Problem Set 1

Opener

1. On a piece of paper, write the results of flipping a coin 120 consecutive times, 10 flips per row, writing heads as 1 and tails as 0.

   But wait: **don't flip coins**. *Fake* the data. Your goal is to write a highly believable fake with no help from anyone or anything. Don't use anything to generate the flips other than your brain. Make it look organized. Write your full name on the page.

One possible row is 1111000010: four heads, then four tails, then one head, then one tail.

2. On a different piece of paper, write the results of flipping a coin 120 consecutive times, 10 flips per row, writing heads as 1 and tails as 0.

   But wait: **flip coins**. Write down the actual results of flipping a coin 120 consecutive times. Don't use anything to generate the flips other than your coin. Make it look equally organized. Write your full name on this page, too, in the same way.

The goal is to make the two lists indistinguishable, except for the different flips.

3. Suppose you were handed two lists of 120 coin flips, one real and one fake. Devise a test you could use to decide which was which. Be as precise as possible.

4. **a.** Mark one of your real or fake pages with the letter A. Mark the other with letter B. *Be sure you know which is which.*

   b. Exchange lists with someone who wasn't working with you. Use the test you developed in Problem 3 to decide which list is real and which is fake. Then, find out if you were right.

   c. Continue to exchange lists, improving your test as you go.

5. Talk about what happened. What went wrong with the fakes? What tests were accurate, and what tests weren't? What kind of fake would you make next time?

Important Stuff

There is more than one way to do this! Keep track of the assumptions you make. Consider comparing the different options here, and differences in corresponding results.

6. **a.** Find the probability that when you pick two integers between 1 and 5 (inclusive), they do not share a common factor greater than 1.
   b. Repeat for picking between 1 and 6, 1 and 7, 1 and 8, 1 and 9.

7. Tina says that if a set of 120 flips has exactly 60 heads and 60 tails, it's *obviously* fake.
   a. **In 10 seconds or less** estimate the proportion of fake data sets that have exactly 60 heads and 60 tails.
   b. **In 10 seconds or less** estimate the proportion of real data sets that have exactly 60 heads and 60 tails.

8. In 120 coin flips, what do you think would be some plausible values for the longest "run" of heads or tails? Explain briefly.

Neat Stuff

9. What's the probability that an integer picked from 1 to n is a perfect square if
   a. $n = 20$?
   b. $n = 200$?
   c. $n = 2000$?
   d. $n = 20000$?
   e. What is happening "in the long run" (as n grows larger)?

10. Faynna offers you these two games:

    Game 1: You roll a die four times. If you roll a six any of the four times, you win.

    Game 2: You roll a pair of dice 24 times. If you roll boxcars (double sixes) any of the 24 times, you win.

    Aside from the fact that Game 2 takes longer to play, which of these games would you rather play to win? Or do both games have the same chance of winning?

Problem 10 lies at the foundation of probability theory, and was originally solved by Pascal.

11. Brian, Brian, Marla, Jennifer, Moe, and Elmer go to dinner every night and play "credit card roulette": the waiter picks one of their six credit cards at random to pay for the meal.

Problem 11 was probably not posed by Pascal. He always paid in cash.

- **a.** What is the minimum number of meals it will take before each of them has paid at least once, and what is the probability of this occurring?
- **b.** What is the maximum number of meals it will take before each of them has paid at least once? Uh oh.
- **c.** What is the *mean* number of meals it will take before each of them has paid at least once?

Tough Stuff

12. In Yahtzee, you get three rolls and you're looking to get all 5 dice to be the same number. You can "save" dice from one roll to the next. There are other goals, but it's not called Yahtzee for nothing.

- **a.** Find the probability that if you try for it, you will get a Yahtzee of all 6s by the end of your third roll.
- **b.** (*harder*) Find the probability that if you try for it, you will get some Yahtzee by the end of your third roll. Assume that you always play toward the nearest available Yahtzee: if your first roll is 2-3-3-5-6, keep the 3s and roll the rest.

13. What is the mean number of meals it will take before the six friends from Problem 11 each pay at least *twice*? At least n times?

