Chapter 1

Motivation and preliminaries

1.1. The heat equation in equilibrium

In this chapter we discuss a number of motivations for the study of harmonic functions, with examples taken from physics to complex analysis. We start in this section with a deduction of the *heat equation in equilibrium*, using the original argument given by Joseph Fourier in his seminal work *Analytical Theory of Heat* [**Fou55**].

Consider the propagation of heat through a solid in space. For example, you can consider a potato in the oven, receiving heat in part of its peel. If you wait sufficiently long, the temperature inside the potato will be in equilibrium; though not necessarily constant in its interior, it will not depend on time.

Let Q be a small cube inside this solid, which we describe with edges parallel to the axes in $\mathbb{R}^3$. Suppose two of its opposite vertices are given by (x_0, y_0, z_0) and $(x_0+\varepsilon, y_0+\varepsilon, z_0+\varepsilon)$ for some small $\varepsilon > 0$, and we consider the propagation of heat in Q, with temperature function $u(x, y, z, t)$. Since we are assuming the system is in equilibrium, the temperature does not depend on time, so it is then a function $u(x, y, z)$ in Q. We also assume u is a smooth function in a neighborhood of Q (that is, an open set that contains Q).

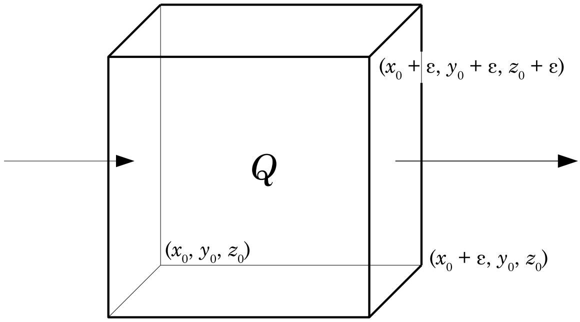

Figure 1.1. The small cube Q, with heat propagating in the x direction.

By Newton's law of heat flow, the amount of heat that enters through the side $x = x_0$ of Q (the left side in Figure 1.1) is proportional to the change in temperature, from hotter to colder, in the x direction on this side, so it is given by

$$-K\varepsilon^2 \frac{\partial u}{\partial x}(x_0, y_0, z_0),$$

where ε^2 is the surface area of the left side and the proportionality constant $K > 0$, which depends only on the material of the solid, is called the *conductivity* constant. The amount of heat that exits through the side $x = x_0 + \varepsilon$ of Q is then given by

$$-K\varepsilon^2 \frac{\partial u}{\partial x}(x_0 + \varepsilon, y_0, z_0).$$

The quantity of heat accumulated in Q as a consequence of propagation in the x direction is the difference between these two quantities,

$$\begin{aligned} -K\varepsilon^2 \frac{\partial u}{\partial x}(x_0, y_0, z_0) - \Big(-K\varepsilon^2 \frac{\partial u}{\partial x}(x_0 + \varepsilon, y_0, z_0)\Big) \\ = K\varepsilon^2 \Big(\frac{\partial u}{\partial x}(x_0 + \varepsilon, y_0, z_0) - \frac{\partial u}{\partial x}(x_0, y_0, z_0)\Big). \end{aligned}$$

By the mean value theorem, there exists $0 < \delta < \varepsilon$ such that

$$\frac{\partial u}{\partial x}(x_0 + \varepsilon, y_0, z_0) - \frac{\partial u}{\partial x}(x_0, y_0, z_0) = \varepsilon \frac{\partial^2 u}{\partial x^2}(x_0 + \delta, y_0, z_0),$$

so the propagation of heat through Q in the x direction is then

$$K\varepsilon^3 \frac{\partial^2 u}{\partial x^2}(x_0 + \delta, y_0, z_0).$$

Similarly, there exist $0 < \eta, \theta < \varepsilon$ so that the propagation of heat through Q in the y and z directions is given by

$$K\varepsilon^3 \frac{\partial^2 u}{\partial y^2}(x_0, y_0 + \eta, z_0) \qquad \text{and} \qquad K\varepsilon^3 \frac{\partial^2 u}{\partial z^2}(x_0, y_0, z_0 + \theta),$$

respectively, and the total propagation is then given by

$$K\varepsilon^3\Big(\frac{\partial^2 u}{\partial x^2}(x_0 + \delta, y_0, z_0) + \frac{\partial^2 u}{\partial y^2}(x_0, y_0 + \eta, z_0) + \frac{\partial^2 u}{\partial z^2}(x_0, y_0, z_0 + \theta)\Big).$$

Since the system is in equilibrium, the total propagation must be equal to 0. As we are assuming that u is a smooth function, its partial derivatives are continuous, so we obtain, as $\varepsilon \to 0$, the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.1}$$

at the point (x_0, y_0, z_0).

Equation (1.1) is called the *Laplace equation*. We can also write it as $\Delta u = 0$, where the differential operator Δ is given by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Δu is called the *Laplacian* of u.

The solutions of equation (1.1) are called *harmonic functions*.

1.2. Holomorphic functions

In this section we observe that harmonic functions also appear in complex analysis. Recall that f is holomorphic (or analytic) in an open set $D \subset \mathbb{C}$ if, for each $z \in D$, its derivative

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{1.2}$$

exists.

If we write the holomorphic function f as $f(z) = u(x, y) + iv(x, y)$, where $z = x+iy$ and u and v are its real and imaginary parts, respectively, then u and v satisfy the *Cauchy–Riemann* equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{1.3}$$

These equations follow directly from the differentiability of f. Indeed, if we take the limit in (1.2) by approaching $h \to 0$ with real numbers, we obtain

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y).$$

Meanwhile, if we approach $h \to 0$ with purely imaginary numbers, we get

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y}(x, y) + \frac{\partial v}{\partial y}(x, y) = \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}(x, y).$$

As these two expressions for $f'(z)$ must be equal, we obtain (1.3).

Assuming u and v are smooth functions,[1] we can differentiate the Cauchy–Riemann equations and get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \text{and} \qquad \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}.$$

All mixed derivatives are continuous, so they must be equal and thus

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}.$$

Therefore u is a harmonic function. We can similarly verify that v is harmonic.

As v is the imaginary part of a holomorphic function of which u is the real part, we say that the function v is a *conjugate harmonic* function to u. Note that conjugate harmonic functions are not unique, because adding any constant to v will give another conjugate harmonic function. Also, observe that $-u$ is the conjugate harmonic function to v.

Under appropiate conditions on the set D, one can prove that every harmonic u has a conjugate harmonic function v. See Exercise (3) for the special case of the complex plane $D = \mathbb{C}$. We will dedicate Chapter 9 to the study of the properties of conjugate harmonic functions in the upper half-plane.

