
Preface

To see form where form is yet unmade,
and shape in the ephemeral air.

...

In whose presence things dare not be incoherent,
this inspired mind is the visionary eye.¹

The idea for this book goes back at least forty years when the senior author was a post-doctoral fellow at the Tata Institute of Fundamental Research in Bombay, India. He gave a course of lectures on the circle method from which the first draft of a monograph was produced. For various reasons, this first draft lay dormant for many years until 1987. That year, being the birth centenary of the legendary Indian mathematician, Srinivasa Ramanujan, many research institutes in India had organized international conferences to commemorate the occasion.

¹These lines are from the poem 'The Visionary Eye' by V. Kumar Murty.

The senior author was at such a conference in the city of Chennai (then known as Madras), but as fate would have it, on Ramanujan's actual birthday December 22, some civil unrest prevented the conference from proceeding and attendees had to stay in their allotted hotel and guest house rooms. It was then that he gave a series of improvised lectures on the circle method of Ramanujan to a group of Indian mathematicians who were not number theorists, but who were together at the same guest house. It was also at that time that the dormant manuscript was resurrected and started showing new signs of life.

But the momentum to write a monograph was lacking partly due to the fact that in the meantime Robert C. Vaughan had published his Cambridge tract titled "The Hardy–Littlewood method", as the circle method is referred to by some. As explained by Atle Selberg in his 1987 lecture at the Ramanujan centenary, the circle method was actually introduced by Ramanujan in one of his letters to Godfrey H. Hardy, and was developed and applied by Hardy and Ramanujan to a number of problems in additive number theory, most prominently to study the partition function. One might therefore have expected this method to be called "The Ramanujan–Hardy method", but mathematical nomenclature seems to follow its own byzantine logic.

The present monograph is the outcome of an endeavour to write about the circle method in a way that is accessible to undergraduate students who are familiar with real and complex analysis, but are learning number theory for the first time. This sets it apart from the monograph of Vaughan which is aimed at graduate students. This text grew out of two mini-courses given by the authors in 2010–2011 at the Indian Institute of Science Education and Research (IISER), Kolkata to undergraduate students. We literally started from scratch. Our goal was to show the students the elegance of the circle method and at the same time give a complete solution of the famous Waring problem as an illustration of the method. Subsequently, the authors gave semester-long courses on the circle method to undergraduate students at Queen's University and IISER Pune. This motivated us to write the text in a self-contained manner so that it is accessible to an undergraduate student who wants to learn the circle method, but has no previous knowledge of number theory.

This book is suitable for a one-semester undergraduate course. Students familiar with elementary number theory can read Chapter 1 for

an overview of the contents of the book, and jump directly to Chapter 5 where we derive some classical theorems of analytic number theory used later in the book. Students who have not seen any number theory before can read Chapters 2–4 for a quick introduction to topics in elementary number theory that will be used later in the textbook. Chapters 6–10 formed the bulk of our short courses. Chapter 6 contains a solution of Waring’s problem using some ideas of Joseph H. Schnirelmann and Yuri V. Linnik. Chapters 7–10 describe the application of the circle method to Waring’s problem and the ternary Goldbach conjecture. Finally, in Chapter 11, we provide the reader a lightning view of the origins of the circle method and describe the underlying philosophy of the method in a general way. We also indicate future directions and avenues of further study, along with several references for the student to explore this topic in greater depth. This chapter is aimed at the advanced student who wants to have a panoramic understanding of the method after having studied the more technical aspects treated in Chapters 6 and beyond. This chapter can also be read by the non-expert to gain a cursory understanding of the method without too many technicalities.

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