Problem Set 2

Opener

On the unpopular game show *Dice Rolling For Dollars*, a lucky contestant has the chance to win an *unlimited* amount of money.

In front of them is a huge number of standard dice. The contestant will pick a specific number of dice, then roll them all at once. The goal for the contestant is to avoid rolling a six.

If the contestant avoids a six, they earn **$1000 for every die they rolled**. If the contestant rolls any sixes, they win nothing.

1. How many dice should the contestant pick?

Important Stuff

2. Describe, in complete detail, a test you could perform on a set of 120 coin flips that would help you decide whether it is real or fake. Try to revise or improve your test from Problem Set 1.

3. *Use your test* on your choice of four of these seven data sets to decide whether they are real or fake.

In each of these, read from left to right across the rows. Set (a) begins with two heads, a tail, three heads, two tails, then eight heads in a row.

a.	1101110011	1111110000	1111111111
	1111111110	1110001111	1110000111
	0001111111	0001111100	1110010101
	1001001111	1100000111	0000011111
b.	1001001110	1011000011	0010011100
	1011100010	1011100111	1011100011
	1001101010	0010011011	0110101101
	0111001100	1100110110	0110010011
c.	0001011010	1001011111	0110110011
	0000000000	1000011110	0100100011
	1010100011	1111111010	0101100110
	1001011000	1111001110	1100101011
d.	1101101000	1111110010	1010101100
	1011001100	1001101011	1000110110
	1011010010	0000100000	0111111000
	0010001011	1011100000	0000000100

e.

0010010110	0111000110	1001101011
0100011010	1010001000	0100000101
0000111010	1111010001	0000011010
1011110111	0000010011	1111111111

f.

1100100011	1010111001	1111001010
1001110110	0011100000	1010101001
0011000101	1101011001	1001001001
1100001101	0011101101	1010011000

g.

0011010000	0101101101	0110010001
0110011100	1111110100	1100111101
0010011001	1000111010	1000111010
1011000111	0100000000	0111011001

4. **a.** What is the probability of throwing four heads on four consecutive coin flips?

b. What is the probability of throwing four coins and having them all come up "the same"?

c. What is the probability of throwing 10 coins and having them all come up "the same"?

5. 70 people each try the experiment of flipping 120 coins. Estimate the longest consecutive "run" of heads or tails among these 70 people, and explain how you came up with this estimate.

6. Find the probability that two positive integers do not share a common factor greater than 1, given that . . .

a. . . . one of the numbers is 1.

b. . . . one of the numbers is 3.

c. . . . one of the numbers is 5.

d. . . . one of the numbers is 9.

e. . . . one of the numbers is 6.

7. Find the probability that two integers between 1 and 10 (inclusive) have no common factor greater than 1. There is more than one way to do this: how do the possible answers compare?

Neat Stuff

8. **a.** Trang rolls four dice. What is the probability that she will *avoid* rolling a six on any of them?

b. Bryce also rolls four dice. What is the probability that he will *hit* at least one six?
c. Kitty rolls two dice. What is the probability that she will roll two sixes together?
d. Usha rolls two dice 24 times. What is the probability that she never rolls two sixes together?

9. Gail gives you a (potentially real or fake) list of 120 coin flips. It turns out to have exactly 60 heads and 60 tails. Does this suggest the list is real or fake? How strong would you consider this evidence to be?

10. In 120 truly random coin flips, what should be the average number of "runs" of flips? For example, the following flipping sequence has 7 runs:

011100011010

11. Participants in a version of this course generated 70 sets of real coin flips, and 70 sets of fake coin flips. 15 of the 140 data sets had an exactly 60 heads and 60 tails, and 8 of those 15 were fake. Reconsider your answer to Problem 9 in light of this information.

12. 44 of the 140 data sets had no "runs" of 6 or longer. Of these 44, 33 were fake. If Jessica gives you a data set with no runs of 6 or longer, does this suggest her data set is real or fake? How strong would you consider this evidence to be?