1.3. Know thy calculus

Before moving on, let's dedicate a section to set the notation used in this text, and review some of the results from advanced calculus that we'll need later on. This will just be a quick summary of these results, so we

[1] It is a fact, proven in any basic complex analysis text (see [**Gam01**], for example), that both u and v are smooth functions whenever f is holomorphic.

invite the reader to consult advanced calculus texts, such as [**Fle77**] or [**Spi65**] for the details and proofs.

We denote the d-dimensional Euclidean space by $\mathbb{R}^d$. Thus

$$\mathbb{R}^d = \{x = (x_1, x_2, \ldots, x_d) \,:\, x_i \in \mathbb{R}\}.$$

We will usually denote the points in the plane $\mathbb{R}^2$ and in the space $\mathbb{R}^3$ by (x, y) and (x, y, z), respectively. We denote the Euclidean norm of a vector $x \in \mathbb{R}^d$ by $|x|$. Thus

$$|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_d^2}.$$

For $x \in \mathbb{R}^d$, x' is the point in $\mathbb{R}^{d-1}$ formed by the first $d-1$ coordinates of x. We can thus write $x = (x', x_d)$. If we need to explicitly distinguish the last coordinate, then we refer to

$$\mathbb{R}^{d+1} = \{(x, t) \,:\, x \in \mathbb{R}^d, t \in \mathbb{R}\}.$$

We denote by $\mathbb{R}^{d+1}_+$ the *upper half-space* of points $(x, t) \in \mathbb{R}^{d+1}$ with $t > 0$.

The open ball of radius $r > 0$ centered at x_0 is given by

$$B_r(x_0) = \{x \in \mathbb{R}^d \,:\, |x - x_0| < r\}.$$

If $x_0 = 0$, we simply denote it by B_r. If, in addition, $r = 1$, we denote it by $\mathbb{B}$. The sphere of radius $r > 0$ centered at x_0 is given by

$$S_r(x_0) = \{x \in \mathbb{R}^d \,:\, |x - x_0| = r\},$$

and we denote it by S_r if $x_0 = 0$, and by $\mathbb{S}$ if we also have $r = 1$.

For an open set $\Omega \in \mathbb{R}^d$, $C(\Omega)$ is the space of continuous functions in Ω, and $C(\bar{\Omega})$ the space of continuous functions on its closure. We denote by $C^k(\Omega)$ the space of k-continuously differentiable functions in Ω, and by $C^\infty(\Omega)$ the space of smooth functions. Note that

$$C^\infty(\Omega) = \bigcap_{k \geq 1} C^k(\Omega).$$

We denote by $C_c^\infty(\Omega)$ the space of smooth functions with compact support in Ω. That is, $f \in C_c^\infty(\Omega)$ if f is smooth in Ω and there exists a compact subset $K \subset \Omega$ such that $f(x) = 0$ for all $x \notin K$. In particular, we say that f is zero "close to the boundary."

We denote the partial derivative of f with respect to x_i either by $\dfrac{\partial f}{\partial x_i}$, by $\partial_{x_i} f$ or simply by $\partial_i f$, if there is no confusion. The gradient of a function f is given by the vector

$$\nabla f = (\partial_1 f, \partial_2 f, \dots, \partial_d f).$$

Note that its norm is given by

$$|\nabla f| = \left(|\partial_1 f|^2 + |\partial_2 f|^2 + \dots + |\partial_d f|^2\right)^{1/2}.$$

If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ is a multi-index, where each $\alpha_i \in \mathbb{N}$, we define x^α as the monomial

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$$

and $\partial^\alpha f$ as the higher order derivative

$$\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d} f.$$

The order of the multi-index α is given by $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$.

A *hypersurface* in $\mathbb{R}^d$ is a differentiable manifold S of dimension $d-1$. Locally, for each $x_0 \in S$, there exists an open set U that contains x_0 such that $U \cap S$ is the solution set to the equation

$$\phi(x) = 0, \tag{1.4}$$

for some continuously differentiable function ϕ in U with $\nabla\phi \neq 0$. By the implicit function theorem, and relabeling the coordinates if needed, we can assume ϕ is of the form $\phi(x) = x_d - \psi(x')$, and thus $U \cap S$ is given by

$$x_d = \psi(x'). \tag{1.5}$$

We say that S is a C^k-hypersurface if the function ϕ above is in $C^k(U)$ (and hence ψ is a C^k function in its domain).

A *domain* in $\mathbb{R}^d$ is an open and connected subset $\Omega \subset \mathbb{R}^d$. The domain Ω is a C^k-domain if its boundary $\partial\Omega$ is a C^k-hypersurface.

If Ω is a C^1-domain and $x_0 \in \partial\Omega$, then the normal vector at x_0 is the unit vector $\nu(x_0)$ orthogonal to the hypersuface $\partial\Omega$ pointing outwards of Ω (Figure 1.2). Thus,

$$\nu = \pm\frac{\nabla\phi}{|\nabla\phi|},$$

where ϕ is a function that describes $\partial\Omega$ locally near x_0, as in (1.4).

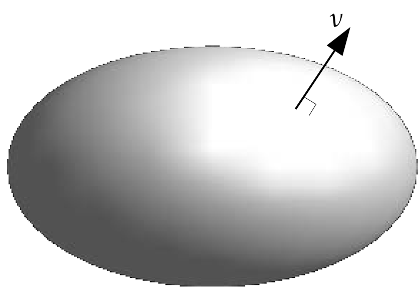

Figure 1.2. The normal vector ν at a point in the boundary of Ω.

Example 1.6. The open ball $B_R(x_0)$ of radius R and centered at $x_0 \in \mathbb{R}^d$ is a C^1-domain (in fact, a C^k-domain for every k), with boundary equal to the sphere $S_R(x_0)$. Note that $\mathbb{S}$ is the solution set to the equation

$$x_1^2 + x_2^2 + \ldots + x_d^2 = 1.$$

Hence, for $x \in \mathbb{S}$,

$$\nu(x) = x.$$

The surface measure on a hypersurface S is denoted by $d\sigma$. Locally, if ψ is as in (1.5), we have that

$$d\sigma = \sqrt{1 + |\nabla\psi|^2} dx'.$$

1.7. One can integrate over $\mathbb{R}^d$ (or over a subset with rotational symmetry) by using *spherical coordinates*. If we write a point $x \neq 0$ in $\mathbb{R}^d$ as $x = r\xi$, where $r = |x| > 0$ and $\xi = x/|x| \in \mathbb{S}$, then

$$\int_{\mathbb{R}^d} f(x)dx = \int_0^\infty \int_{\mathbb{S}} f(r\xi)d\sigma(\xi)r^{d-1}dr.$$

1.8. The area of the unit sphere in $\mathbb{R}^d$ is denoted by ω_d. Thus,

$$\omega_d = \int_{\mathbb{S}} d\sigma.$$

We leave it as an exercise (Exercise (7)) to prove that

$$\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

where $\Gamma(s)$ is the gamma function given by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt,$$

for every $s > 0$.

