w + 120 = 0$
c. $w^2 - 22w + 117 = 0$ **d.** $w^2 - 22w + 119 = 0$
e. $w^2 - 22w + 100 = 0$ **f.** $w^2 - 22w + 2 = 0$
g. $w^2 - 22w - 408 = 0$ **h.** $w^2 - 22w + 130 = 0$

We like shortcuts, but we *especially* like shortcuts that do not involve the phrase "$4ac$".

4. Two numbers add up to 200 and their product is 9,991. What are the numbers?

5. Find three rectangles whose perimeter, in units, is the same number as the rectangle's area, in square units.

6. Triangle SAM has points $S(2,1)$, $A(4,1)$, and $M(4,6)$.

a. Draw triangle SAM in the plane.

b. New points are created from the points in triangle SAM according to the rule

$$(x, y) \mapsto (-y, x)$$

Draw the new triangle created in this way in the same plane, and describe how this triangle is related to the original.

7. Transform triangle SAM according to the rule

$$(x, y) \mapsto (-x, -y)$$

Draw the new triangle and describe how this triangle is related to the original.

8. **a.** Show, beyond a reasonable doubt, that $(14, -4)$ is *not* equidistant from $(-2, -5)$ and $(4, 11)$.

b. Again for $(-10, 7)$.

c. Build a rule you could use to decide whether (x, y) is equidistant from $(-2, -5)$ and $(4, 11)$.

9. Cuong repeats the transformation in Problem 6 a whole bunch of times.

a. What happens after the transformation is applied twice?

b. What happens after the transformation is applied three times?

c. . . . four times?

d. . . . five times?

e. . . . thirteen times?

f. . . . 101 times?

Neat Stuff

10. For any rectangle you can assign a point (l, w) in a coordinate plane, defined by the length and width of the rectangle.

a. Plot four points that all correspond to rectangles with area 20.

b. How many rectangles are there with area 20? Plot them all.

c. Plot all the rectangles with perimeter 20.

d. Is there a rectangle with perimeter 20 *and* area 20?

11. Repeat Problem 10 for rectangles with area 12 and perimeter 12. What happens?

12. **a.** Put a point Q on the number line. Define $f(P)$ to be the distance from point P on the number line to point Q. What does the graph of $f(P)$ look like?

b. Put point R on the number line, and now define $f(P)$ as the total distance from any point P to the two given points. What does the graph of $f(P)$ look like?

c. Put point S on the number line, and do all that stuff we said again.

13. Find three triangles that have the same numeric value for their perimeter and area.

14. Let $f(x) = x^3 + 3x^2 - 8x - 80$. Use long division to find the remainder when $f(x)$ is divided by each of the following.

a. $(x - 1)$
b. $(x - 2)$
c. $(x - 3)$
d. $(x - 4)$
e. $(x - 5)$

15. Complete this table for $f(x) = x^3 + 3x^2 - 8x - 80$.

x	$f(x)$
0	
1	
2	
3	
4	
5	

16. Find the remainder when $f(x) = x^{12} + 3x - 1$ is divided by each of the following.

a. $(x - 1)$ **b.** $(x - 2)$ **c.** $(x - 10)$ **d.** $(x + 1)$

17. Jessica takes triangle SAM from Problem 6 and applies a wacky transformation:

$$(x, y) \mapsto (x + y, -3x + 7y)$$

a. Draw this new triangle JES. Is it even a triangle anymore?

b. What is the area of this new shape? How does JES compare (in area) to SAM?

18. Find all the ways to write $\frac{1}{10}$ as the sum of two unit fractions. Here's one for free:

$$\frac{1}{10} = \frac{1}{20} + \frac{1}{20}$$

A *unit fraction* has numerator 1 and positive integer denominator.

Tough Stuff

19. There's a point inside most triangles that forms three 120° angles with segments to the three vertices. A *Matsuura triangle* is a triangle whose side lengths are all integers, *and* whose three interior segment lengths from the 120° point are also integers. Find some Matsuura triangles, or prove they do not exist.

20. Complete this long division problem, where all the missing digits are marked with an X. (There is no remainder.)

```
              X X 8 X X
        ---------------
X X X ) X X X X X X X X
          X X X
        -------
            X X X X
              X X X
            -------
                X X X X
                X X X X
                -------
```

CHAPTER 2 Facilitator Guide

These facilitator notes are designed to be used as needed. Each problem set has two components:

1. **Goals of the Problem Set:** here we lay out what the principal ideas of each problem set are.
2. **Notes on Selected Problems:** we identify a few problems that are worth going over in a whole group discussion.

We will put our emphasis on the main goals of each lesson, drawn from the problems in the "Important Stuff."

