Problem Set 1

Opener

1. Check out this floor from a church in Seville, Spain:

Taken from http://en.wikipedia.org/wiki/File:Semi-regular-floor-3464.eps

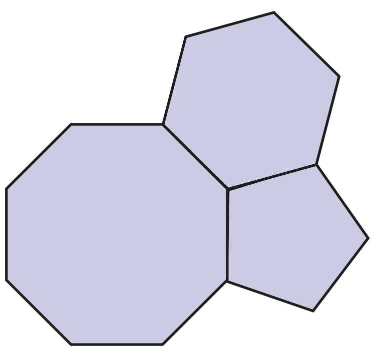
Imagine the pattern continuing forever in all directions. Well, all directions besides "up" and "down".

Recreate this pattern using your cut-out regular polygons. If you extended your pattern in every direction forever, what would be the ratio of the number of triangles to squares to hexagons that you'd need?

Hopefully you still have your cut-outs, or it's time to cut some more . . .

Important Stuff

2. Paul put a regular pentagon, hexagon and octagon together so they share a vertex. Does this work out? Explain why or why not.



3. Look at all the arrangements of regular polygons that you made on Set 1. For each arrangement, calculate $V - E + F$, where V is the number of vertices, E is the number of edges, and F is the number of polygon faces in the arrangement.

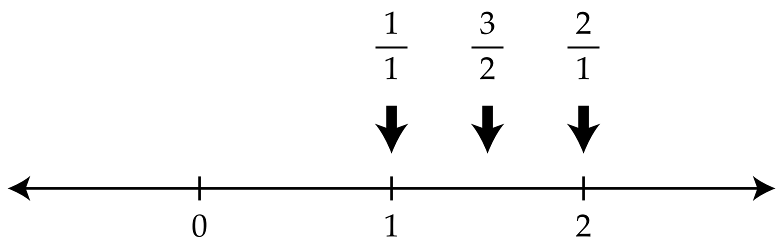
Include interior edges and vertices.

4. Start with 0 and 1, then keep adding two terms to get the next:

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

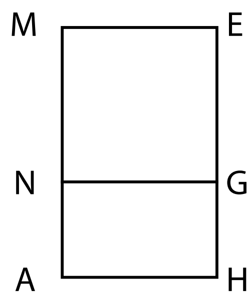
Eventually these numbers start multiplying like rabbits. Wait, no, they're adding like rabbits.

Use consecutive Fibonacci numbers (value/previous value) to make fractions and place these fractions on this number line. Keep placing! What do you notice?



5. You can take any non-square rectangle and chop a square out of it! In the diagram below, $MEGN$ is a square chopped out of rectangle $AHEM$.

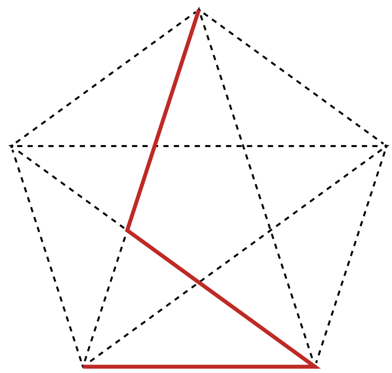
$AHEM$! May I have your attention please! Thanks, carry on.



- **a.** Suppose $EH = 2$ and $AH = 1$. Is rectangle $GHAN$ a scaled copy of the original rectangle?
- **b.** Suppose $EH = 3$ and $AH = 2$. Is $GHAN$ a scaled copy of $AHEM$ this time?
- **c.** What if $EH = 5$ and $AH = 3$? Well, shoot.
- **d.** If $EH = x$ and $AH = 1$, write a proportion that would have to be true if $GHAN$ is a scaled copy of $AHEM$.
- **e.** If $AH = 1$, estimate the length of EH to two decimal places.

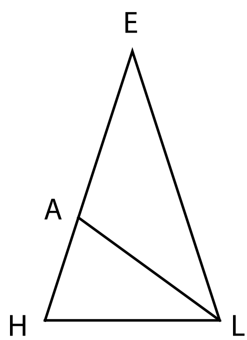
What's x? EH.

6. Bob measured these three marked segments in a regular star pentagram. He couldn't decide which was longest. Which one is longest? Explain how you know.



7. Here's the center triangle from Problem 6.

In Set 1, this was Triangle 2. Now, it's got a triangle, too!



a. Which triangle is a scaled copy of which other triangle? Be as precise as you can.

b. Give a brief explanation of *why* these triangles are scaled copies.

c. Replace these question marks in a way that must form a true proportion:

Two objects are *similar* if they are scaled copies. Congruent objects are similar, with scale factor 1.

$$\frac{\mathrm{EH}}{??} = \frac{??}{??}$$

d. If $\mathrm{LH} = 1$, estimate the length of EH to two decimal places.

Wait, which length is that? EH.

Neat Stuff

8. The golden ratio ϕ can be defined several ways. One way is that it's a positive number so that $\phi^2 = \phi + 1$.

This letter is pronounced like either "fee" or "fie".

a. Using the definition above, without using or figuring out the value of ϕ, show that $\phi^3 = \phi(\phi + 1)$.