We will also make use of Theorem 1.9.

Theorem 1.9 (Divergence theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^1-domain and F a continuously differentiable vector field defined in a neighborhood of $\bar{\Omega}$. Then*

$$\int_\Omega \nabla \cdot F dx = \int_{\partial\Omega} F \cdot \nu d\sigma. \tag{1.10}$$

If $F = (F^1, F^2, \ldots, F^d)$, then

$$\nabla \cdot F = \partial_1 F^1 + \partial_2 F^2 + \ldots + \partial_d F^d$$

is called the *divergence* of F, and is also denoted by $\operatorname{div} F$.

Note that the divergence theorem is a multi-dimensional version of the fundamental theorem of calculus. Indeed, in the line $\mathbb{R}$, Ω is just an open interval, say $\Omega = (a, b)$, the normal vector at its boundary is given by $\nu(a) = -1$, $\nu(b) = 1$, and $\nabla \cdot F$ is the derivative of F, so (1.10) is

$$\int_a^b F'(x) dx = -F(a) + F(b).$$

1.11. Taking $F = (0, 0, \ldots, uv, \ldots, 0)$, where the nonzero component is the ith term, we obtain the formula for *integration by parts*:

$$\int_\Omega \frac{\partial u}{\partial x_i} v dx = \int_{\partial\Omega} uv\nu_i d\sigma - \int_\Omega u \frac{\partial v}{\partial x_i} dx.$$

Theorem 1.12 (Green's identities)**.** *Let Ω be a bounded C^1-domain in $\mathbb{R}^d$.*

(1) *If u is continuously differentiable and v is twice continuously differentiable in a neighborhood of $\bar{\Omega}$, then*

$$\int_\Omega (u\Delta v + \nabla u \cdot \nabla v) dx = \int_{\partial\Omega} u \partial_\nu v d\sigma, \tag{1.13}$$

where $\partial_\nu v = \nabla v \cdot \nu$ is the normal derivative of v at the boundary of Ω.

(2) *If u and v are twice continuously differentiable in a neighborhood of $\bar{\Omega}$, then*

$$\int_{\Omega} (u\Delta v - v\Delta u) dx = \int_{\partial\Omega} (u\partial_\nu v - v\partial_\nu u) d\sigma. \tag{1.14}$$

Theorem 1.12 follows almost immediately from the divergence theorem (or 1.11), and we leave it as an exercise (Exercise (9)).

1.4. The Dirichlet principle

The Green identities provide us with another motivation for the study of harmonic functions: they are *minimizers of energy*. Let Ω be a C^1-domain. We define the *energy form* on Ω as the bilinear form

$$\mathcal{E}(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx, \tag{1.15}$$

for smooth functions u and v in a neighborhood of $\bar{\Omega}$ (we denote the space of such functions as $C^\infty(\bar{\Omega})$). The energy of the function u, denoted simply as $\mathcal{E}(u)$, is then given by

$$\mathcal{E}(u) = \mathcal{E}(u, u) = \int_{\Omega} |\nabla u|^2 dx.$$

We now consider the following question: can we find the function u that minimizes $\mathcal{E}(u)$, given its values at the boundary $\partial\Omega$ of Ω? That is, given a function f defined on $\partial\Omega$, find $u \in C^\infty(\bar{\Omega})$ such that $u|_{\partial\Omega} = f$ and

$$\mathcal{E}(u) = \min\{\mathcal{E}(v) : v \in C^\infty(\bar{\Omega}) \text{ and } v|_{\partial\Omega} = f\}.$$

It is clear that we cannot expect the above problem to always have a solution u. First, as we require $u \in C^\infty(\bar{\Omega})$, f cannot be arbitrary because it must be the restriction of such a function to $\partial\Omega$. Moreover, although it is true that the set

$$\{\mathcal{E}(v) : v \in C^\infty(\bar{\Omega}) \text{ and } v|_{\partial\Omega} = f\}$$

is bounded from below, because $\mathcal{E}(v) \geq 0$ for any smooth function v, it is not clear whether it has a minimum or not.

However, we have the following fact: in the case when $\mathcal{E}$ takes its minimum at u, then *u is a harmonic function* in Ω, that is a function that satisfies the equation

$$\Delta u = 0$$

in Ω.

To prove this, suppose $\mathcal{E}$ takes its minimum at u for given values at $\partial\Omega$. Now, for any function $v \in C_c^\infty(\Omega)$ and any $t \in \mathbb{R}$, the function $u + tv$ is smooth and has the same values as u at the boundary. Since $\mathcal{E}(u)$ is minimal, we have that

$$\mathcal{E}(u) \leq \mathcal{E}(u + tv).$$

Hence, as a function of t, the function $I(t) = \mathcal{E}(u + tv)$ takes its minimum value at $t = 0$, and thus

$$I'(0) = 0. \tag{1.16}$$

Now

$$\begin{aligned} I(t) = \mathcal{E}(u + tv) &= \int_\Omega \nabla(u + tv) \cdot \nabla(u + tv) dx \\ &= \int_\Omega |\nabla(u)|^2 dx + 2t \int_\Omega \nabla u \cdot \nabla v dx + t^2 \int_\Omega |\nabla v|^2 dx, \end{aligned}$$

so

$$I'(t) = 2 \int_\Omega \nabla u \cdot \nabla v dx + 2t \int_\Omega |\nabla v|^2 dx$$

and (1.16) implies

$$\int_\Omega \nabla u \cdot \nabla v dx = 0.$$

By the Green identity (1.13), we thus have

$$\int_\Omega v \Delta u dx = \int_{\partial\Omega} v \partial_\nu u d\sigma = 0, \tag{1.17}$$

because v is zero near the boundary. Moreover, since (1.17) holds for every $v \in C_c^\infty(\Omega)$, we conclude $\Delta u = 0$, and thus u is harmonic.

We have left many open questions in the discussion above. We have mentioned that we cannot expect to have a minimizer u of the energy form satisfying that $u|_{\partial\Omega} = f$ for any function f. However, as a minimizer is a harmonic function, this leads to the following problem: *given a domain Ω and a function f defined on $\partial\Omega$, find a harmonic function u in Ω such that it is equal to f on the boundary*. This is known as the *Dirichlet problem*, in honor of the french mathematician Lejeune Dirichlet. It opens a handful of questions, such as the following:

- For which domains Ω can we solve the Dirichlet problem? For which functions on its boundary does the solution exist?