13. On *Extreme Dice Tossing* players are paid $1,000 multiplied by the total *value* of the dice they throw, as long as they avoid the pesky six. (Previously, the contestant won $1,000 multiplied by the *number* of dice thrown.)
a. Why might the show use the six as the "bankrupting" number in the game?
b. Find the contestant's best strategy, and the amount of money the show should expect to give out per contestant.

14. Jack is chosen to play a coin-flipping game, in which he gets $1 every time he flips heads. But, if he flips tails he is in "danger" and must flip heads next. If he flips tails twice in a row, he will "bust" and lose all his money (but continue playing). The game lasts 10 flips.

Perhaps Danger is Jack's middle name?

a. Find the probability that Jack survives all 10 flips without busting even once.

b. Determine the average amount of money Jack could expect after 10 flips.

c. What would happen in a longer game? Will the average payout increase or decrease?

Tough Stuff

15. On the game show “Tic Tac Dough”, the bonus game consists of nine squares:

Two squares containing TIC and TAC
Six money squares worth 100, 150, 250, 300, 400, 500
The dreaded DRAGON, which ends the round

The player wins if they can hit TIC and TAC, *or* if they can collect $1000 from the money squares, before hitting the dragon. Find the probability that the player wins the game.

16. In a set of 120 real coin flips, what is the probability of getting at least one “run” of 7 consecutive flips (either heads or tails)? What is the probability of a run of at least 8? 9?

CHAPTER

2 Facilitator Notes

The facilitator notes are designed to be used as needed. Each session has two components:

1. **Goals of the Problem Set:** here we lay out what the principal ideas of each session are.
2. **Notes on Selected Problems:** we identify a few problems that are worth going over in a whole group discussion.

We will put our emphasis on the main goals of each lesson, drawn from the problems in the "Important Stuff."

Problem Set 1

Goals of the Problem Set

The course begins simply; there is little or no assumed knowledge of probability and statistics. This problem set's opener is one that will be followed up several times, and shows the difficulty of trying to "fake" randomness.

Problem Set 1 is also intended as an introduction to the style of the course. Consider having participants read the Introduction to learn about the course expectations. Remember that participants are not expected to move through the entire problem set, and many will not move past "Important Stuff". That's fine; things in "Neat Stuff" are generally extensions or previews.

Notes on Selected Problems

In this opening activity, participants create two sets of data: a fake set of 120 coin flips, and a real set of 120 coin flips that comes from actually flipping coins. The purpose is to develop tests that can be used to determine whether or a set is real or fake.

A long-term goal is for participants is to realize that no test can be 100% accurate. Statistics is not about 100% accuracy, but about testing the likelihood of something. A 99% chance of something happening is strong compared to an 80% chance of something happening, for example.

This first activity gives students an inroad into the statistical environment. Give participants a lot of time to construct fakes; the revelation of how real data behaves, notably the lengths of strings of consecutive heads or tails, should be revealing. A full-group discussion could make guesses about the longest run of heads or tails for the entire room, checked against the actual data (Problem 5 in Set 2 will ask a similar question).

Be sure participants know that they need to write the entire sequence in order of the flips. Also, watch out for participants flipping multiple coins at once, since they may be writing down these results in an order that removes the randomness. If either of these things happens, try to catch it quickly, since the experiment needs to be restarted and takes a while. Some participants may desire to simulate the experiment with software, but we feel the reality of the coin-flipping is important for its impact.

Also, keep a record of the tests participants generate. These tests will be used later for work on statistics and conditional probability. Look particularly for tests involving the number and length of "runs" and a test such as "exactly 60 heads and 60 tails is fake". Problems 7 and 8 speak to this track. Later, participants will compare the compiled real and fake data to answer conditional questions such as "For all sequences that have exactly 60 heads and 60 tails, what proportion are fake?"

These are some tests teachers developed during this session at PCMI 2013. In general, dive into the simpler tests, ones which you are certain all participants can understand or duplicate.

- Heads per row (too close to 50% consistently = fake).
- Number of times there was a switch; more switches = fake.
- If there were mostly streaks of 3s in a row (and no longer), that was the fake.
- Heads per row – was the distribution bell-shaped?