ϕ^3 is not pronounced "foe".

b. Show that $\phi^3 = \text{blah}\phi + \text{bleh}$. You figure out the blahnks, but they're integers.

c. Show that $\phi^4 = \text{blih}\phi + \text{blöh}$, again without evaluating ϕ.

ϕ^4 is not pronounced "fum".

d. Show that $\phi^5 = \text{bluh}\phi + \text{blyh}$.

e. Describe a general rule for ϕ^n. Awesome!!

9. There are some ways to fit 3 regular polygons together perfectly to surround a vertex. Find as many as you can and write them as ordered triples (a, b, c) where $a \leqslant b \leqslant c$ are the number of sides in the three polygons.

10. For each triple you found in Problem 9, there's something interesting about the sum of the numbers, the product, or the sum of the reciprocals. One of those.

This problem is deliberately vague. Sorry about that!

11. Use what you found in Problem 10 to make a complete list of *all* the ways that 3 regular polygons can fit together at a vertex.

12. About what proportion of that Seville floor's area is made up of hexagons? squares? triangles?

13. Here's an interesting fact about ϕ^{-10}:

$$\phi^{-10} = a \cdot \phi + b$$

where a and b are integers. Find a and b by yourself, then verify the result with technology. What does this tell you about the value of ϕ?

Tough Stuff

14. A regular pentagon has side length 1. How far is it from one vertex to the midpoint of the opposite side?

Tough Stuff problems are (usually) significantly harder than the rest, but they're still fun to tackle.

15. **a.** Find the three regular polygons that come the closest to fitting together at a vertex, but don't because they overlap.

b. Find the three regular polygons that come the closest to fitting together at a vertex, but don't because they leave a super-tiny gap.

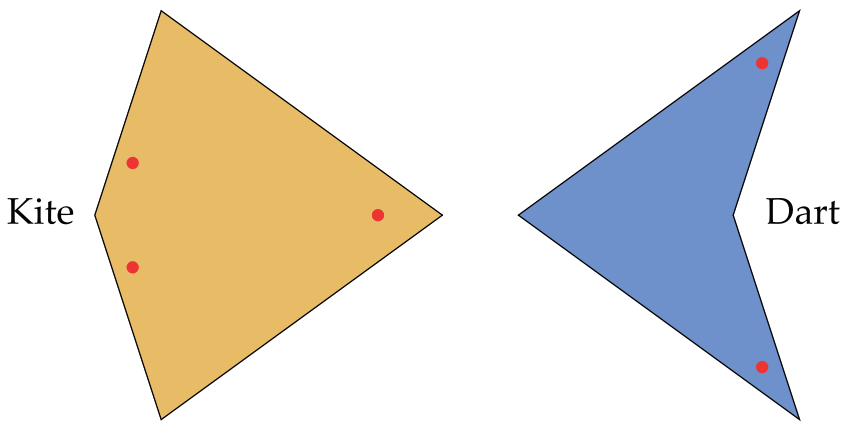
Problem Set 2

Opener

1. Cut out the polygons on the handout from page 15. The larger polygon is called a *kite* and the smaller polygon is called a *dart*.

Did you remember to go back?

Be careful not to call the smaller polygon a *kart*.



For 12 minutes, use the kites and darts to create tilings (arrangements with no holes or overlapping pieces). But, you must adhere to two rules.

Rule #1: Vertices can only touch other vertices, not the middle of edges. If two pieces share an edge, they must share the entire edge.

Rule #2: Edges must line up so that their "dots" are together.

Start a new tiling whenever you want. Write down your observations, especially any "recurring themes" you notice while tiling.

This problem is more difficult in New England, where "dart" and "dot" are the same word.

When 12 minutes are up, move on to Problem 2.

Important Stuff

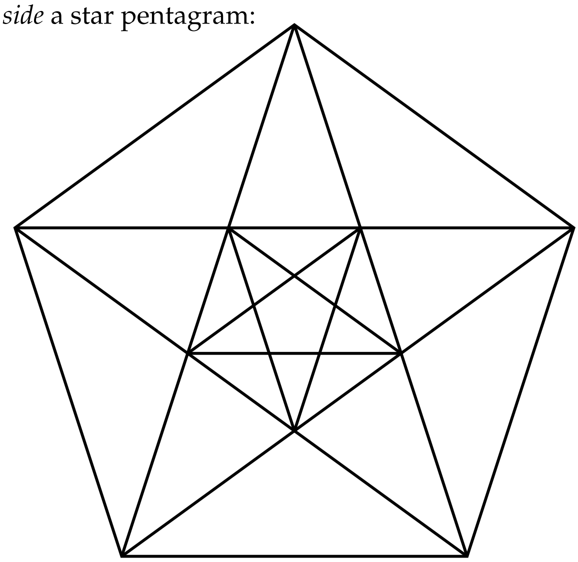
2. There are only 7 different ways that a vertex can be surrounded with kites and/or darts, while adhering to all of the rules. Find them! What is true of all 7 ways?

3. Get a piece of $8\frac{1}{2}$-by-$5\frac{1}{2}$ paper. (Fold letter-size paper in half.) Cover the paper with kites and darts (following all of the rules) until you can't see the paper any more. Count how many kites and darts are touching the paper. Gather some data and

If you're feeling adventurous, start over and cover an entire piece of paper, but you may need more tiles to do it!

make conjectures about the ratio between darts and kites in a tiling.