- If u is a harmonic function in Ω, what can we say about its behavior at the boundary of Ω? Does it extend continuously to $\partial\Omega$?

Throughout this text we will be discussing results related to the previous questions. In particular, we will focus our attention to harmonic functions in the domains $\Omega = \mathbb{B}$, the unit ball, and $\Omega = \mathbb{R}^{d+1}_{+}$, the upper half-space, and the behavior of such harmonic functions at the boundaries of their domains.

The fact that the minimizers of the energy form $\mathcal{E}$ are harmonic functions is called the *Dirichlet principle*. The first to give it this name was Bernard Riemann in [**Rie51**], who used this fact to prove the result in complex analysis that we now know as the *Riemann mapping theorem*. See, for example, [**Ull08**] for a study of the Riemann mapping theorem and its relation to the Dirichlet problem.

Exercises

(1) Let R be a rotation in the plane.
   (a) Consider the change of variables $(\xi, \eta) = R(x, y)$. Then
   $$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$
   (b) If u is harmonic, then $u \circ R$ is also harmonic.

(2) Let (r, θ) be the polar coordinates of the plane. Then
$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

(3) Let u be a harmonic function in $\mathbb{R}^2$. Then there exists a conjugate harmonic function v to u. (*Hint:* Consider a line integral of the 1-form
$$-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.)$$

(4) If v_1 and v_2 are conjugate to u in the plane, then $v_1 - v_2$ is constant.

(5) (a) If 0 is conjugate to u in the plane, then u is constant.
   (b) If f is holomorphic in $\mathbb{C}$ and real valued, then f is constant.

(6) Let $\Gamma(s)$ be the gamma function.

(a) Integrate by parts to verify the identity

$$\Gamma(s+1) = s\Gamma(s).$$

(b) For every $n \in \mathbb{Z}_+$, $\Gamma(n) = (n-1)!$.

(7) (a) Use polar coordinates to verify the identity

$$\int_{\mathbb{R}^2} e^{-\pi|x|^2} dx = 1.$$

(b) For every dimension d,

$$\int_{\mathbb{R}^d} e^{-\pi|x|^2} dx = 1.$$

(c) Use spherical coordinates to verify

$$\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

(8) Use integration in spherical coordinates, fact 1.7, to prove that the volume of the unit ball $\mathbb{B}$ is given by

$$\int_{\mathbb{B}} dx = \frac{\omega_d}{d}.$$

(9) Prove Theorem 1.12.

(10) Consider the unit interval and, for smooth functions in $[0,1]$, define the form

$$\mathcal{E}(f) = \int_0^1 f'(x)^2 dx.$$

(a) The minimizers of this form, given the values of f at $x = 0$ and $x = 1$, are the linear functions $f(x) = ax + b$.

(b) If $\mathcal{E}(f)$ is a minimum, then

$$\mathcal{E}(f) = (f(1) - f(0))^2.$$

Chapter 2

Basic properties

2.1. The mean value property

Let $\Omega \subset \mathbb{R}^d$ be an open set. As discussed in Chapter 1, we say that a twice differentiable function u is *harmonic* in Ω if it satisfies

$$\Delta u = \partial_1^2 u + \partial_2^2 u + \ldots + \partial_d^2 u = 0$$

in Ω.

Example 2.1. Any linear function $u = a_1x_1 + a_2x_2 + \ldots + a_dx_d$ is harmonic in $\mathbb{R}^d$, as all of its second derivatives are zero. Observe that, in the case $d = 1$, the linear functions $u(x) = ax + b$ are precisely the functions that satisfy $u''(x) = 0$, so the only harmonic functions in $\mathbb{R}$ (or in any interval in the real line) are the linear functions.

Example 2.2. The quadratic polynomial $u(x, y) = x^2 - y^2$ is harmonic in $\mathbb{R}^2$, as its second derivatives are equal to $\partial_1^2 u = 2$ and $\partial_2^2 u = -2$. Note that u is the real part of the holomorphic function $f(z) = z^2$. As observed in Section 1.2, the real and imaginary parts of an analytic function are harmoinc. The imaginary part of z^2, and thus a conjugate harmonic to u, is $v(x, y) = 2xy$. Similarly, the functions

$$u(x, y) = \Re\big((x + iy)^n\big) \qquad \text{and} \qquad v(x, y) = \Im\big((x + iy)^n\big),$$

the real and imaginary parts of z^n, are harmonic in $\mathbb{R}^2$ for each $n \in \mathbb{N}$.

Example 2.3. The function $u(x,y) = \sin x \sinh y$ is harmonic in $\mathbb{R}^2$, because

$$\partial_1^2 u = -u \qquad \text{and} \qquad \partial_2^2 u = u.$$

Note that u is the imaginary part of the holomorphic function $-\cos z$.

It is easy to see that the harmonic functions in any open set in $\mathbb{R}^d$ form a vector space because, if u and v are harmonic, then any linear combination

$$\alpha u + \beta v$$

of u and v is also a harmonic function. Moreover, the space of harmonic functions in $\mathbb{R}^d$ is invariant under translations and orthogonal transformations (see Exercises (1) and (2)).

We observed in Example 2.1 that the linear functions are harmonic functions and, in fact they are the only harmonic functions in $\mathbb{R}$. We now make a rather immediate observation: if u is linear, say in the interval $[a,b]$, then its value at the midpoint $(a+b)/2$ is the average of its values at a and b,

$$u\Big(\frac{a+b}{2}\Big) = \frac{u(a)+u(b)}{2}.$$

It turns out that this is true in every dimension.

Theorem 2.4 (Mean value property). *Let u be a harmonic function in a neighborhhod of the closed ball $\bar{B}_r(x_0)$. Then*

$$u(x_0) = \frac{1}{|S_r(x_0)|}\int_{S_r(x_0)} u(\xi)\,d\sigma(\xi). \tag{2.5}$$

In other words, the average of the values of u over any sphere around x_0 is equal to $u(x_0)$.

In the identity (2.5), $d\sigma$ is the surface measure on the sphere $S_r(x_0)$, and $|S_r(x_0)|$ is its surface area,

$$|S_r(x_0)| = \omega_d r^{d-1}.$$

With an appropiate change of variables, we can also write (2.5) as

$$u(x_0) = \frac{1}{\omega_d}\int_{\mathbb{S}} u(x_0 + r\xi)\,d\sigma(\xi), \tag{2.6}$$

where $\mathbb{S}$ is the unit sphere around the origin.