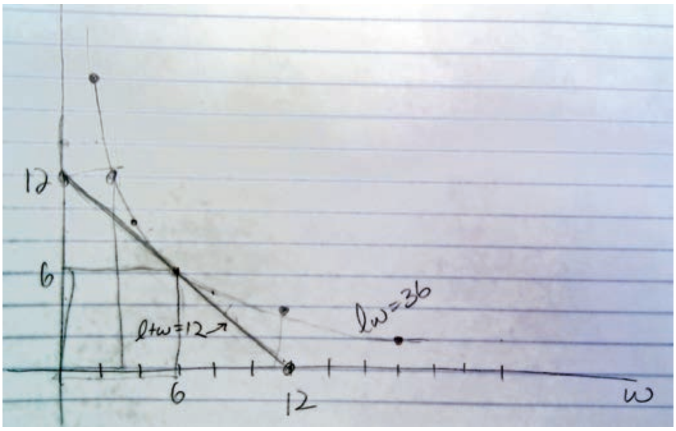
Problem Set 1

Goals of the Problem Set

This course is about representation and parameterization, and the problems here are intended to introduce some of the key questions and concepts that will be seen throughout.

A key problem is introduced: whether or not the surface area and volume of a box can uniquely define its dimensions. This problem will be revisited and refined a number of times through the course. A similar but more straightforward question is asked first: whether or not the perimeter and area of a rectangle uniquely define its dimensions. Other problems then introduce the correspondence between the geometric problem and an algebraic problem, and a similar tactic will be used to analyze the surface area problem.

The other key problem introduces geometric representation, with each point inside triangle MAL corresponding to a number, the total distance to the three vertices. A short-term goal is to discover the property of the minimizing point. A long-term goal is for participants to use this tactic on other problems. For example, all rectangles with perimeter 24 can correspond to the points on the line with equation $x + y = 12$ where x and y are the length and width of the rectangle, and all rectangles with area 36 correspond to the points on the curve whose equation is $xy = 36$:



Problem Set 1 is also intended as an introduction to the style of the course. Consider having participants read the Introduction to learn about the course expectations.

Notes on Selected Problems

The opener is delayed in order to give participants opportunity to get geometry software working. If all participants have working geometry software, it is reasonable to have them work on the opener (at the end of the Important Stuff section) before working on the other problems, but it is not necessary.

In problem 1, try to press participants to give a better answer than "two variables and two equations". Give an example such as $x + y = 10, 3x + 3y = 30$ where this is not enough information, or an example such as $x + y = 10, x^2 + y^2 = 1$ where there is no solution.

Watch for participants making connections between problems 2 and 3, especially on part **e** where the solution is not an integer. Encourage participants to solve the problems without using the quadratic formula, especially if some participants do not have much experience with it.

In problem 5, watch carefully for participants who answer "no" by saying "three variables and two equations". This would only apply if the equations were linear, and these are not. Give an example such as $x^2 + y^2 + z^2 = 0$ where an equation can have a unique solution with more than one variable. Overall, encourage participants to spend a lot of time on this problem, even if you think they are not making much progress. One potential method is to pick a value for one of the dimensions, then proceed. It's likely this problem will remain unresolved in this set, and that is a good thing.

In the airport problem, show participants how to draw line segments, measure lengths, and calculate values based on measurements. When showing how to measure lengths, point out that other measurements are available (angle, slope, area, etc). The property of point X, that it forms three 120-degree angles from the segments, is verified by measuring the angles; this can also be used to better place the point. Some participants will look for the usual "centers" of a triangle:

- centroid: intersection of the medians
- orthocenter: intersection of the altitudes
- incenter: intersection of the angle bisectors
- circumcenter: intersection of the perpendicular bisectors

All of these centers are wrong. Participants should not be discouraged to find these centers, but should be encouraged to first find the "best" placement of point X for a specific triangle. It is likely that someone with no knowledge of these centers will take less time to find X and its property than someone with a lot of knowledge of the centers.

Problem Set 2

Goals of the Problem Set

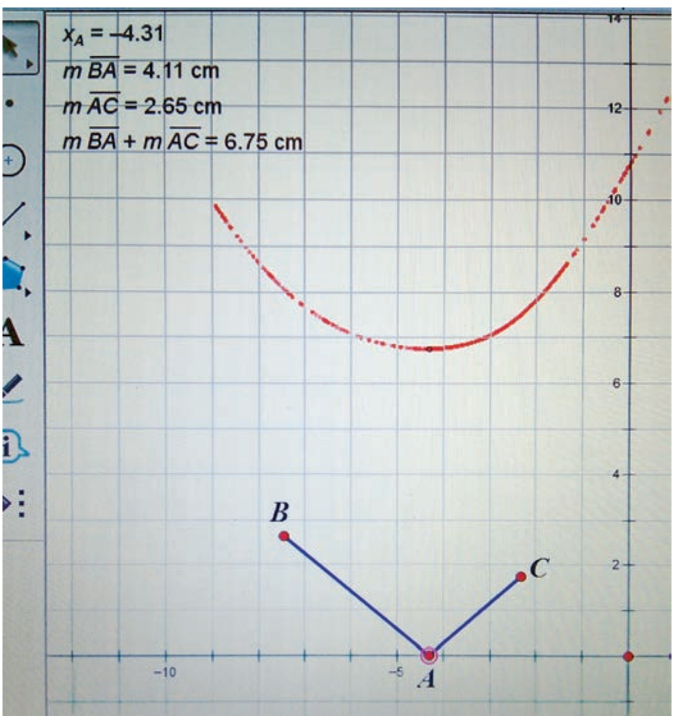
Problem Set 2 cements the connection between rectangles and quadratics, offering a means of solving any quadratic equation by using "difference of squares". By knowing the maximum product for a given sum, you can calculate how far away the answers should be compared to the perfect pair.