4. Lindsay says it's cool to put a star pentagram *inside* a star pentagram:



The kite and dart polygons are in Lindsay's diagram! Find them. Use the diagram and/or your work from Problem 2 to find the measures of all angles in the kite and dart polygons.

5. Draw a line connecting the word "kite" with the word "dart" on Problem 1, dividing the kite and dart in half. Where have we seen the resulting triangles before? What are their angle measures?

Neat Stuff

6. Kim wants to build a huge kite-and-dart tiling, extending in every direction forever. Estimate the ratio of the number of kites to darts she would need.

7. For each shape, build a tessellation using copies of that shape, or explain why it can't be done.

This is just a fancy word for *tiling*.

   a. A non-square rectangle
   b. An otherwise unremarkable parallelogram
   c. A rhombus
   d. A right triangle
   e. Any triangle
   f. A trapezoid
   g. The Z-lookin' piece from Tetris

A *rhombus* is a quadrilateral where all four sides have the same length. The plural is *rhombuseses*.

8. In Problem Set 2, you looked at the ratios of consecutive Fibonacci numbers. What happens when you begin with a different starting pair? For example, the *Lucas numbers* follow the same rule as the Fibonacci numbers, but start with 2 and 1:

$$2, 1, 3, 4, \ldots$$

What happens to the ratio of consecutive Lucas numbers? Try some other starting pairs and see what develops.

9. Here's an interesting function.

$$f(x) = 1 + \frac{1}{x}$$

The output of a function is determined by what you put in. For example,

$$f(3) = 1 + \frac{1}{3} = \frac{4}{3}$$

a. What's $f(1)$?
b. Find the exact values of $f(2)$ and $f(\frac{3}{2})$.
c. Carla starts with $x = 1$, then she runs the function repeatedly on outputs. What happens?
d. Diana starts with $x = \frac{1}{2}$ then does what Carl did. Notice anything?
e. Escher starts with $x = 0$. What happens?

You might notice more here by using exact values instead of decimals.

10. Problem Set 2 asked about the ways that three regular polygons could fit at a vertex. Find some ways to fit *four* regular polygons at a vertex, and look for any relationship among the quadruples (a, b, c, d) giving the number of sides of the regular polygons used.

It's okay to walk back to any previous problem at any time.

11. Find every possible way 4, 5, 6, or 7 regular polygons can fit at a vertex.

Using a shoehorn does not count as a possible way.

12. Kites and darts aren't regular polygons, but they have known angle measures. Look for a way to use the style of Problem 10 on kites and darts. If you didn't have to follow the matching rules, what are all possible ways to surround a vertex with kites and darts?

13. Start with a 203-by-77 rectangle. Chomp off the largest possible square from this rectangle, then continue chomping squares. What happens? Try again starting with different rectangles and see what you come up with.

14. Suppose the shortest side of each kite and dart has unit length. For both the kite and dart, find all side lengths, the area, and the length of the two diagonals.

15. In Problem 4, Lindsay drew a pentagram inside a pentagram. The larger pentagram is how many times larger (in area) than the smaller pentagram? Give your answer in terms of integers and Greek letters only, please.

Tough Stuff

16. In Problem 8 you noticed something. Find a nonzero starting pair for which it does *not* happen that the ratio of consecutive terms never approaches ϕ. Yes, they do exist.

And yes, they're nonzero!

17. For a lot of positive integers $N > 1$, there is an integer solution to

$$\frac{4}{N} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

a. Find solutions for $N = 2$ through $N = 10$.
b. Find the smallest N for which there is no solution, or prove that a solution exists for every N.

18. What is the ratio of kites to darts in an infinite kite-and-dart tiling? Prove it!

CHAPTER

2 Facilitator Notes

The facilitator notes are designed to be used as needed. Each session has two components:

1. **Goals of the Problem Set:** here we lay out what the principal ideas of each session are.
2. **Notes on Selected Problems:** we identify a few problems that are worth going over in a whole group discussion.

We will put our emphasis on the main goals of each lesson, drawn from the problems in the "Important Stuff."

Problem Set 1

Goals of the Problem Set

The course begins simply; there is little assumed knowledge of geometry, other than the knowledge that the angle measures in a triangle add to 180°. The main goal of the problem set is for participants to understand and discuss the ways regular polygons might meet at a vertex to completely cover it, in preparation of Set 2's introduction of tessellations.

Problem Set 1 is also intended as an introduction to the style of the course. Consider having participants read the Introduction to learn about the course expectations. Remember that participants are not expected to move through the entire problem set, and many will not move past "Important Stuff". That's fine; things in "Neat Stuff" are generally extensions or previews.