2.7. Integrating over the ball $B_r(x_0)$ in spherical coordinates, it follows from (2.5) that, under the same assumptions of Theorem 2.4,

$$u(x_0) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u(x)dx = \frac{d}{\omega_d}\int_{\mathbb{B}} u(x_0 + rx)dx,$$

where $\mathbb{B}$ is the unit ball centered at the origin. We leave this as an exercise (Exercise (3)). Again, this identity is immediate in $\mathbb{R}$ (Exercise (4)).

Proof of Theorem 2.4. We only need to prove the case $d \geq 2$, by the observations made before the statement of the theorem. By translating by x_0, we can assume $x_0 = 0$ (see Exercise (1)). For $0 < \varepsilon < r$, let $\Omega = B_r \setminus \bar{B}_\varepsilon$, where $B_r = B_r(0)$ and $B_\varepsilon = B_\varepsilon(0)$, as in Figure 2.1. Define

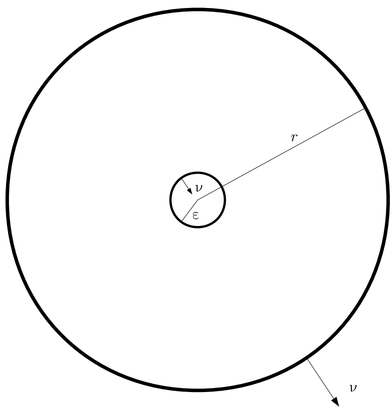

Figure 2.1. The domain $\Omega = B_r \setminus \bar{B}_\varepsilon$. On a point in S_r, the normal vector ν points away from the origin, while on a point in S_ε points towards the origin.

the function v in $\mathbb{R}^d \setminus \{0\}$ by

$$v(x) = \begin{cases} \log|x| & d = 2 \\ |x|^{2-d} & d \geq 3. \end{cases}$$

Note that, for any $x \in \Omega$, $\nabla v(x) = c_d \dfrac{x}{|x|^d}$, where $c_d = 1$ if $d = 2$ and $c_d = 2 - d$ if $d \geq 3$. Also, $\partial\Omega = S_r - S_\varepsilon$, where we have written S_r and S_ε for $S_r(0)$ and $S_\varepsilon(0)$, respectively (as oriented manifolds,[1] see Figure 2.1),

[1] We are only interested, at this moment, in the fact that the normal vectors on $-S_e$ point opposite to those at S_r.

and

$$\nu(x) = \begin{cases} \dfrac{x}{r} & \text{on } S_r \\ -\dfrac{x}{\varepsilon} & \text{on } -S_e. \end{cases}$$

Thus

$$\partial_\nu v = \begin{cases} \dfrac{c_d}{r^{d-1}} & \text{on } S_r \\ -\dfrac{c_d}{\varepsilon^{d-1}} & \text{on } -S_e. \end{cases}$$

We can also verify explicitly that $\Delta v = 0$ (Exercise (5)), and hence

$$\int_\Omega (u\Delta v - v\Delta u)dx = 0.$$

Applying Green's identity (1.14), and the previous explicit calculations, we obtain

$$\begin{aligned} 0 &= \int_{\partial\Omega} (u\partial_\nu v - v\partial_\nu u)d\sigma \\ &= \int_{S_r} \Big(u\frac{c_d}{r^{d-1}} - v_d(r)\partial_\nu u\Big)d\sigma - \int_{S_e} \Big(u\frac{c_d}{e^{d-1}} - v_d(\varepsilon)\partial_\nu u\Big)d\sigma, \end{aligned}$$

where

$$v_d(s) = \begin{cases} \log s & d = 2 \\ \dfrac{1}{s^{d-2}} & d \geq 3, \end{cases}$$

for $s = d$ or $s = \varepsilon$, which are constant over S_r and S_ε. Thus, since the surface integral of $\partial_\nu u$ over a sphere is zero (Exercise (6)), we obtain

$$\frac{c_d}{r^{d-1}} \int_{S_r} u d\sigma = \frac{c_d}{\varepsilon^{d-1}} \int_{S_\varepsilon} u d\sigma$$

for any $\varepsilon > 0$. Since u is continuous we obtain, taking $\varepsilon \to 0$ (see Exercise (7)),

$$\frac{1}{\omega_d r^{d-1}} \int_{S_r} u d\sigma = u(0).$$

□

If a continuous function u in $\mathbb{R}$ satisfies the mean value property, that is,

$$u\Big(\frac{x+y}{2}\Big) = \frac{u(x) + u(y)}{2}$$

for all $x, y \in \mathbb{R}$, then u must be a linear function. Indeed, let $a = u(1) - u(0)$ and $b = u(0)$, so we have $u(1) = a + b$ and $u(0) = b$. Since $u(1) = (u(0) + u(2))/2$, we see that

$$u(2) = 2u(1) - u(0) = 2a + b,$$

and we can verify, inductively, that

$$u(n) = an + b \tag{2.8}$$

for every $n \in \mathbb{N}$. We can similarly prove that (2.8) holds for negative integers n. Now, for every $n \in \mathbb{Z}$,

$$\begin{aligned} u\Big(\frac{2n+1}{2}\Big) = \frac{u(n) + u(n+1)}{2} &= \frac{an + b + a(n+1) + b}{2} \\ &= a\Big(\frac{2n+1}{2}\Big) + b, \end{aligned}$$

and similarly for every number of the form $k/2^n$, for every $k \in \mathbb{Z}$ and every $n \in \mathbb{N}$. Since such numbers are dense in $\mathbb{R}$ and u is continuous, we conclude that

$$u(x) = ax + b$$

for every $x \in \mathbb{R}$. As we have stated above, the linear functions are the harmonic functions in $\mathbb{R}$ so, therefore, the continuous functions that satisfy the mean value property are precisely the harmonic functions. This is also true in higher dimensions.

Theorem 2.9 (Converse to the mean value property). *Let $\Omega \subset \mathbb{R}^d$ be open and u a continuous function on Ω that satisfies that, whenever $\bar{B}_r(x) \subset \Omega$,*

$$u(x) = \frac{1}{\omega_d} \int_{\mathbb{S}} u(x + r\xi)\, d\sigma(\xi). \tag{2.10}$$

Then $u \in C^\infty(\Omega)$ and u is harmonic in Ω.

As harmonic functions must be twice differentiable, we must prove that a function u that satisfies (2.10) is at least twice differentiable before proving that $\Delta u = 0$ in Ω. However, the conclusion of Theorem 2.9 is much stronger: u is actually an infinitely differentiable function. We thus conclude Corollary 2.11, which follows by applying Theorems 2.4 and 2.9.