Some problems in this set help participants toward understanding and recognizing complex numbers, both algebraically (problems 2d, 3h) and geometrically (problem 6). These are still previews and will be revisited when necessary, but look for participants making these connections early.

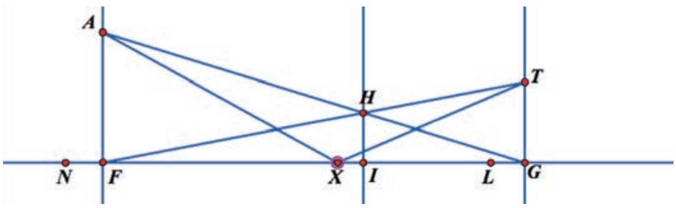
Notes on Selected Problems

For the opener, participants will need to learn how to place a point "on" an object (X on NL) and it may be useful to learn how to create a line instead of a segment. As before, measurement determines the property of point X (it forms equal angles at AXN and TXL). Look for a large number of potential explanations:

- By reflecting point T across NL to T′, the length AX+XT is minimized when AX+XT′ is minimized, but that can be a straight line. This is probably the "best" way to explain it, and can be hinted as by talking about an equivalent problem where T is on the other side of the river. Then, properties of vertical angles and congruent triangles prove the desired result.
- The total distance can be written as the sum of two square root functions and modeled on a graphing calculator or within the geometry software. As with the second solution, this is not a proof but a means of finding the correct placement of X.



- By dropping perpendiculars from A and T, point X is "best" when the two triangles formed are similar. This is not a proof but can explain or find the correct X for a new situation.
- After dropping perpendiculars, connect A to the foot of the altitude from T to NL, and connect T to the foot of the altitude from A. The intersection point of these segments is directly above the correct placement of point X. In this case it can be proven that the two triangles formed are similar, then use this information to prove that the other two triangles (that include X) must also be similar.



Carefully watch participants work on problem 2b. Those who struggle should be directed to problem 1, which gives the answer. Participants may not be used to working this way or actively looking for connections between problems.

Look for the methods participants are using in problems 3f–3h. As the sidenote says, discourage use of the quadratic formula even for those who know it well, and especially discourage teaching the quadratic formula from one participant to another. Some participants have used this problem as a jumping-off point to derive the quadratic formula, or at least a version of it for monic quadratics (with no leading coefficient).

Problem 4 is a quick check to see whether participants know the method of difference of squares, and a similar problem is found in Problem Set 3.

Problem 5 is important, since it encourages participants to think about dimensions other than integers. The 4-by-4 and 3-by-6 rectangles are easily found but the third is tougher, and many participants will discover a useful method for future work: pick a value for one of the dimensions then use that value to solve the rest of the problem. This is one means of attacking the surface area and volume problem when it returns.

Problems 6 and 7 are a preview of work with complex numbers. Ideally, participants should also work on problem 9, so they can see what the repeated transformation looks like. In particular, the application of the transformation twice will eventually be linked to the identity $i^2 = -1$ and the application of the transformation four times will be linked to $i^4 = 1$. It's unlikely these connections will be noticed here, though, so just watch for recognition of the 90-degree counterclockwise transformation.

Problem Set 3

Goals of the Problem Set

This problem set continues to focus on parameterization. Geometrically, parameterization can lead to the recognition of invariants, as it does in this set's opener, or the recognition of optimal and extreme cases, as in the previous sets' openers. Coordinates can be used in several ways, including verification of geometric properties (problem 3), visualization of dimensions in new ways (problem 4), and the building of functions to model situations (problem 6).

These concepts will be brought to bear on the surface area problem in the long term, but in the short term the concept can be used to find more rectangles like the ones asked about in Problem Set 2 whose perimeter and area are the same numeric value.

Notes on Selected Problems

The opener will take a long time to build. Lead participants through the construction of an equilateral triangle by building a line segment and two circles; from segment AN the first circle has center A and radius point N, and the second has center N and radius point A. Be especially careful that participants use the points when building these circles instead of "eyeballing" it: ideally, make the error yourself so that others can watch out for it.