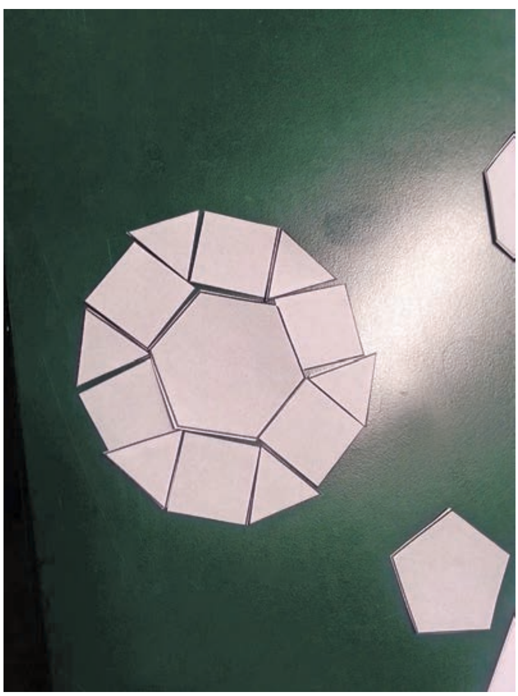
Notes on Selected Problems

Do not cut out the polygons before starting Set 1. The process of cutting out the polygons as a group encourages conversation and ice-breaking in a way that is equitable.

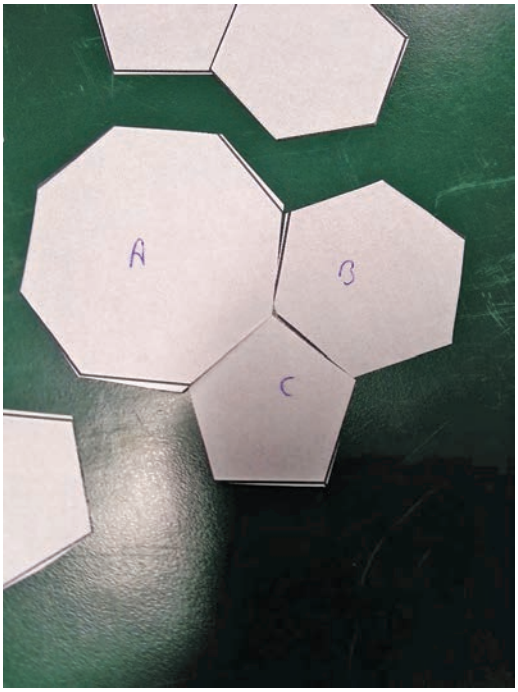
The opener is intended to solidify participants' understanding and recognition of the terms vertices, edges, and faces, which will be used throughout the course. Expect some pushback against counting vertices, edges, or faces that are "interior" to the shapes formed, even though the opener's sidenote is clear. If participants are still having trouble, ask them to count the pictured shape until they can count the correct number of vertices, edges, and faces given. Ultimately this is not critical, but an understanding of vertices, edges, and faces as part of an infinite tessellation must focus entirely on its interior (since there is no exterior).

The other, more mathematical, purpose of the opener is to find some combinations of shapes that completely surround a vertex. Use this language, rather than discuss specific angle measures, unless participants are all comfortable discussing the problem in this way.

Actively look for two specific patterns that participants may build, seen below. The first will become the pattern used in Set 2's opener:



The second is a near-miss, with an octagon, pentagon, and hexagon. In fact, we give the pentagons with the express hope of this happening, and it's happened every time we run the course:



If this is produced, try not to decide whether it is acceptable (it isn't), because Problem 2 of Set 2 will address this precise combination of shapes. Some participants may get annoyed that the pentagon is not more useful: this is good! This will make the pentagonal symmetry of Penrose tiles more surprising.

Save the cutouts participants produce! They will be used again several times.

Problems 3 and 4 will deepen this understanding: in particular, Problem 3 may encourage participants to talk about squares having 90-degree angles, and hexagons having 120-degree angles. Do not introduce or discuss a general rule for the angle measures in a regular polygon; participants should know the hexagon's angle by saying that 3 of them fit around a vertex. This also leads to the ideal solution to Problem 4: because 4 squares fit, and 3 hexagons fit, no number of pentagons can fit.

Problem 5 then helps participants find the exact angle measures in a pentagon, by thinking of it as 3 triangles (each of which has an angle sum of 180°). The diagram of this problem will be used again, since it is helpful in leading to one definition of the golden ratio. These triangles will appear many times (they are known as Robinson triangles); they are the triangles used to form the kites and darts of the Penrose tiling, as seen in Set 3. For now, keep the explanations as simple as possible, and watch out for participants having trouble understanding others' explanations due to different levels of incoming experience.

Problem Set 2

Goals of the Problem Set

The main purpose of this problem set is to introduce the golden ratio, by way of several different problems that each lead to the same number. In particular, these problems will be referenced later to help participants learn and prove that the golden ratio is irrational. For now, recognition that this ratio emerges in multiple problems, along with a rough estimate of the value of the ratio, is the target.

The secondary purpose, which is connected, is the tessellation work of the opener. The goal is to build an understanding that any tessellation comes from a replication from a core set of tiles, with a fixed ratio between the types of pieces involved. This turns out *not* to be true, even though it was believed to be true for thousands of years! This is true only for periodic tilings. Participants will discover the aperiodic Penrose tiling in Set 4.