- Strings of 8 or 9 or 10 heads in a row were plausible. Once you had 11 or 12 in a row that meant it was fake.
- Look at columns instead of rows.
- Length of "runs" of 0s or 1s, found mean and standard deviation of length. Higher standard deviation = real.

Problem Set 2

Goals of the Problem Set

A major goal here is to move participants toward developing a yes-or-no condition for determining whether a coin flipping set is real or fake. Participants will probably not yet understand that there is no such test, and many tests will succeed in correctly categorizing the sequences listed in the set. In the end, these tests will give evidence (perhaps very strong evidence) of whether a coin flipping set is real or fake, but no guarantee is possible.

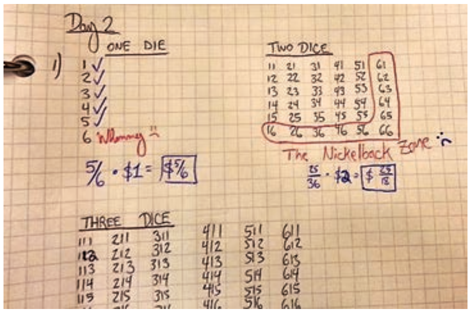
The other major goal is an informal understand of expected value and an exploration of participants' incoming knowledge of probability. The problem set starts with a game that should lead to a debate, and it is not as important to resolve that debate as it is to seed the ideas for what is coming.

Notes on Selected Problems

We strongly suggest playing the Opener game in front of participants before handing out the problem set. At PCMI 2013 this game was played twice for $1 per die, with extra time before the contestant's decision so that participants could know what is controllable.

Problem 1 should lead to some discussion about what is best for the contestant; try to focus discussion toward participants stating what they are measuring and why. There is no need to introduce "expected value" here since that will come in later problem sets. Some participants will argue that 1 die is best, because it gives the greatest chance of victory, while others will argue that a very large number of dice is best, because it gives the chance of a big victory. Watch for participants' use of probability theory or expected value here: do they use $\frac{5}{6}$? Do they recognize that the probability for multiple dice is multiplicative? Do they raise $\frac{5}{6}$ to a power? What sort of representation do they build? Keep it very informal here, and especially try not to overwhelm new participants.

Here's a sample of something that might be worth showing to the full group, because it does not overwhelm participants who are new to these concepts.



In Problem 3 participants will apply the test built in Problem 2; some will want to revise their test when it breaks. The difference between this activity and the one in Set 1 is that it is a single test rather than a comparison.

Problems 6 and 7 move toward the goal of determining the probability that any two positive integers do not share a common factor greater than 1. For now, these problems will feel disconnected from the others.

Problem Set 3

Goals of the Problem Set

Two main purposes are achieved in this problem set. First, this set introduces the concept of expected value, and the introduction of conditional probability problems. Expected value will be used throughout the course in the games, and then in the work on election math: the expected number of dollars won, or the expected number of electoral votes won, for example. Several key properties of expected value will be explored, including its linearity.

Second, this set asks participants to explore some problems requiring work in conditional probability: given some information, the probability of smething changes. This ties to statistics, specifically Bayesian analysis, where a probability is affected by new information. This will tie back to the coin flipping tests: *if* a coin flipping sequence has more than 70 of 120 heads, how likely is it to be fake?

CHAPTER 3 Solutions

Problem Set 1

1. Answers will vary. One set of fake data:

Flips 01-10:	1 0 0 1 1 1 0 0 1 1
Flips 11-20:	1 0 0 1 0 1 1 1 0 0
Flips 21-30:	0 1 1 0 1 0 1 0 1 0
Flips 31-40:	0 0 0 1 1 1 0 1 1 0
Flips 41-50:	1 0 1 1 0 0 1 1 1 0
Flips 51-60:	0 0 1 0 1 1 1 0 0 0
Flips 61-70:	0 0 1 0 1 1 1 0 1 1
Flips 71-80:	0 0 0 0 1 1 1 1 0 1
Flips 81-90:	1 0 0 1 0 0 1 1 0 1
Flips 91-100:	1 0 0 0 1 1 0 1 1 0
Flips 101-110:	0 0 0 0 1 1 1 0 1 1
Flips 111-120:	1 0 0 1 1 0 1 1 0 1