This is also the best time to teach the concept of "hiding", which will be necessary both for hiding the circles and the perpendicular lines from point X. Show participants how to build one of the perpendicular segments by building the perpendicular line, marking the intersection with the base, hiding the line, then drawing the segment from X to the marked point. Participants should be able to finish the diagram.

Encourage participants to show all measurements in the diagram so that when the sum of the three lengths is found to be constant, it is not judged to be "broken".

CHAPTER

3 Solutions

Problem Set 1

1. Yes. There are many possible ways to think about this. One is to look at the graphs of perimeter (a line) and area (a curve) in the coordinate plane of lengths and widths. These graphs can have at most two intersections, and those intersections correspond to the same rectangle.

Another is to use the quadratic formula to solve the equations $P = 2l + 2w$ and $A = lw$ for either l or w, giving a rule that will determine the specific dimensions for given perimeter and area.

Note that not all options are valid: for example, there is no rectangle with perimeter 10 and area 100.

2.
- **a.** 6 by 6
- **b.** 7 by 5
- **c.** 8 by 4
- **d.** 9 by 3
- **e.** This one is tougher. By using the pattern or by finding the solution to the system of equations, the dimensions are $6 + \sqrt{2}$ and $6 - \sqrt{2}$.
- **f.** There is no rectangle with these dimensions.

Note: When the perimeter is n less than 36, the dimensions are $6 \pm \sqrt{n}$.

3. The answers here correspond to the answers in Problem 2.

- **a.** 6
- **b.** 7 and 5
- **c.** 8 and 4
- **d.** 9 and 3

e. $6+\sqrt{2}$ and $6-\sqrt{2}$
f. No real solution exists. Some participants may give the answers $6+i$ and $6-i$ but that is not required yet.

4. We can use the results from a previous problem here. Notice we found rectangles with perimeter 24, which was double 12, the coefficient of x. Thus we want numbers that add to $\frac{36}{2}=18$. The product of such numbers would then give us the area. Possibilities are:
17×1, $A=17$
16×2, $A=32$
15×3, $A=45$
14×4, $A=56$
13×5, $A=65$
12×6, $A=72$
11×7, $A=77$
10×8, $A=80$
9×9, $A=81$
In addition, rational and irrational values may be used, such as: 13.5×4.5, $A=60.75$. Note that the area values will fall in the interval $0<A\leqslant 81$.

5. If we know the surface area, S, and the volume, V, then we have the equations $S=2(lw+lh+wh)$ and $V=lwh$. There are too many unknowns! If you knew one of the dimensions, then you could solve for the other two. In addition, there could be multiple answers. It is possible to find specific boxes with the property.

Though you cannot necessarily determine the dimensions with confidence, you could find some boxes that meet Chance's given surface area and volume . . . so we will not make him sad!

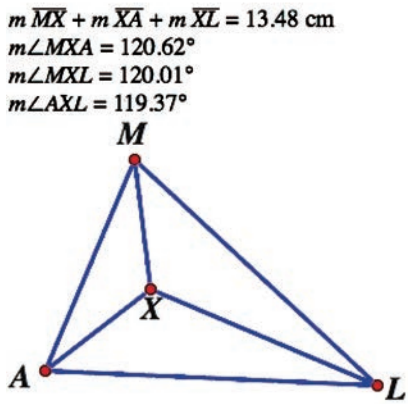
Don't worry, we will see this problem again soon!

Opener

Explore this problem! Try out everything that comes to mind. Utilize GSP's measuring capabilities - segments, angles, sum of distances, bisectors, etc.

Note that the lengths of the triangles sides will vary depending on your construction.

Interestingly the sum is smallest when each angle around vertex X right around 120°. Hmmm . . .



6. **a.** The distance from $(7,1)$ to $(7,10)$ is $\sqrt{81} = 9$
The distance from $(-2,9)$ to $(7,10)$ is $\sqrt{82}$, so the distance is not the same.

b. The distance from $(7,1)$ to $(-1,1)$ is $\sqrt{64} = 8$
The distance from $(-2,9)$ to $(-1,1)$ is $\sqrt{65}$, so the distance is not the same.

c. The distance from $(7,1)$ to $(-17,-17)$ is $\sqrt{900}$ $= 30$
The distance from $(-2,9)$ to $(-1,1)$ is $\sqrt{901}$, so the distance is not the same.

d. The distance from $(7,1)$ to $(-2,0)$ is $\sqrt{82}$
The distance from $(-2,9)$ to $(-2,0)$ is $\sqrt{81} = 9$, so the distance is not the same.