Notes on Selected Problems

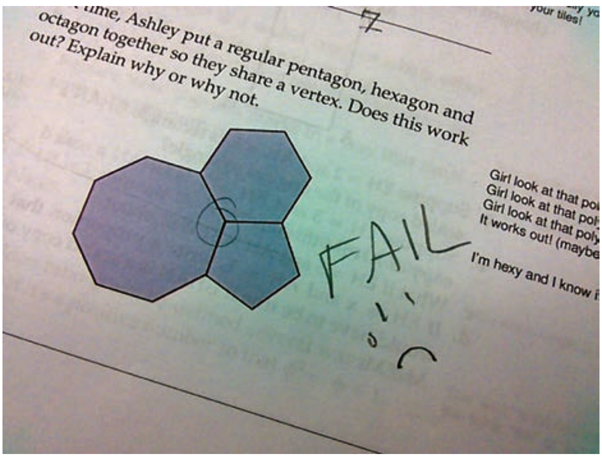
Expect the Opener to take a long time, and provide the tiles participants cut out during Set 1. Many participants will begin counting the tiles in larger zones of the floor, in order to estimate the ratios; encourage this, because participants can get accurate approximations for the ratios of the polygons, but remind them that this cannot work because the tessellation would continue forever in all directions.

Look for participants who find a single "stamp" that can be used to produce the pattern.

Some of the "stamps" involve cutting parts of shapes; the most typical are rhombus and triangle stamps. Once the ratios are calculated, participants should be encouraged to find a "stamp" that uses only whole polygons, specifically 1 hexagon, 2 triangles, and 3 squares. There is more than one possible configuration, but the typical one is seen here:

Watch for participants focusing on the area of shapes instead of the number of shapes involved in the pattern; participants may have trouble understanding why there are fewer hexagons.

Problem 2 follows up on a likely configuration from Set 1. This is the right time to discuss the angle measures of regular polygons, which are necessary to show that this configuration is invalid. Participants should have found the angle measures of the pentagon and hexagon during Set 1, and the angle measures of the octagon can be found in several ways that do not involve a general formula for the angle measures of all regular polygons.



In Problems 4 and 5, watch for participants who may have previous understanding of the connections between Fibonacci numbers and the golden ratio; it is common to see participants attempting to teach one another here, when it is meant to be discovered. In general, if you hear anyone talking about the golden ratio, stop them – unless you are certain all participants are comfortable with the conversation. The golden ratio is defined in Problem 8, but this name is not needed to move forward.

Generally throughout the Important Stuff, it will never be necessary for participants to know how to solve a quadratic equation, or to produce the exact value of the golden ratio. You should only present about the exact value of the golden ratio if you feel every participant has the right incoming experience to follow along.

Some participants may have trouble understanding the meaning of "scaled copy" in Problem 5; rather than give a formal definition, give either an example or redirect to another participant who you expect might be helpful but won't explain away the concept. Examples serve better here. One goal in both Problems 5 and 7 is for participants to figure out the correct correspondence, which is very helpful in understanding the setup of the equations involved.

If participants are having trouble with Problem 7, redirect them to Problem 6; they may not have noticed the diagram for Problem 7 embedded in the pentagram.

Problem Set 3

Goals of the Problem Set

The primary goal of this problem set is for participants to get used to the tessellations of kites and darts (Penrose tiles), and to work toward estimating the ratio of kites to darts in a large tessellation, similar to what happened in Set 2's opener. In Set 4, the ratio will be revealed to be the number that participants worked with during Set 2.

There is also an opportunity to revisit the pentagram in relation to the kites and darts, a diagram that can ultimately be used to prove the irrationality of the golden ratio (in Set 9's Neat Stuff).

Notes on Selected Problems

In general, use "kites and darts" and not "darts and kites". The reason to do this is so that the ratio participants seek is the ratio of kites to darts, which will be the golden ratio, instead of its reciprocal.

As with Set 1, the time spent cutting the shapes is useful as an icebreaker; at PCMI 2014, participants moved into new groups at the beginning of this session, so it was an opportunity to meet and talk while working toward the math.

Stick to the timing in the Opener: 12 minutes. The biggest reason to do so is that participants are likely to make many errors following the rules (notably the rule that dots must align). Problem 2 will help participants figure out what is and is not possible, but without the time spent doing poorly, Problem 2 won't be appreciated. Overall, expect Problems 1-3 to take an hour or more in total, but the time is useful and necessary.