2. Answers will vary. One set of real data:

Flips 01-10:	1 1 1 1 1 1 0 0 1 1
Flips 11-20:	0 1 1 1 1 1 1 1 0 0
Flips 21-30:	0 0 0 0 1 0 1 0 1 1
Flips 31-40:	1 1 1 1 0 1 1 0 1 1
Flips 41-50:	1 0 0 1 1 1 1 1 0 0
Flips 51-60:	0 0 1 0 1 1 1 1 1 1
Flips 61-70:	0 0 1 0 1 1 1 1 1 0
Flips 71-80:	0 0 1 1 1 1 1 0 1 1
Flips 81-90:	0 1 0 1 1 1 1 1 0 0
Flips 91-100:	1 1 1 0 0 0 0 1 0 1
Flips 101-110:	1 1 0 0 1 0 1 0 1 1
Flips 111-120:	0 1 0 1 1 0 0 0 0 1

3. Some possible aspects that might indicate fake:

- Close to 50% heads in each row.
- More switches between 0s and 1s.
- A "string" of heads (back-to-back heads) is not usually longer than 3.
- Though a string of 8 or 9 heads is plausible, a string of 11 or 12 could indicate fake.

What else do you see that might indicate fake?

4. Get to work! What else do you see that might help you refine your test? You can get complicated with your test!

5. Bring your ideas together as a group. Can you refine and improve your test? Can you come up with a set of tests? What did you discover?

6. There's more than one way to do this. Let's look at the case where doubles are allowed, and (1,2) is "the same" as (2,1).

a. Possible Outcomes:

				5,5
			4,4	4,5
		3,3	3,4	3,5
	2,2	2,3	2,4	2,5
1,1	1,2	1,3	1,4	1,5

The orange pairs have no shared common factor other than 1. 10 orange pairs out of a total of 15 pairs, which appears to give a probability of $= \frac{10}{15} = \frac{2}{3}$

					6,6
				5,5	5,6
			4,4	4,5	4,6
		3,3	3,4	3,5	3,6
	2,2	2,3	2,4	2,5	2,6
1,1	1,2	1,3	1,4	1,5	1,6

The orange pairs have no shared common factor other than 1. 12 orange pairs out of a total of 21 pairs, which appears to give a probability of $= \frac{12}{21} = \frac{4}{7}$

Integers between 1 & 7:

						7,7
					6,6	6,7
				5,5	5,6	5,7
			4,4	4,5	4,6	4,7
		3,3	3,4	3,5	3,6	3,7
	2,2	2,3	2,4	2,5	2,6	2,7
1,1	1,2	1,3	1,4	1,5	1,6	1,7

The orange pairs have no shared common factor other than 1. 18 orange pairs out of a total of 28 pairs, which appears to give a probability of $= \frac{18}{28} = \frac{9}{14}$

Integers between 1 & 8:

							8,8
						7,7	7,8
					6,6	6,7	6,8
				5,5	5,6	5,7	5,8
			4,4	4,5	4,6	4,7	4,8
		3,3	3,4	3,5	3,6	3,7	3,8
	2,2	2,3	2,4	2,5	2,6	2,7	2,8
1,1	1,2	1,3	1,4	1,5	1,6	1,7	1,8

The orange pairs have no shared common factor other than 1. 22 orange pairs out of a total of 36 pairs, which appears to give a probability of $= \frac{22}{36} = \frac{11}{18}$

Integers between 1 & 9:

								9,9
							8,8	8,9
						7,7	7,8	7,9
					6,6	6,7	6,8	6,9
				5,5	5,6	5,7	5,8	5,9
			4,4	4,5	4,6	4,7	4,8	4,9
		3,3	3,4	3,5	3,6	3,7	3,8	3,9
	2,2	2,3	2,4	2,5	2,6	2,7	2,8	2,9
1,1	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9