7. Notice that if a rectangle has the same value for both its perimeter P and area A, we have $2l + 2w = lw \Rightarrow 2w - lw = -2l \Rightarrow w = \frac{-2l}{2-l} = \frac{2l}{l-2}$.
You can chose a value for l or w and then solve for the other. Make sure you choose values $l > 2$.
Also notice that $2l + 2w = lw \Rightarrow \frac{2(l+w)}{2lw} = \frac{lw}{2lw} \Rightarrow \frac{1}{w} + \frac{1}{l} = \frac{1}{2}$. Wow! That's Fancy Nice! The reciprocals of the dimensions sum to $\frac{1}{2}$. So . . . *Let's get ready to rectangle!*
4×4. $A = P = 16$
3×6. $A = P = 18$
$8 \times \frac{8}{3}$ gives $A = P = \frac{64}{3}$
$10 \times \frac{5}{2}$ gives $A = P = 25$
$7 \times \frac{14}{5}$ gives $A = P = \frac{98}{5}$
$18 \times \frac{9}{4}$ gives $A = P = \frac{81}{2}$
$20 \times \frac{20}{9}$ gives $A = P = \frac{400}{9}$
and so on . . .

8. This is similar to the previous problem, but we have another dimension. Some solutions can be found by picking values for two of the dimensions then solving for the remaining dimension. Some may notice that the equation $lwh = 2(lw + lh + wh)$ can be rewritten as $\frac{1}{2} = \frac{1}{h} + \frac{1}{w} + \frac{1}{l}$.

Some specific boxes that work are 6-by-6-by-6, 8-by-8-by-4, and 12-by-12-by-3.

9. No it is not. There could be more than one triangle meeting Rina's criteria. For example, if Rina said she had a triangle with area $A = 210$ and perimeter $P = 70$, triangles with sides 17, 25, 28 and 29, 20, 21 both will have said area and perimeter.

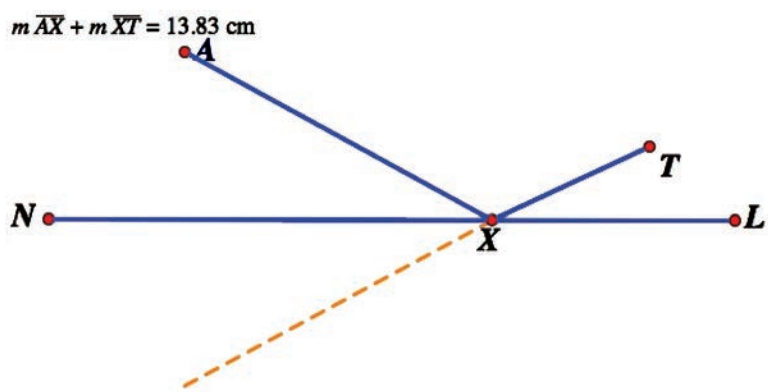
Problem Set 2

Opener

In GSP you can create this picture in a way so that you can move the point X around, which is way cool!

Hopefully you notice that as you move X from the left to the right you will see the distance decrease and then begin to increase again. The key is to find the *sweet spot* for X. Let's not bring in calculus or other heavy powered tools.

Isn't the shortest distance between any two points a straight line? Hmmm . . .



1. **a.** $2(11) = 22$
b. $11^2 - (\sqrt{2})^2 = 121 - 2 = 119$
c. $11^2 - x^2$

2. Each part of this problem is like a problem from Set 1. We want 2 numbers that sum to $22 = \frac{1}{2}(44)$ and multiply to be 117, 119, 100, 130. Use the square 11×11 to start, since $11 + 11 = 22$ and $11 \times 11 = 121$.

a. $121 - 117 = 4$, so the dimensions are $11 \pm \sqrt{4} = 11 \pm 2 = 13, 9$

b. $121 - 119 = 2$, so the dimensions are $11 \pm \sqrt{2}$

c. $121 - 100 = 21$, so the dimensions are $11 \pm \sqrt{21}$

d. $121 - 130 = -9$, so the dimensions are $11 \pm \sqrt{-9} = 11 \pm 3i$. Nonreal answer, so no such rectangle exists.

3. We can tie this problem to the previous problem. The previous problem wanted rectangles with perimeter $P = 2(l + w) = 44 \Rightarrow l + w = 22$ and area $A = 117, 119, 100, 130$. Here we are solving for the same kinds of numbers. We want numbers that sum to 22 and multiply to be 121, 120, 117, etc.