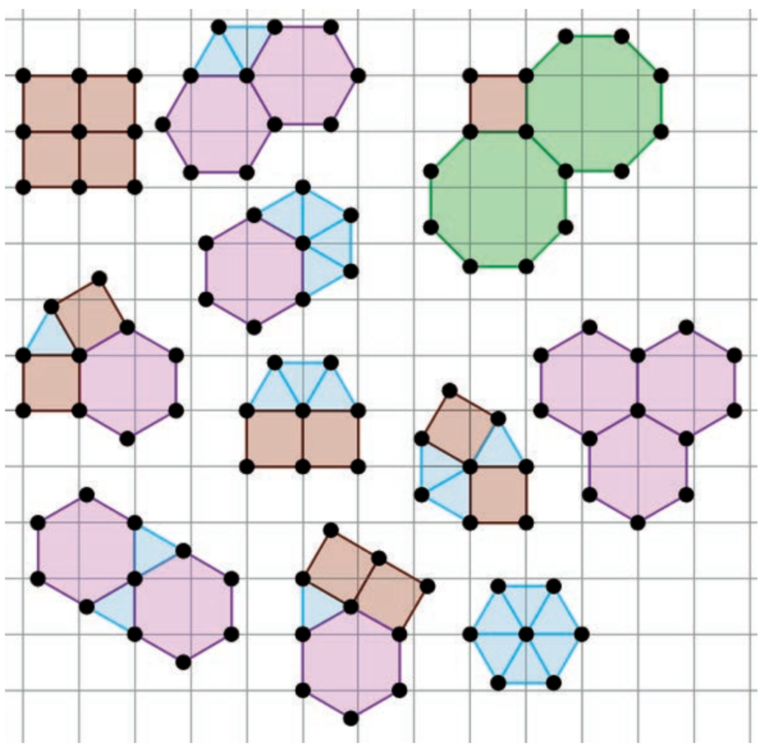
In Problem 2, look for participants attempting to determine the angle measures of the shapes by the ways they can be placed together. This is good, and can be made

CHAPTER

3 Solutions

Problem Set 1

1. Check this out!



We can do three hexagons (hexagons have six sides, so I will abbreviate: 6-6-6)

We can do four squares (4-4-4-4).

We can throw down six triangles (3-3-3-3-3-3).

A square and two octagons (4-8-8).

And so on... 3-3-6-6, 3-6-3-6, 3-4-4-6, 3-4-6-4, 3-3-3-3-6, 3-3-3-4-4, 3-3-4-3-4

2. Well, the shapes need to fill in all 360° around a vertex, so the interior angles need to add to 360°. We have seen, for example, that six triangles can surround a vertex. All the interior angles are 60°, with six of them, that adds up to 360°.

We cannot do that, however, with three squares and one triangle; this would leave a small gap. On the other hand, five triangles and a square would cause an overlap.

3. Squares, triangles, and hexagons, Oh my!

4. Since four squares completely surround a vertex and three hexagons completely surround a vertex, how many pentagons could completely surround a vertex? It would have to be more than 3 and less than 4. So, it can't happen!

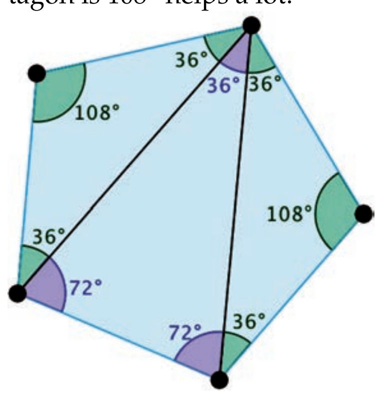
5. Go Britni! Slice the pentagon!

a. The angle measures in Triangle 1 add to 180°. Hey, that's true for all the triangles . . .

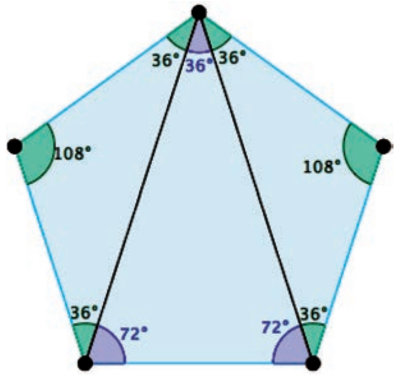
b. Since there are three triangles, the angle sum is 540°.

c. All three triangles are isosceles. Triangles 1 and 3 are congruent, but Triangle 2 is not congruent to Triangles 1 and 2.

d. Ba-da-boom! Knowing each angle of the pentagon is 108° helps a lot.



6. To make the star, we just split the pentagon five times as in problem 5.



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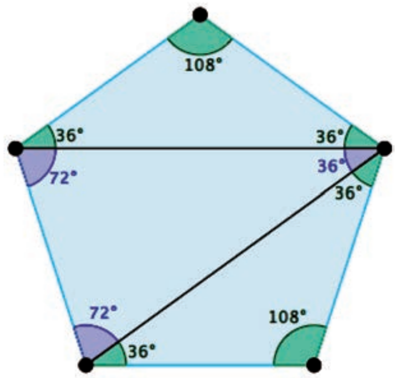
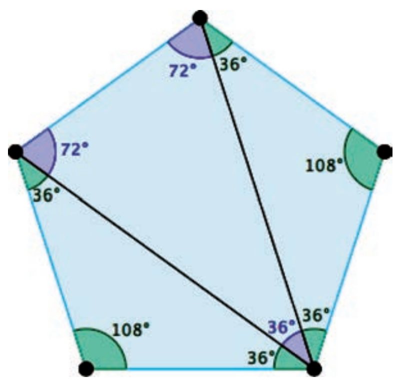
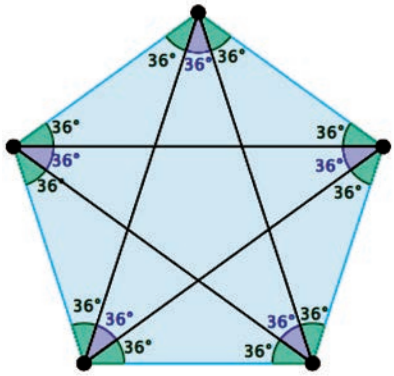


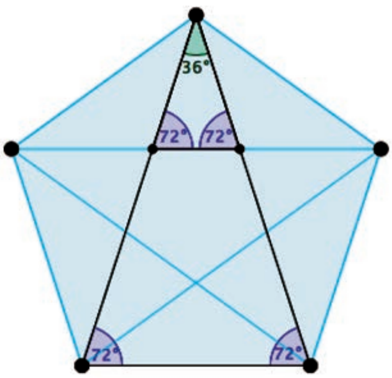
and again...

