Preface

"To Think Deeply about Simple Things" was a motto of Arnold Ross, founder of the Ross Mathematics Program. Ross's philosophy has provided inspiration for many content-based professional development programs, each with its own personality and design, including the Summer School Teacher Program (SSTP) at the Park City Mathematics Institute (PCMI) and the PROMYS for Teachers Program (PfT) at Boston University.

More information about the SSTP can be found at `projects.ias.edu/pcmi/hstp/resources.html`. For PfT, see `www.promys.org/pft/`

PCMI, sponsored by the Institute for Advanced Study with major funding from Math for America, is a three-week summer program for those involved in mathematics: research mathematicians, graduate students, undergraduate faculty, undergraduate students, and precollege teachers.

The SSTP has been an integral part of PCMI from the beginning, and one of its features is a mathematics course two hours per day for three weeks, designed by a collaborative of teachers, educators, and mathematicians representing Boston University and Harvey Mudd College.

The goal of PCMI is to improve the quality of mathematics education and foster world-class mathematics research. To that end, PCMI has five core programs, each designed for a different group of people: secondary school mathematics teachers, graduate students in mathematics, undergraduate students studying mathematics, mathematics faculty interested in the teaching at the undergraduate level, and research mathematicians. The mathematics content course for the SSTP is coordinated with PCMI's mathematical research theme each year, which in 2018 was Harmonic Analysis. The goal of the SSTP course is not to delve deeply into the research theme but to take up mathematics that helps teachers put the topics in their curriculum into the broader landscape of mathematics as a scientific discipline. A lot of work in harmonic analysis involves overlapping and intersecting waveforms, as well as finding good-quality, simpler approximations for complex mathematical objects. For these reasons the 2018 course, published here as *Fractions: To Be Continued*, worked through concepts involving greatest common divisors, continued fractions, and finding approximations for rational and irrational numbers.

The Design of the Course

The central feature of the SSTP course is a set of intricately sequenced questions that engage participants in doing mathematics in ways that exemplify mathematical habits of mind, also prominently seen in the Common Core Standards for Mathematical Practice. What's important here is the structure of the program, because the program is not a "course" in the traditional sense. The materials provide participants with the opportunity for authentic mathematical discovery. Participants build mathematical structures by investigating patterns, use reasoning to test and formalize their ideas, offer and negotiate mathematical definitions, and apply their theories and mathematical machinery to solve problems. Through this experience, participants develop habits of mind for thinking about and doing mathematics, deepening their mathematical intuition, sense-making, and reasoning skills.

Each fall and spring, a team drawn from the PfT community meets regularly to create two or three major themes for the upcoming SSTP course, and for each theme, it creates a "soup" of potential problems and investigations that might be used at PCMI. Once in Park City, instructors Bowen Kerins and Darryl Yong create daily problem sets, revised each night to reflect what happened in the day's session. After the course, the problem sets are revised once more, solutions and hints are written, and the course is prepared for publication.

The problem sets are separated into three sections: Important Stuff, Neat Stuff, and Tough Stuff. The problems in Important Stuff contain the fundamental concepts that should enable everyone to move forward. The problems in Neat Stuff and Tough Stuff are there for those who are curious or looking for a challenge.

The distinguishing features of the program have stayed constant over the years:

Teachers as professionals. These materials are designed and implemented by practicing teachers in collaboration with mathematicians and mathematics educators. Experienced teachers at PCMI mentor teachers new to the program by acting as "table leaders." The connections between the program and the teaching profession are real, because teachers are involved at every level.

Serious mathematics connected to secondary teaching. Each experience is designed to connect to the mathematics teachers use in their professional lives. This "applied" mathematics is sometimes around underpinnings that will end up in the hands of students. But it may also take up mathematics that helps teachers put the topics in their curricula into the broader mathematical landscape.

In one summer at the SSTP, teachers studied how the geometry and algebra of complex numbers are related to each other, to Pythagorean triples, and to other non-real number systems. In another summer, teachers studied the connections between polynomials, probability, statistical Bernoulli trials, and the Central Limit Theorem.

Experience before formality. Participants experience firsthand the effectiveness of struggling with new ideas and connections *before* they are brought to closure. The role of the instructor is to pull together the participants at several points to collate conjectures, logical arguments, and extensions. With help from those around them, teachers refine and prove their own conjectures, sometimes over the course of several days. This style of learning, emblematic of the intent of the Standards for Mathematical Practice, has had an immense effect on how teachers approach their own classes and how they view the discipline they teach.