Corollary 2.11. *If u is harmonic in an open set Ω, then it is infinitely differentiable in Ω.*

In order to prove Theorem 2.9, we will make use of Lemma 2.12.

Lemma 2.12. *There exists a smooth radial function ϕ on $\mathbb{R}^d$ such that it is supported in $\mathbb{B}$ and $\int \phi = 1$.*

Proof. Consider the function ψ on $\mathbb{R}$ given by

$$\psi(x) = \begin{cases} e^{\frac{1}{(4t-1)(2t-1)}} & 1/4 < t < 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

It is a standard calculus exercise to verify that ψ is a smooth function in $\mathbb{R}$, supported in $[1/4, 1/2]$. Indeed, it is infinitely flat at the points 1/4 and 1/2 (see Figure 2.2). Now, we define on $\mathbb{R}^d$ the function $\phi(x) = c\psi(|x|)$,

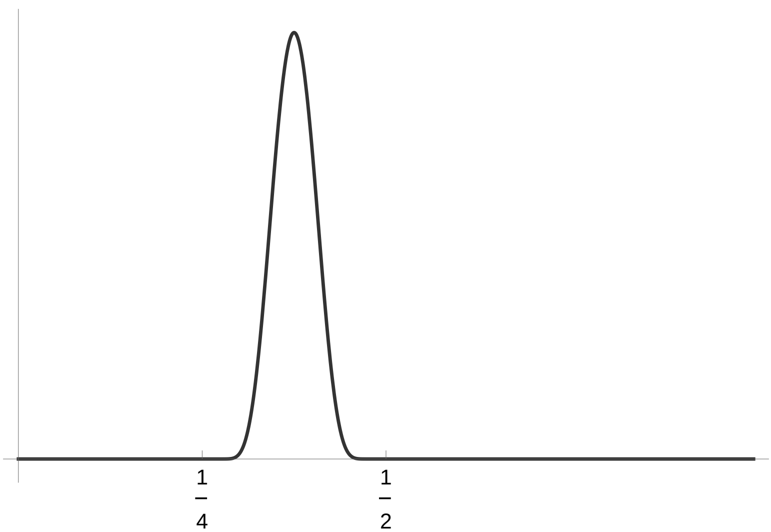

Figure 2.2. The cut-off function $\psi(t)$. Note that it is supported in $[1/4, 1/2]$, and infinitely flat at the points 1/4 and 1/2.

where c is such that

$$\int_{\mathbb{R}^d} \phi(x)dx = 1.$$

Such c exists because ψ is nonnegative, and thus $\int_{\mathbb{R}^d} \psi(|x|)dx > 0$. Now, ψ is C^∞ and supported away from zero, and hence $\phi \in C^\infty(\mathbb{R}^d)$, because

$x \mapsto |x|$ is smooth away from zero. Finally, as $\psi(t) = 0$ unless $1/4 < t < 1/2$, then $\phi(x) = 0$ unless $1/4 < |x| < 1/2$, and thus

$$\operatorname{supp} \phi \subset \mathbb{B}.$$

□

Proof of Theorem 2.9. Let u be a continuous function that satisfies (2.10) for every $\bar{B}_r(x) \subset \Omega$. Let $x_0 \in \Omega$, and $\varepsilon > 0$ such that $\bar{B}_{2\varepsilon}(x_0) \subset \Omega$.

Let $\phi(x) = \tilde{\phi}(|x|)$ as in Lemma 2.12, and define the function $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(\varepsilon^{-1}x)$. Note that $\phi_\varepsilon \in C^\infty(\mathbb{R}^d)$, it is supported in $B_\varepsilon(0)$, and

$$\int_{\mathbb{R}^d} \phi_\varepsilon(x)dx = 1.$$

In particular, for any $x \in B_\varepsilon(x_0)$, the function $y \mapsto \phi_\varepsilon(x-y)$ is supported in $B_{2\varepsilon}(x_0) \subset \Omega$. Hence, we observe that, for $x \in B_\varepsilon(x_0)$,

$$\begin{aligned}\int_{\mathbb{R}^d} u(y)\phi_\varepsilon(x-y)dy &= \int_{\mathbb{R}^d} u(x-y)\phi_\varepsilon(y)dy = \int_{B_\varepsilon(0)} u(x-y)\phi_\varepsilon(y)dy \\ &= \int_0^\varepsilon \int_{\mathbb{S}} u(x-r\xi)d\sigma(\xi)\cdot \varepsilon^{-d}\tilde{\phi}(\varepsilon^{-1}r)r^{d-1}dr.\end{aligned}$$

Using (2.10) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} u(y)\phi_\varepsilon(x-y)dy &= u(x)\cdot\omega_d \int_0^\varepsilon \varepsilon^{-d}\tilde{\phi}(\varepsilon^{-1}r)r^{d-1}dr \\ &= u(x)\int_0^\varepsilon \int_{\mathbb{S}} \varepsilon^{-d}\tilde{\phi}(\varepsilon^{-1}r)d\sigma(\xi)r^{d-1}dr \\ &= u(x)\int_{B_\varepsilon(0)} \phi_\varepsilon(y)dy = u(x).\end{aligned}$$

Note that the function $u(y)\phi_\varepsilon(x-y)$ is C^∞ in x and continuous with compact support in y, so we can differentiate under the integral the function

$$x \mapsto \int_{\mathbb{R}^d} u(y)\phi_\varepsilon(x-y)dy$$

as many times as we want, and thus we conclude $u \in C^\infty(B_\varepsilon(x_0))$. In particular, Δu is a continuous function in $B_\varepsilon(x_0)$.

Now, for any $x \in B_\varepsilon(x_0)$ and $0 < r < \varepsilon$ such that $\bar{B}_r(x) \subset B_\varepsilon(x_0)$,

$$u(x) = \frac{1}{\omega_d}\int_{\mathbb{S}} u(x+r\xi)d\sigma(\xi),$$

so if we differentiate with respect to r we obtain

$$\begin{aligned} 0 &= \frac{d}{dr}\int_{\mathbb{S}} u(x + r\xi)d\sigma(\xi) = \int_{\mathbb{S}} \nabla u(x + r\xi) \cdot \xi d\sigma(\xi) \\ &= \frac{1}{r^{d-1}}\int_{S_r(x)} \partial_\nu u d\sigma = \frac{1}{r^{d-1}}\int_{B_r(x)} \Delta u dx. \end{aligned}$$

In the last equality we have used Green's identity (1.14) with $v = -1$. Hence, the integral of Δu over any ball in $B_\varepsilon(x_0)$ is zero. By the continuity of Δu, $\Delta u = 0$ in $B_\varepsilon(x_0)$.

