Preface

This text grew out of the lecture notes for the course *Introduction to Harmonic Analysis* given at the Undergraduate Summer School during the 2018 Park City Mathematics Institute (IAS/PCMI). The twelve-hour course contained the basic properties of harmonic functions and the explicit solutions to the Laplace equation in special cases. It also contained a study of the behavior of harmonic functions at the boundary of domains, and introduced the Hilbert transform. The last lectures were dedicated to the construction of harmonic functions in fractals.

The lectures were then extended to a full semester course at the University of Colima, aimed to junior and senior undergraduate mathematics majors. Besides a more detailed discussion on the previous topics, the course also included a discussion on Fourier series and their convergence, an introduction to Lebesgue measure and integration, the Hardy–Littlewood maximal function, the Fourier transform and a more extended study of analysis on fractals, in particular the Laplacian on the Sierpiński gasket and the construction of its eigenfunctions.

The purpose of both the minicourse at IAS/PCMI and the course at Colima is to introduce the modern ideas and problems of harmonic analysis to undergraduate students, from the point of view of harmonic functions. Most books on harmonic and Fourier analysis are too advanced to be appropriate at the undergraduate level, and undergradute textbooks, as those by Thomas William Körner [**K88**] and Elias M. Stein and Rami Shakarchi [**SS03**], focus on Fourier series and their convergence, rather than on harmonic functions and their behavior at boundary domains. So this text starts with a motivation to study harmonic functions and the Dirichlet problem. The first chapter discusses the solution to the heat equation in equilibrium, the real and imaginary parts of holomorphic functions, and the minimizing functions of energy, all of which are harmonic functions.

This book is intended for junior and senior undergraduates with a basic knowledge of real analysis. It requires familiarity with the properties of complete metric and normed spaces, uniform convergence and density. The needed results are reviewed in Appendix A. It does not require knowledge of measure theory, as the text includes two chapters on measure theory, Lebesgue integration, and approximation theorems. It does requires knowledge of linear algebra, in particular familiarity with vector spaces, subspaces, linear operators and orthogonality. Complex analysis is not required but it is recommended, as a couple of calculations of the Fourier transform of some functions are easily done using line integrals and the residue theorem.

The text can be roughly partitioned in three parts. The first part, Chapters 1–4, discusses the basic properties of harmonic functions and the problem of their behavior at the boundary of their domains. Basic properties as the mean value property, the maximum principle or the classification of singularities are discussed in Chapter 2. Explicit solutions to the Dirichlet problem, in terms of Poisson integrals, are discussed for the ball (Chapter 2) and the half-space (Chapter 4). We also discuss the problem of the behavior at the boundary of these domains in those chapters, for the case of continuous boundary values. In Chapter 3, we discuss the solution of the Dirichlet problem in the disk using Fourier series, so in this chapter we also discuss the problem of their convergence. In particular, we discuss Abel means, Fejér's theorem and mean-square convergence. We end Chapter 3 with the construction of an example of a continuous function with divergent Fourier series at a point.

The second part, Chapters 5–9, discusses the problem of the behavior of harmonic functions at the boundary of their domains, in particular the half-space, for noncontinuous functions. This requires the use of measure theory, and thus is developed in Chapters 5 and 6. This is not intended to be a comprehensive course in measure theory, but a brief introduction to the main ideas of Lebesgue integration and the basic properties of the Lebesgue spaces L^1 and L^2. In Chapter 7 we discuss the Hardy–Littlewood maximal function and the problem of almost everywhere convergence of Poisson integrals of integrable functions at the boundary of the half-space. In Chapter 9 we introduce the Hilbert transform, which describes the limits at the boundary of the conjugate function to the Poisson integral. We discuss the L^1 theory of the Hilbert transform and the concept of operators of weak type. In order to study the L^2 theory of the Hilbert transform, we introduce the Fourier transform in the previous Chapter 8, where we discuss its basic properties, as the Riemann–Lebesgue lemma for integrable functions and the Plancherel theory of square integrable functions.

In the third part, Chapters 10–13, we discuss the theory of harmonic functions on fractals. We start with the fundamental ideas of self-similarity and Hausdorff dimension in Chapter 10, and then proceed to the study of harmonic analysis, the Laplacian and its eigenfunctions, on the Sierpiński gasket, in Chapters 11 and 12. We discuss in these chapters the construction of a harmonic structure and the harmonic functions by interpolation. We describe the construction of the Laplacian and an algorithm to construct its eigenfunctions. In particular, we explicitly describe the Dirichlet eigenfunctions on the Sierpiński gasket. We discuss in Chapter 13 harmonic functions on more general self-similar sets.

This text can be used in an introductory course on Harmonic Analysis in several ways. In Colima, semesters are sixteen weeks long, so one has enough time to cover almost all of the material,[1] but for shorter semesters we can choose accordingly to the interests and previous knowledge of the audience. Chapters 1–9 provide an introduction to classical harmonic analysis, and students who already took a course on measure theory may skip Chapters 5 and 6. For a course on analysis on fractals, one may choose Chapters 1–3 and then move on to Chapters 10–13, as the first chapters serve as motivation for the study of harmonic functions and eigenfunctions of the Laplacian. Each chapter has a list of exercises and bibliographic and historical notes.

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[1] The Fall 2020 course was transmitted online and is available at the page `https://www.facebook.com/HarmonicAnalysis`