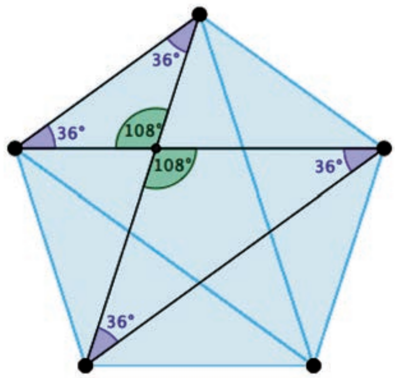


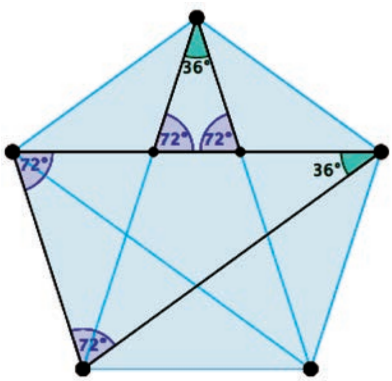
Five splits. All together now!

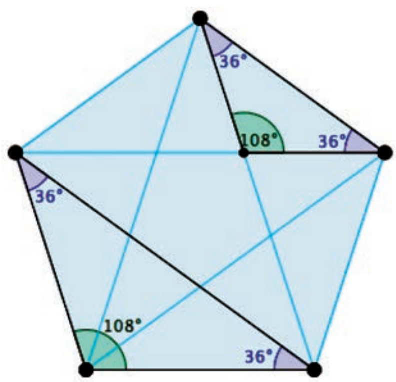


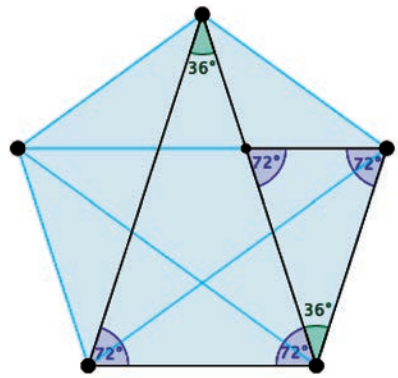
So what do you think? Four pairs of similar triangles... Here are a few:







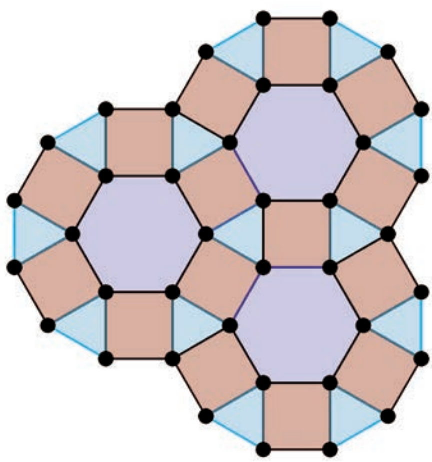




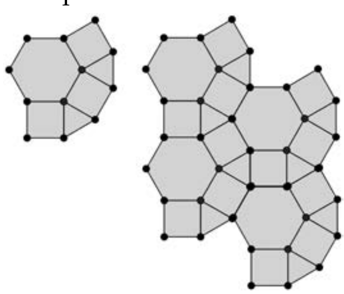
Problem Set 2

1. Hmmm.... let's explore this. What surrounds every vertex? 2 squares, 1 hexagon, 1 triangle... is that useful info?

How can I create this pretty picture over and over and over?



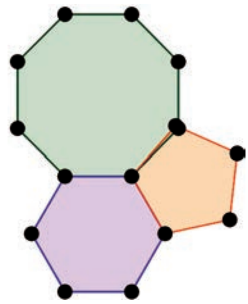
Stamp it!



2 Triangles: 3 Squares: 1 Hexagon

Stamp and repeat.

2. This looks sketchy. I think Paul is trying to trick us! Let's add up the interior angles.



Pentagon interior angles: 108°

Hexagon interior angles: 120°

Octagon interior angles: 135°

Sum = 363°. Too much! Girl, it doesn't work out!

3.

Arrangement	Vertices, V	Edges, E	Faces, F	$V - E + F$
Six △s	7	12	6	1
Four □s	9	12	4	1
Two □s, One △, One ⬡ (all)	10	13	4	1
Two △s, Two ⬡s (all)	11	14	4	1
Four △s, One ⬡	9	13	5	1
Three ⬡s	13	15	3	1
Three △s, Two □s (all)	8	12	5	1
One □, Two ⯃s	8	12	5	1

What?!