a. $w^2 - 22w + 121 = (w - 11)^2 = 0 \Rightarrow w = 11$

b. $121 - 120 = 1$, so $w = 11 \pm \sqrt{1} = 11 \pm 1 = 12, 10$

c. $121 - 117 = 4$, so $w = 11 \pm \sqrt{4} = 11 \pm 2 = 13, 9$

d. $121 - 119 = 2$, so $w = 11 \pm \sqrt{2}$

e. $121 - 100 = 21$, so $w = 11 \pm \sqrt{21}$

f. $121 - 2 = 119$, so $w = 11 \pm \sqrt{119}$

g. $121 - (-408) = 529$, so $w = 11 \pm \sqrt{529} = 11 \pm 23 = 34, -12$

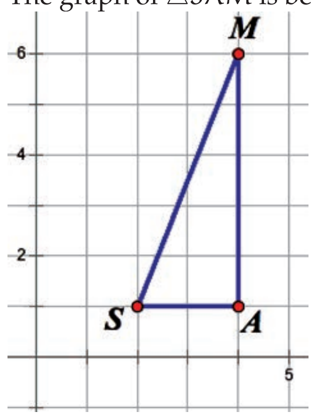
h. $121 - 130 = -9$, so $w = 11 \pm \sqrt{-9} = 11 \pm 3i$

4. To sum to 200, you can write your two numbers in the form $(100 + x)$ and $(100 - x)$. Their product is $(100 + x)(100 - x) = 9991 \Rightarrow 9 = x^2 \Rightarrow \pm 3 = x$
So our two numbers are 103 and 97.
We can also relate this to the previous problem, if we want . . . You can view this as trying to solve the polynomial $x^2 - 200x + 9991 = 0$.
Let's use previous ideas! $x^2 - 200x + 10000 = (x - 100)^2$ $10000 - 9991 = 9$, so the solution to $x^2 - 200x + 9991 = 0$ is $x = 100 \pm \sqrt{9} = 100 \pm 3$ $= 103, 97$.

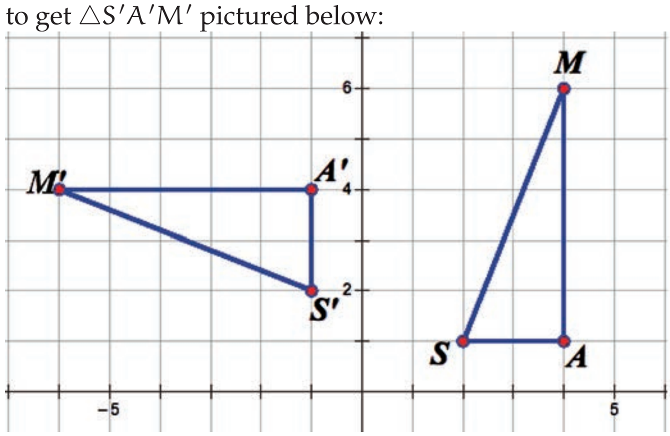
5. So we want three rectangles such that:
$P = A \Rightarrow 2(l+w) = lw \Rightarrow l = \frac{2w}{w-2}$, $w > 2$. A value for w can be chosen, and then you can find the value of l. Also notice, though, that $2(l+w) = lw \Rightarrow \frac{1}{2} = \frac{1}{w} + \frac{1}{l}$. So you can find two numbers such that their reciprocals sum to $\frac{1}{2}$. Some possible rectangles include:

4×4 gives $A = P = 16$
3×6 gives $A = P = 18$
$8 \times \frac{8}{3}$ gives $A = P = \frac{64}{3}$
$10 \times \frac{5}{2}$ gives $A = P = 25$
$7 \times \frac{14}{5}$ gives $A = P = \frac{98}{5}$
$18 \times \frac{9}{4}$ gives $A = P = \frac{81}{2}$
$20 \times \frac{20}{9}$ gives $A = P = \frac{400}{9}$
and so on . . .

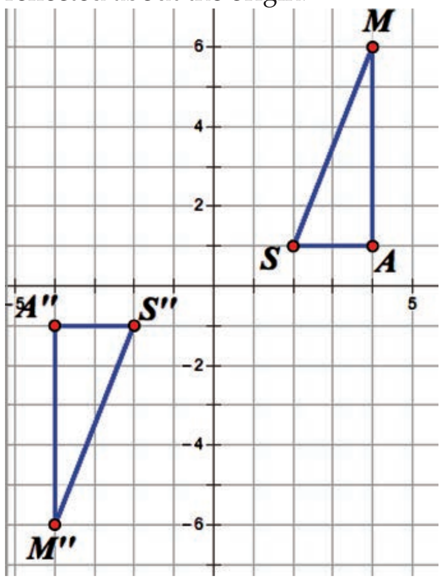
6. a. The graph of $\triangle SAM$ is below:



b. $\triangle SAM$ was rotated about the origin by 90° to get $\triangle S'A'M'$ pictured below:



7. You can see from the graph below that $\triangle SAM$ was reflected about the origin.



8. **a.** The distance from $(14,-4)$ to $(-2,-5)$ is $\sqrt{257}$, and the distance from $(14,-4)$ to $(4,11)$ is $\sqrt{325}$.
The distances are *not* the same.