Since $x_0 \in \Omega$ is arbitrary, we conclude u is harmonic in Ω. □

2.2. The maximum principle

From the mean value property 2.4 we obtain another basic property of harmonic functions, the *maximum principle*.

Corollary 2.13 (Maximum principle). *If $\Omega \subset \mathbb{R}^d$ is a domain and u is harmonic in Ω, then u does not take a maximum nor a minumum in Ω, unless u is constant.*

This is easy to see in the case $d = 1$, where harmonic functions coincide with linear functions: if u is linear in the interval (a, b), then it clearly does not take neither a maximum or a minimum, because u is either strictly increasing or strictly decreasing, unless it is constant.

Proof of Corollary 2.13. Suppose that u takes its maximum M at some $x_0 \in \Omega$, so $u(x_0) = M$. Let

$$U = \{x \in \Omega : u(x) = M\}$$

be the set of points in Ω where u takes the value M. Note that $U \neq \emptyset$ because $x_0 \in U$. We prove that $U = \Omega$.

First, U is closed in Ω because $U = u^{-1}(\{M\})$ and u is continuous, so it is the pre-image of a closed set under a continuous function.[2] Now let $x \in U$. Since Ω is open, there exists $r > 0$ such that $\bar{B}_r(x) \subset \Omega$. By the mean value property,

$$u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy.$$

[2]See Sections A.2 and A.4 for a summary of results from topology in Euclidean spaces.

As we are assuming $x \in U$, this integral must equal M. Now $u(y) \leq M$ for all $y \in \Omega$, because M is the maximum of u. Hence, if at some $y \in B_r(x)$ we had $u(y) < M$, this integral would be smaller than M, because u is continuous. Thus $u(y) = M$ for all $y \in B_r(x)$. Therefore $B_r(x) \subset U$, and U is also open in Ω.

Since Ω is connected, we conclude $U = \Omega$, and therefore u is the constant function $u(x) = M$.

By taking $-u$, we also see that u takes its minimum in Ω only if it is constant. □

The maximum principle implies that if Ω is bounded, so $\bar{\Omega}$ is compact, and u is harmonic in Ω and continuous on $\bar{\Omega}$, then u takes its maximum (and its minimum) at the boundary of Ω (Exercise (8)). Again, this is clear in the case of a linear function in a closed interval.

The maximum principle also implies uniqueness of harmonic functions in a bounded domain Ω, given their values on the boundary. See Exercise (9) for details.

The maximum principle states that harmonic functions, unless they are constant, do not take their maxima nor minima in their domains. If the domain Ω is bounded, a nonconstant harmonic function in Ω may be bounded, of course, and in that case it may be possible to extend it to the boundary of Ω, and hence its extrema would be achieved in $\partial\Omega$. Even if the domain Ω is unbounded, we may have bounded harmonic functions, as we will see later on.

However, in the case where the domain of the harmonic function is all of the Euclidean space, we have the following result.

Theorem 2.14 (Liouville). *If u is harmonic and bounded in $\mathbb{R}^d$, then it is constant.*

Proof. Suppose u is harmonic and $|u(x)| \leq M$ for all $x \in \mathbb{R}^d$. We prove that $u(x) = u(0)$ for all $x \in \mathbb{R}^d$.

Fix $x \in \mathbb{R}^d$ and let $R > |x|$. By the mean value property in balls, 2.7, we have that

$$u(x) - u(0) = \frac{1}{|B_R(x)|} \int_{B_R(x)} u(y)dy - \frac{1}{|B_R|} \int_{B_R} u(y)dy,$$

where $B_R = B_R(0)$. Since $|B_R(x)| = |B_R| = \omega_d R^d/d$, we can write this difference as

$$u(x) - u(0) = \frac{d}{\omega_d R^d}\Big(\int_{B_R(x)} u(y)dy - \int_{B_R} u(y)dy \Big).$$

Now, if A is the annulus

$$A = \{y \in \mathbb{R}^d : R - |x| \le |y| \le R + |x|\},$$

we see that the symmetric difference of the balls $B_R(x)$ and B_R satisfies

$$B_R(x) \triangle B_R \subset A.$$

(See Figure 2.3) Indeed, if $y \in B_R(x) \setminus B_R$, then $|y - x| < R$ and $|y| \ge$

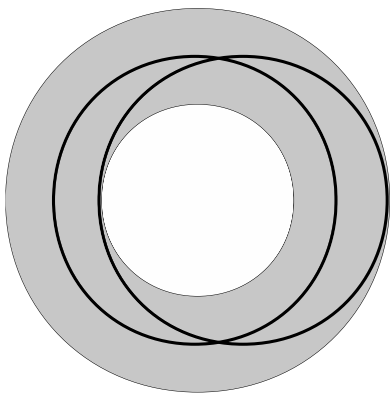

Figure 2.3. The annulus A containing the symmetric difference of the balls $B_R(x)$ and B_R.

$R \ge R - |x|$, and further

$$|y| \le |y - x| + |x| < R + |x|;$$

similarly, if $y \in B_R \setminus B_R(x)$, then $|y| < R \le R + |x|$ and $|y - x| \ge R$, so

$$|y| \ge |y - x| - |x| \ge R - |x|.$$

If we integrate using spherical coordinates, we obtain

$$\begin{aligned} |u(x) - u(0)| &\le \frac{d}{\omega_d R^d} \int_A |u(y)|dy \le \frac{d}{\omega_d R^d} \int_{\mathbb{S}} \int_{R-|x|}^{R+|x|} M r^{d-1} dr d\sigma \\ &= \frac{d}{\omega_d R^d} \cdot M\omega_d \cdot \frac{(R + |x|)^d - (R - |x|)^d}{d} \le \frac{C_x M}{R}, \end{aligned}$$

where we have used the fact that $|u(y)| \leq M$ for all $y \in \mathbb{R}^d$, and the constant C_x only depends on d and $|x|$. As x is fixed and R is arbitrary, we conclude that $|u(x) - u(0)| = 0$, and therefore $u(x) = u(0)$. $\square$

We have seen above that, if $f(z)$ is holomorphic, then it is real and imaginary parts are harmonic functions. Therefore, Theorem 2.14 implies that, if f is a holomorphic function in $\mathbb{C}$ (such a function is called an *entire function*) and is bounded, then f must be constant (this is also know as *Liouville's theorem*). This fact provides a proof for the fundamental theorem of algebra: *If $p(z)$ is polynomial over $\mathbb{C}$ of degree at least* 1*, then it has a root in $\mathbb{C}$.* Indeed, if $p(z)$ is a polynomial over $\mathbb{C}$ with no roots, then $1/p(z)$ is an entire bounded function, and thus constant, so $p(z)$ is a constant polynomial. See Exercise (10) for the details.