4. Fibonacci! Fibonacci!

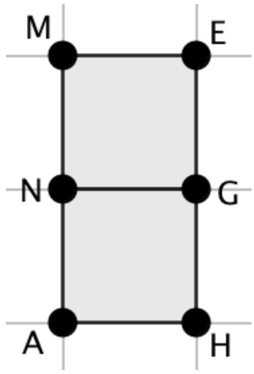
You should be placing:

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, ...$

$$\text{Ratios: } \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21} ...$$

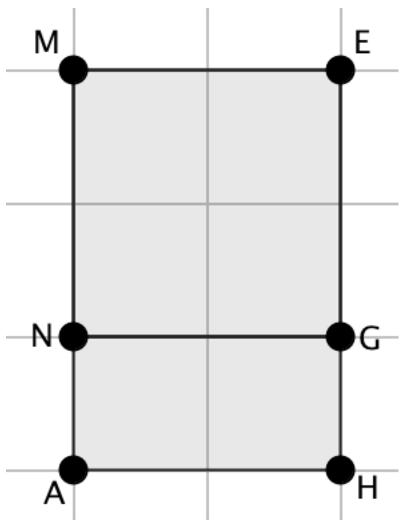
The numbers are staying around 1.6 and some change.

5. **a.** EH = 2 and AH = 1



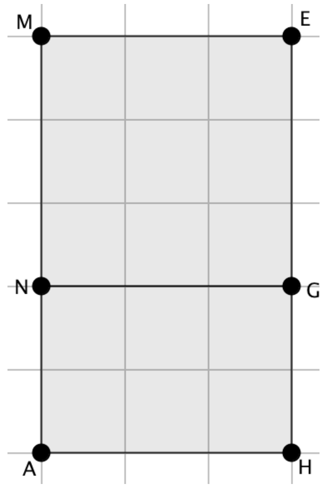
GHAN is a square, so it is not a scaled copy of AHEM.

b. EH = 3 and AH = 2



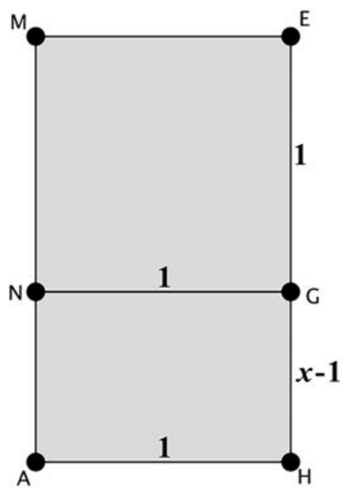
GHAN is 2×1, AHEM is 3×2. Nope, not a scaled copy!

c. EH = 5 and AH = 2



GHAN is 3 × 2, AHEM is 5 × 3. Nadda. Zip. Zilch.

d. So what needs to be true in order to have a scaled copy of AHEM?
EH = x and AH = 1



$$\frac{x}{1} = \frac{1}{x-1}$$

e. If AH = 1, then EH is about 1.6 and some change.

6. They are all the same! The isosceles triangles in the pentagram help explain this.

7. **a.** △ALH is a scaled copy of △HEL

b. Well, they are both isosceles, which of course isn't enough, but it's a start. And then we

know the angles. Yeah, the angles! The base angles are 72°. "Why?", you ask? Well, I'll tell ya "why"!
From Problem set 1 we know:

- $\angle EHL = \angle ELH = 72^\circ$ AND $\angle AEL = \angle ALE = 36^\circ$
- $HL \cong AL \cong AE$
- So $\angle EAL = 108^\circ$, which means that $\angle HAL = 72^\circ$

BOOM! Isosceles with same angles, yo! You do the math!

c. $\frac{EH}{LA} = \frac{LH}{HA}$

d. Well, if I must.
Ratios! Ratios! Lots of fun with ratios!

$$\frac{x}{1} = \frac{1}{x-1}$$

Say this looks familiar.

e. EH is about 1.6 and some change . . .

8.

$$\phi^2 = \phi + 1$$

a. $\phi^3 = \phi \cdot \phi^2 = \phi(\phi + 1)$

b. $\phi^3 = \phi^2 + \phi = \phi + 1 + \phi = 2\phi + 1$
blah=2, bleh=1

c. $\phi^4 = \phi \cdot \phi^3 = \phi(2\phi + 1) = 2\phi^2 + \phi = 2(\phi + 1) + \phi = 3\phi + 2$
blih=3, blöh=2

d. $\phi^5 = \phi \cdot \phi^4 = \phi(3\phi + 2) = 3\phi^2 + 2\phi = 3(\phi + 1) + 2\phi = 5\phi + 3$
bluh=5, blyh=3

e. Notice that $\phi^2 = \phi + 1 = (1 + 0)\phi + 1$
$\phi^3 = (1 + 1)\phi + 1$
$\phi^4 = (2 + 1)\phi + 2$
$\phi^5 = (3 + 2)\phi + 3$
Based on this pattern, $\phi^6 = (5 + 3)\phi + 5$
and $\phi^6 = (8 + 5)\phi + 8$
So how do you think ϕ^n should be written?