b. The distance from $(-10,7)$ to $(-2,-5)$ is $\sqrt{208}$, and the distance from $(-10,7)$ to $(4,11)$ is $\sqrt{212}$.
The distances are *not* the same.

c. The distance from some point (x,y) to $(-2,-5)$ is $\sqrt{(x+2)^2+(y+5)^2}$ and the distance from ome point (x,y) to $(4,11)$ is $\sqrt{(x-4)^2+(y-11)^2}$.
If these distances were to be the same, then we would want to know when $(x+2)^2+(y+5)^2=(x-4)^2+(y-11)^2$.
Hey! These are circles! The first one is a circle centered at $(-2,-5)$ and the second one is a circle centered at $(4,11)$. Let's expand everything out and see what happens!

$$x^2+4x+4+y^2+10y+25=x^2-8x+16+y^2-22y+121$$

$$12x+32y=108 \Rightarrow y=-\frac{3}{8}x+\frac{27}{8}$$

All points (x,y) on the line $y=-\frac{3}{8}x+\frac{27}{8}$, and *only* those points, will be equidistant from $(-2,-5)$ and $(4,11)$. Geometrically, this line is the perpendicular bisector of the segment between the two points.

9. **a.** Applying the transformation twice maps $(x, y) \mapsto (-x, -y)$, which will reflect $\triangle SAM$ about the origin.

b. Applying the transformation three times maps $(x, y) \mapsto (y, -x)$, which will rotate $\triangle SAM$ $+270°$ or $-90°$ about the origin.

c. Applying the transformation four times maps $(x, y) \mapsto (x, y)$, which keeps $\triangle SAM$ in place. It will not move. We call this an identity map!

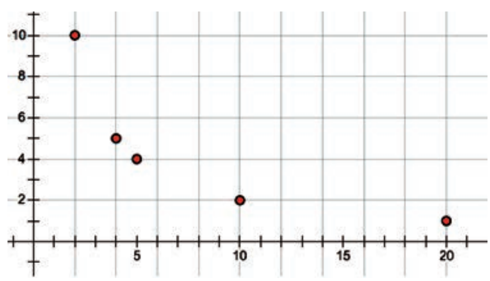
d. Applying the transformation five times will be the same as applying the transformation once, since the fourth time does not do anything to $\triangle SAM$.

e. Applying the transformation thirteen times will be the same as applying the transformation once, since performing the transformation in multiples of 4 does not do anything to $\triangle SAM$.

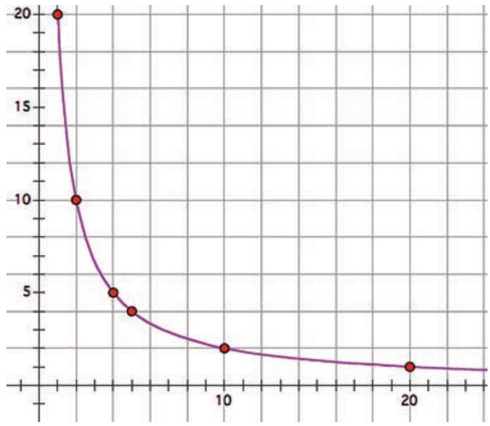
f. 101 divided by 4 is 25 with a remainder of 1. Thus applying the transformation 101 times will be the same as applying the transformation once.

If you remember your modular arithmetic, you can say $101 \bmod (4) = 1$.

10. **a.** Some possible points are plotted below:



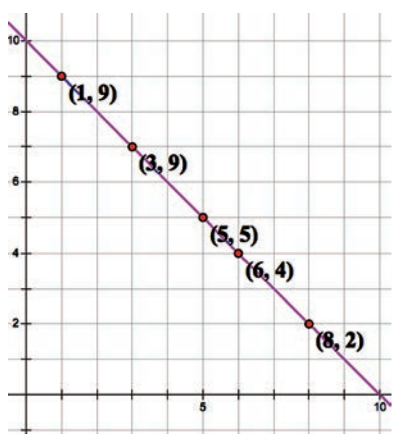
b. If our length and width can be any positive real numbers, then we can have infinitely many rectangles with area 20. $20 = lw \Rightarrow \frac{20}{l} = 2$. The graph of $\frac{20}{l}$ is below:



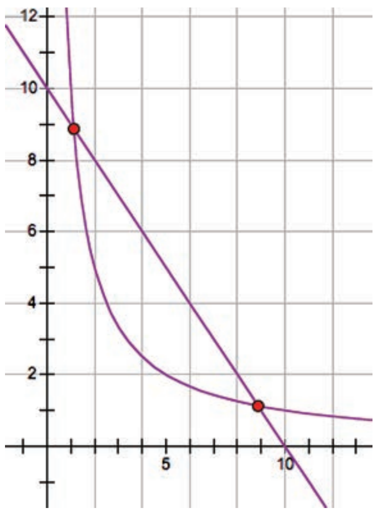
We can see that as values of l approach 0, values of w get larger so that the area is 20. Similarly, as the values of l get larger, the values of w get smaller and approach 0.

c. If our length and width can be any positive real numbers, then we can have infinitely many rectangles with perimeter 20. However, as you can see from the graph below, all the values must lie between 0 and 10 to ensure that the perimeter is 20.

