

Contents

Preface	ix
1 Differential Equations	1
1.1 Classification of Differential Equations	4
1.2 Can we write solutions explicitly?	6
1.3 Differential Equations as Models of Reality	7
1.4 Named Equations	8
1.5 When are two equations equivalent?	9
1.6 Evolution in Time	12
Problems	19
Suggested Reading	23
2 Developing PDE Intuition	25
2.1 The Structure of Linear Equations	25
2.2 Examples of Linear Equations	32
2.3 Examples of Nonlinear Equations	38
Problems	43
Suggested Reading	46
3 The Story of Solitons	47
3.1 The Observation	47
3.2 Terminology and Backyard Study	49
3.3 A Less-than-enthusiastic Response	49
3.4 The Great Eastern	51
3.5 The KdV Equation	51
3.6 Early 20th Century	54
3.7 Numerical Discovery of Solitons	55
3.8 Hints of Nonlinearity	59

3.9	Explicit Formulas for n -soliton Solutions	61
3.10	Soliton Theory and Applications	62
3.11	Epilogue	64
	Problems	65
	Suggested Reading	68
4	Elliptic Curves and KdV Traveling Waves	69
4.1	Algebraic Geometry	69
4.2	Elliptic Curves and Weierstrass \wp -functions	71
4.3	Traveling Wave KdV Solutions	86
	Problems	93
	Suggested Reading	96
5	KdV n-Solitons and τ-Functions	97
5.1	Introductory Remarks on KdV n -solitons	97
5.2	KdV τ -Functions	98
5.3	KdV 1-solitons and their τ -functions	101
5.4	KdV 2-solitons and their τ -functions	104
5.5	The 2-soliton Phase Shift	112
5.6	KdV n -Solitons and their τ -functions	118
5.7	Predicting the Appearance of an n -soliton	122
5.8	Proofs of the Main Claims in This Chapter	130
5.9	There's Something about KdV	133
	Problems	133
	Suggested Reading	137
6	Multiplying and Factoring Differential Operators	139
6.1	Differential Algebra	139
6.2	Factoring Differential Operators	147
6.3	Almost Division	150
6.4	Application to Solving Differential Equations	151
6.5	Producing an ODO with a Specified Kernel	153
	Problems	156
	Suggested Reading	160
7	Eigenfunctions and Isospectrality	161
7.1	Isospectral Matrices	161
7.2	Eigenfunctions and Differential Operators	166
7.3	Dressing for Differential Operators	168
	Problems	173

Suggested Reading	176
8 Lax Form for KdV and Other Soliton Equations	179
8.1 KdV in Lax Form	180
8.2 Finding Other KdV-like Soliton Equations	185
8.3 The Non-commutative KdV Equation	190
8.4 Scalar Equations with Matrix Lax Operators	193
8.5 Connection to Algebraic Geometry	198
Problems	200
Suggested Reading	205
9 The KP Equation and Bilinear KP Equation	207
9.1 The KP Equation	207
9.2 The Bilinear KP Equation	215
Problems	227
Suggested Reading	229
10 $\Gamma_{2,4}$ and the Bilinear KP Equation	231
10.1 Wedge Products	231
10.2 Decomposability and the Plücker Relation	234
10.3 The Grassmann Cone $\Gamma_{2,4}$ as a Geometric Object	237
10.4 Bilinear KP as a Plücker Relation	239
10.5 Geometric Objects and Nonlinear PDEs	243
Problems	249
Suggested Reading	252
11 Pseudo-Differential Operators and the KP Hierarchy	253
11.1 Motivation	253
11.2 The Algebra of Pseudo-Differential Operators	254
11.3 Ψ DOs are Not Really Operators	259
11.4 Application to Soliton Theory	259
Problems	266
Suggested Reading	268
12 $\Gamma_{k,n}$ and the Bilinear KP Hierarchy	269
12.1 Higher Order Wedge Products	269
12.2 The Bilinear KP Hierarchy	275
Problems	281
Suggested Reading	284
13 Concluding Remarks	287

13.1	Soliton Solutions and their Applications	287
13.2	Algebro-Geometric Structure of Soliton Equations	288
A	Mathematica Guide	293
A.1	Basic Input	293
A.2	Some Notation	295
A.3	Graphics	299
A.4	Matrices and Vectors	301
A.5	Trouble Shooting: Common Problems and Errors	303
B	Complex Numbers	305
B.1	Algebra with Complex Numbers	305
B.2	Geometry with Complex Numbers	307
B.3	Functions and Complex Numbers	308
	Problems	310
C	Ideas for Independent Projects	313
D	References	329
E	Glossary of Symbols	339
F	Index	343