The goal of the PCMI teacher program is to provide teachers with opportunities to:

- deepen their understanding of mathematics,
- reflect on the practice of teaching, and
- serve as mathematical resources for their colleagues.

All of this has worked. Exit reviews of the summer programs and in-depth interviews by external evaluators of teachers from varying backgrounds and school systems have provided evidence that the program has helped hundreds of teachers become more effective in—and more satisfied with—their professional lives. According to one evaluation report, many participants have been influenced by PCMI to revise their roles as teachers, acting more as facilitators, rebalancing how much time they allow students to talk versus talking themselves, and giving students more responsibility for their own learning. They consciously change the ways they question students and answer questions, and make reasoning and sense making a core goal of their instruction.

Teachers meet after each mathematical session to discuss the work of teaching. One summer focused on how to make classrooms a place where questioning is central to learning. Another summer, participants considered how to manage discourse that made students the center of the discussion. Activities are designed around artifacts of practice such as student work, classroom videos, assessment, or lesson design.

While these materials were developed for use at the SSTP, they have been used in numerous other settings, such as a capstone course for preservice mathematics teachers, a summer institute for teachers, an elective in mathematics and mathematics education programs, and a course for high school students. While the course does not replace standard number theory or group theory courses, it prepares students to take such courses, with the problem sets building towards a mathematical conclusion. In settings where credit is offered, instructors often assign projects or reports on specified readings related to the work.

We do not expect all users of these materials to use all sessions, but we encourage beginning from Set 1 and running until one of three endpoints. Set 4 comes to a provable conclusion that the golden ratio is irrational. Set 9 comes to an understanding of finding approximations using continued fractions, including the result that $\frac{355}{113}$ is an astoundingly good approximation for π. And the full course continues to include work on finding rational approximations to $\sqrt{k}$ for many k, offering a general method for solving the Pell equation $a^2 - kb^2 = 1$.

Navigating the Problem Sets

At first glance, some of the mathematics in this course (and others in the Teachers Program Series) may appear appropriate only for teachers with a strong mathematical background. However, the course is designed to provide participants with the necessary background as the course continues. Any topic in high school algebra or beyond is approached as though it is newly encountered. For example, the work in this course on the value of the golden ratio does not assume participants come in knowing the quadratic formula or how to find the exact value of ϕ. When generalizations happen, such as the quadratic formula, they emerge from repeated iterations of examples, not from an initial exposure. Many learners find their first answers by testing options, and when an unusual one comes along, they use the patterns from their previous work to figure out what might happen in this new case. The problems are not built to support a lecture, but rather deliberately constructed for students to pursue a general solution over time, with the goal of enabling learners to build their own understanding from the problems as they work through them.

Because the problems are carefully crafted to let the mathematics unfold through the experience of actually doing the work, facilitators are encouraged to do all of the problems themselves, at least in Important Stuff, before engaging participants so as to anticipate their thought processes and to encourage them during explorations and discoveries.

A few things about how the materials are structured:

- The openers are meant to be answered just like the rest of the problems. They are, in general, more important than other problems. Keep looking forward in the problem sets. Solutions to problems in new concepts do not go immediately for proof, but rather the course tends to let ideas sit for a while.
- There are problem categories: *Important Stuff*, *Neat Stuff*, *Tough Stuff*. All the mathematics that is central to the program can be found and developed in the Important Stuff. That's why it's Important Stuff.
- The materials provide experience before formality, where the participant uses examples to build intuition. Definitions and theorems appear as capstones, not foundations.
- The problems should lead to the appropriate mathematics rather than requiring it. The same goes for technology: the problems should lead to the appropriate use of technology rather than requiring it.
- The problem sets typically have multiple points of entry where everyone, regardless of the level of confidence or experience, can begin.
- The materials have a low threshold and a high ceiling—they allow everyone to experience success and are also designed so that participants will feel challenged regardless of skill or experience level.
- The problems explicitly link different content areas and encourage participants to seek multiple representations and solutions.
- A problem presented from one perspective (algebraically, say) may be repeated in another (geometrically).
- The problems often foreshadow key ideas that are not introduced formally until later problem sets.
- The problems repeat and connect throughout all of the sets. A goal is to have the participants look for connections rather than being surprised when they notice relationships.