The orange pairs have no shared common factor other than 1. 28 orange pairs out of a total of 45 pairs, which appears to give a probability of $= \frac{28}{45}$

7. Ready...go!

8. What do you think?

9. **a.** There are four perfect squares between 1 and 20:
1, 4, 9, 16
Thus the probability of choosing a perfect square is $\frac{4}{20} = \frac{1}{5}$

b. There are fourteen perfect squares between 1 and 200:
1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196
Thus the probability of choosing a perfect square is $\frac{14}{200} = \frac{7}{100}$

c. There are 44 perfect squares between 1 and 2000:
1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961, 1024, 1089, 1156, 1225, 1296, 1369, 1444, 1521, 1600, 1681, 1764, 1849, 1936
Thus the probability of choosing a perfect square is $\frac{44}{2000} = \frac{11}{500}$
I guess we didn't really have to list them all, but it was fun, right?!

d. Sheesh! I'm not going to list all of these. Well, I know that $100^2 = 10,000$...150^2 goes too far. Let's just play around here.
$141^2 = 19,881$, so there are 141 perfect squares between 1 and 20,000. Probability $= \frac{141}{20000}$

e. So what is going on?
4 perfect squares between 0 and 20.
Probability $= 0.2$

14 perfect squares between 0 and 200.
Probability $= 0.07$

44 perfect squares between 0 and 2,000.
Probability $= 0.022$

141 perfect squares between 0 and 20,000.
Probability $= 0.00705$

Hmmm...Approaching 0?

447 perfect squares between 0 and 200,000.
Probability $= 0.002235$

1414 perfect squares between 0 and 2,000,000.
Probability $= 0.000707$

4472 perfect squares between 0 and 20,000,000.
Probability $= 0.0002236$

Yes. Approaching 0.

Problem Set 2

1. Well, if only one die is rolled, there is a 1 in 6 (16.67%) chance of losing - or as I like to call it getting a *Whammy*. Boo. Winning chances: $\frac{5}{6}$ (83.33%). But then we would only win \$1,000 at most. Go big or go home!

2 dice!

⚀⚀	⚁⚀	⚂⚀	⚃⚀	⚄⚀	⚅⚀
					Whammy!
⚀⚁	⚁⚁	⚂⚁	⚃⚁	⚄⚁	⚅⚁
					Whammy!
⚀⚂	⚁⚂	⚂⚂	⚃⚂	⚄⚂	⚅⚂
					Whammy!
⚀⚃	⚁⚃	⚂⚃	⚃⚃	⚄⚃	⚅⚃
					Whammy!
⚀⚄	⚁⚄	⚂⚄	⚃⚄	⚄⚄	⚅⚄
					Whammy!
⚀⚅	⚁⚅	⚂⚅	⚃⚅	⚄⚅	⚅⚅
Whammy!	*Whammy!*	*Whammy!*	*Whammy!*	*Whammy!*	*Whammy!*

Chances of *Whammy* $\frac{11}{36}$ (30.56%).
Chances of winning \$2,000: $\frac{25}{36}$ (69.44%).
3 dice!
6 outcomes for each die, so 216 possible outcomes. How many have a 6? Let's refer to the 2 dice table... We get the table above if the first die is a 1, 2, 3, 4, or 5 and the next two dice are what is listed in the table above. Thus there are $11 \times 5 = 55$ possible outcomes that have the first die as a 1, 2, 3, 4, or 5 and one of the next two dice is a 6. Then there are 36 out comes if the first die is a 6. See the next table for a generalized version of 3 dice.
Chances of *Whammy*: $\frac{91}{216}$ (42.13%)
Chances of winning \$3,000: $\frac{125}{216}$ (57.87%)