The function in the graph is $w = 10 - l$. This is the result of working with the perimeter equation. $20 = 2l + 2w \Rightarrow 10 - l = w$.



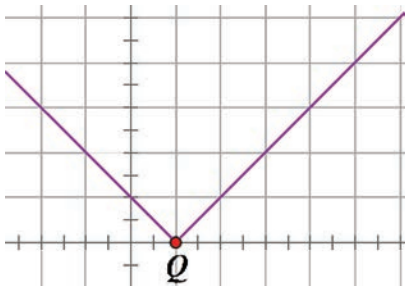
d. If we want a rectangle with both area and perimeter to be 20, then we want to know where our area and perimeter graphs intersect. In other words, when does $\frac{20}{l}$ equal $10 - l$?
$\frac{20}{l} = 10 - l \Rightarrow 20 = 10l - l^2 \Rightarrow l^2 - 10l + 20 = 0$
$l = \frac{10 \pm \sqrt{100-80}}{2} = \frac{10 \pm 2\sqrt{5}}{2} = 5 \pm \sqrt{5}$
The graphs illustrating the solutions are below:



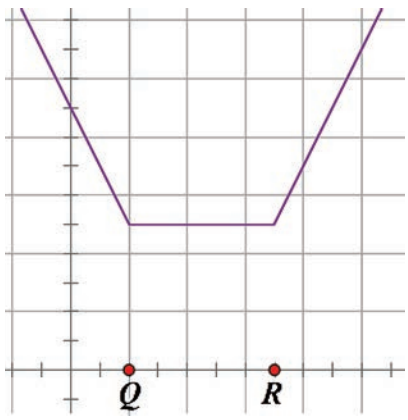
So there are two possible rectangles, one with either a length of $l = 5 + \sqrt{5}$ and width $w = 5 - \sqrt{5}$ or one with a length of $l = 5 - \sqrt{5}$ and width $w = 5 + \sqrt{5}$. Though, one could consider this to be just one rectangle.

11. In the situation where the area and perimeter both equal 12, the two graphs of $2l + 2w = 12$ and $lw = 12$ do not intersect. Therefore there is no rectangle with area 12 and perimeter 12.

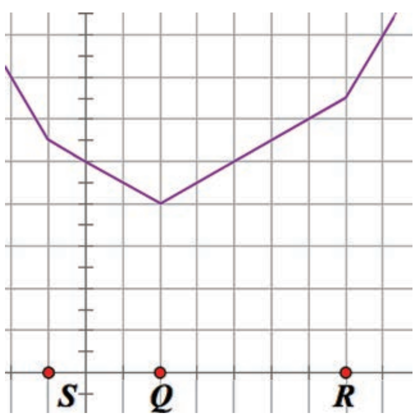
12. **a.** Let P be at the point x on the number line. If Q is an arbitrary point on the number line, then the distance from some point P to Q is $f(P) = |x - Q|$ and the graph would look something like:



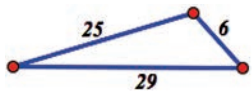
b. For total distance, we sum the distances from P to Q and from P to R and we get $f(P) = |x - Q| + |x - R|$ and the graph would look something like:



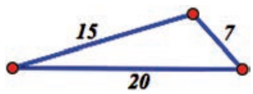
c. For total distance, we sum the distances from P to Q, from P to R, and from P to S and we get $f(P) = |x - Q| + |x - R| + |x - S|$ and the graph would look something like:



13. The triangle below has both an area and perimeter of 60.



The triangle below has both an area and perimeter of 42.



14.
- **a.** The remainder is -84
- **b.** The remainder is -76
- **c.** The remainder is -50
- **d.** The remainder is 0
- **e.** The remainder is 80

15. The remainders from the long division problem are the same as the values in our table! Whoa!

x	$f(x)$
0	-80
1	-84
2	-76
3	-50
4	0
5	80

16. Based on what we saw in problems 14 and 15, we can just evaluate $f(x)$ at values rather than using long division on a 12^{th} degree polynomial.

- **a.** The remainder when dividing $f(x)$ by $x-1$ is equivalent to evaluating $f(1)$. $f(1) = 3$.
- **b.** The remainder when dividing $f(x)$ by $x-2$ is equivalent to evaluating $f(2)$. $f(2) = 4,101$.
- **c.** The remainder when dividing $f(x)$ by $x+1 = x-(-1)$ is equivalent to evaluating $f(-1)$. $f(-1) = -3$.