die #1⚀⚀	die #1⚁⚀	die #1⚂⚀	die #1⚃⚀	die #1 ⚄⚀	die #1⚅⚀
					Whammy!
die #1⚀⚁	die #1⚁⚁	die #1⚂⚁	die #1⚃⚁	die #1⚄⚁	die #1⚅⚁
					Whammy!
die #1⚀⚂	die #1⚁⚂	die #1⚂⚂	die #1⚃⚂	die #1⚄⚂	die #1 ⚅⚂
					Whammy!
die #1⚀⚃	die #1⚁⚃	die #1⚂⚃	die #1⚃⚃	die #1⚄⚃	die #1⚅⚃
					Whammy!
die #1⚀⚄	die #1⚁⚄	die #1⚂⚄	die #1⚃⚄	die #1⚄⚄	die #1⚅⚄
					Whammy!
die #1⚀⚅	die #1⚁⚅	die #1⚂⚅	die #1⚃⚅	die #1⚄⚅	die #1⚅⚅
Whammy!	*Whammy!*	*Whammy!*	*Whammy!*	*Whammy!*	*Whammy!*

Let's roll more dice!!

4 Dice: Total possible outcomes are $6^4 = 1296$. We can break the undesirables into cases like we did with 3 dice. die #1 [3 dice]. If die #1 is a 1, 2, 3, 4, or 5, then there are $91 \times 5 = 455$ undesirable outcomes. If we have die #1=6 [3 dice], then we have 216 undesirable outcomes.
Chances of *Whammy*: $\frac{455+216}{1296} = \frac{671}{1296}$ (51.77%)
Chances of winning \$4,000: $\frac{625}{1296}$ (48.23%)

5 Dice: Total= $6^5 = 7776$
Chances of *Whammy*: $\frac{671*5+1296}{7776} = \frac{4651}{7776}$ (59.81%)
Chances of winning \$5,000: $\frac{3125}{7776}$ (40.19%)

6 Dice: Total= $6^6 = 46656$
Chances of *Whammy*: $\frac{4651*5+7776}{46656} = \frac{31031}{46656}$ (66.51%)
Chances of winning \$6,000: $\frac{15625}{46656}$ (33.49%)

7 Dice: Total= $6^7 = 279936$
Chances of *Whammy*: $\frac{31031*5+46656}{279936} = \frac{201811}{279936}$ (72.09%)
Chances of winning \$7,000: $\frac{78125}{279936}$ (27.91%)

So what kind of risk are you willing to take? Perhaps roll 5 dice? Or do you not like taking risks and want to stick with 4 dice? 7 dice or more seems pretty chancy. You could calculate how much money you'd win, on average, from each choice.

2. Yeah, you know this. Look back at Session 1...

3. Will your test pass muster?

a. Fake
b. Fake
c. Real
d. Real
e. Real
f. Fake
g. Fake

4. **a.** Each coin flip could be (H) or (T).

(H) or (T) — Flip #1 (H) or (T) — Flip #2 (H) or (T) — Flip #3
(H) or (T) — Flip #4

This gives $2 \cdot 2 \cdot 2 \cdot 2 = 16$ possibilities. We are interested in 1 specific outcome:

(H)	(H)	(H)	(H)
Flip #1	Flip #2	Flip #3	Flip #4

Probability=$\frac{1}{16}$. Boom!

b. All the same could be

(H)	(H)	(H)	(H)
Flip #1	Flip #2	Flip #3	Flip #4

or

(T)	(T)	(T)	(T)
Flip #1	Flip #2	Flip #3	Flip #4

Probability=$\frac{2}{16}$.
Can also think of this as there is a probability of $\frac{1}{2}$ for H and T, respectively. With 4 coins being flipped, the chances of getting all H or all T is $\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = \frac{1}{16} + \frac{1}{16} = \frac{2}{16} = \frac{1}{8} = \frac{1}{2^3}$

Neato!

c. Use the same logic from (b):
$\left(\frac{1}{2}\right)^{10} + \left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}} + \frac{1}{2^{10}} = \frac{2}{2^{10}} = \frac{1}{2^9}$

5. Well, looking at the real data in problem 3, the longest I can see there is a string of 12 heads...

6. **a.** Well, if one of the numbers is 1, then it doesn't matter what the other number is! There will be no shared common factor greater than 1.
Probability$= 1$.

b. Every 3^{rd} positive integer is divisible by 3, so will share with 3 a common factor greater than 1.
So $\frac{1}{3} =$ probability of sharing

Giving $\frac{2}{3}$ = probability of not sharing a common factor greater than 1.

c. Every 5^{th} positive integer is divisible by 5...
$\frac{4}{5}$ = probability of not sharing a common factor greater than 1.

d. $9 = 3 \cdot 3$, so any multiple of 3 will share a common factor with 9. Thus Probability of not sharing a common a common factor= $\frac{2}{3}$

e. 6 is divisible by 2 and 3, so we need to be aware of numbers divisible by 2 and 3. Every other positive integer is divisible by 2 and every 3^{rd} is divisible by 3, these numbers will share a common factor greater than 1. Every 6^{th} positive integer will be divisible by 2 and 3, so we need to not double count here.
Probability of sharing= $\frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$
Probability of NOT sharing= $\frac{1}{3}$

7. Different Integers!

Assuming the 2 integers are different, there are 45 possible ways to choose 2 different integers between 1 and 10 (inclusive).

Why are there 45, you ask? Well, we can have:

1&2, 1&3, 1&4, 1&5, ..., 1&10 (9 pairs)

+

2&3, 2&4, 2&5, ..., 2&10 (8 pairs)

+

3&4, 3&5, ..., 3&10 (7 pairs)

+

...

= 9pairs + 8 pairs + ... + 1 pair = 45 pairs

Brute Force (the orange pairs do not share):

								9,10
							8,9	8,10
						7,8	7,9	7,10
					6,7	6,8	6,9	6,10
				5,6	5,7	5,8	5,9	5,10
			4,5	4,6	4,7	4,8	4,9	4,10
		3,4	3,5	3,6	3,7	3,8	3,9	3,10
	2,3	2,4	2,5	2,6	2,7	2,8	2,9	2,10
1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9	1,10

31 out of 45 pairs: $\frac{31}{45}$

Using what we learned in problem 6:

All of the pairs involving 1 and another integer do not share. So there are $\underline{9}$ pairs there that don't share a common factor greater than 1.

With the eight pairs involving 2, $\frac{1}{2}$ of them will not share. There are $\underline{4}$ such pairs.

With the seven pairs involving 3, $\frac{2}{3}$ of them will not share ($\frac{2}{3} \cdot 7 = 4.\overline{6}$). There are $\underline{5}$ such pairs.

With the six pairs involving 4, since 4 is a power of 2... $\frac{1}{2}$ of them will not share. There are $\underline{3}$ such pairs.

With the five pairs involving 5, $\frac{4}{5}$ of them will not share. There are $\underline{4}$ such pairs.

With the four pairs involving 6, $\frac{1}{3}$ of them will not share. There is $\underline{1}$ such pair.

With the three pairs involving 7, $\frac{6}{7}$ of them will not share. There are $\underline{3}$ such pairs.

With the two pairs involving 8, $\frac{1}{2}$ of them will not share. There is $\underline{1}$ such pair.

With the one pair involving 9, namely 9&10, neither share a common factor.

There are a total of 9+4+5+3+4+1+3+1+1=31 pairs that do not share a common factor.

Doubles Allowed!

Now, if you can choose the same integer (i.e. have doubles), not much changes, then there are 10 more possibilities to add to the total of 45, giving 55. Now, all of these new pairs share a common factor greater than 1, namely them selves. Thus there are still 31 pairs that do not, but now the total is out of 55 pairs.

I wonder what other ways we can solve this?

8. **a.** Refer back to ideas from problem 4.
No sixes!! 4 dice!! $\left(\frac{5}{6}\right)^4$

b. $1 - \left(\frac{5}{6}\right)^4$

c. See also dice picture in opener from this problem set.
$\frac{1}{36}$ =probability ⚅⚅

d. Whoa! 24 rolls?! Okay... $\frac{35}{36}$ is the probability of not having ⚅⚅, so to do that in 24 rolls would be $\left(\frac{35}{36}\right)^{24}$.