We can refine Theorem 2.14 to obtain the same conclusion even when u is only bounded from below or from above.

Theorem 2.15. *If u is harmonic and nonnegative in $\mathbb{R}^d$, then it is constant.*

Proof. The proof of Theorem 2.15 follows similarly as the proof of Theorem 2.14, but we now have to be more careful when estimating the difference

$$u(x) - u(0) = \frac{d}{\omega_d R^d}\Big(\int_{B_R(x)} u(y)dy - \int_{B_R} u(y)dy\Big),$$

for $x \in \mathbb{R}^d$ and $R > |x|$. This time, we use the fact that $u(y) \geq 0$ to observe that, if again A is the annulus

$$A = \{y \in \mathbb{R}^d \,:\, R - |x| \leq |y| \leq R + |x|\},$$

then

$$\begin{aligned}|u(x) - u(0)| &\leq \frac{d}{\omega_d R^d}\int_A u(y)dy \\ &= \frac{d}{\omega_d R^d}\Big(\int_{B_{R+|x|}} u(y)dy - \int_{B_{R-|x|}} u(y)dy\Big)\end{aligned}$$

where $B_{R+|x|}$ and $B_{R-|x|}$ are the balls of radii $R + |x|$ and $R - |x|$ centered at 0, respectively. We use again the mean value property and, as before, for some constant C_x that depends only on d and $|x|$,

$$\begin{aligned}|u(x)-u(0)| &\le \frac{d}{\omega_d R^d}\big(|B_{R+|x|}|u(0)-|B_{R-|x|}|u(0)\big)\\ &= \frac{(R+|x|)^d-(R-|x|)^d}{R^d}\cdot u(0)\\ &\le \frac{C_x u(0)}{R}.\end{aligned}$$

We can now conclude again that $u(x)=u(0)$. □

It is clear that we obtain the same conclusion of Theorem 2.15 whenever u is a harmonic function in $\mathbb{R}^d$ and there exists a constant $\alpha \in \mathbb{R}$ such that either $u(x) \ge \alpha$ for all $x \in \mathbb{R}^d$, or $u(x) \le \alpha$ for all $x \in \mathbb{R}^d$.

2.3. Poisson kernel and Poisson integrals in the ball

We have seen that the value of a harmonic function at the center of a sphere is equal to the average of its values over the sphere. One can ask naturally if the value at any other point in the interior of the sphere is similarly determined by the values over the sphere, and if this value corresponds to a perhaps weighted average over such values. This is certainly the case for a linear function in a close interval $[a,b]$: if $t \in (a,b)$ and u is linear, then

$$u(t)=\frac{b-t}{b-a}u(a)+\frac{t-a}{b-a}u(b),$$

which is a convex combination of $u(a)$ and $u(b)$.

We will prove that this is true for harmonic functions in $\mathbb{R}^d$, for $d \ge 2$, as well. The weight function is called the *Poisson kernel*. For $x \in \mathbb{B}$ and $\xi \in \mathbb{S}$, we define

$$P(x,\xi)=\frac{1}{\omega_d}\frac{1-|x|^2}{|x-\xi|^d}. \tag{2.16}$$

The Poisson kernel satisfies the following facts.

2.17. $P(x,\xi)>0$ for any $x \in \mathbb{B}$ and $\xi \in \mathbb{S}$, which is easily seen from (2.16) because $|x|<1$.

2.18. For each fixed $\xi \in \mathbb{S}$, the function $x \mapsto P(x,\xi)$ is harmonic in $\mathbb{B}$. This is followed by explicit differentiation (Exercise (13)).

2.19. For each fixed $x \in \mathbb{B}$,

$$\int_{\mathbb{S}} P(x,\xi)d\sigma(\xi) = 1. \tag{2.20}$$

This is clear if $x = 0$, since

$$P(0,\xi) = \frac{1}{\omega_d} \tag{2.21}$$

for any $\xi \in \mathbb{S}$, and thus

$$\int_{\mathbb{S}} P(0,\xi)d\sigma(\xi) = \int_{\mathbb{S}} \frac{1}{\omega_d} d\sigma(\xi) = 1.$$

For $x \neq 0$, $x/|x| \in \mathbb{S}$ and, since $P(\cdot, x/|x|)$ is harmonic in $\mathbb{B}$, the mean value property implies

$$P\Big(0, \frac{x}{|x|}\Big) = \frac{1}{\omega_d} \int_{\mathbb{S}} P\Big(|x|\xi, \frac{x}{|x|}\Big) d\sigma(\xi), \tag{2.22}$$

because the integral on the right side is the average over the sphere centered at the origin of radius $|x| < 1$, contained in the ball $\mathbb{B}$. By the identity

$$\Big| |x|\xi - \frac{x}{|x|} \Big| = |x - \xi|,$$

known as the *symmetry lemma* (Exercise (14)), we have that

$$P\Big(|x|\xi, \frac{x}{|x|}\Big) = \frac{1}{\omega_d} \frac{1 - \big||x|\xi\big|^2}{\Big||x|\xi - \frac{x}{|x|}\Big|^d} = \frac{1}{\omega_d} \frac{1 - |x|^2}{|x-\xi|^d} = P(x,\xi),$$

and thus

$$P\Big(0, \frac{x}{|x|}\Big) = \frac{1}{\omega_d} \int_{\mathbb{S}} P(x,\xi) d\sigma(\xi).$$

The identity (2.20) follows using (2.21).

Observe that, if $d = 1$, we simply have $\omega_1 = 2$ and hence

$$P(x,-1) = \frac{1}{2} \frac{1 - |x|^2}{|x - (-1)|} = \frac{1}{2}(1 - x)$$

and

$$P(x,1) = \frac{1}{2} \frac{1 - |x|^2}{|x - 1|} = \frac{1}{2}(1 + x)$$

for every $x \in (-1,1)$, since the boundary of the unit interval $(-1,1)$ is the set $\{-1,1\}$ of two points. Note that each function $x \mapsto P(x,\pm 1)$ is linear, so it is harmonic in $(-1,1)$, and (2.20) is just

$$P(x,-1) + P(x,1) = 1.$$

Example 2.23. In the case when $d = 2$, we can write the Poisson kernel $P(x,\xi)$ in polar coordinates. Indeed, if $x = re^{i\theta}$ for some $0 \le r < 1$ and $\xi = e^{i\tau}$, then

$$P(re^{i\theta}, e^{i\tau}) = \frac{1}{2\pi}\frac{1-r^2}{|re^{i\theta} - e^{i\tau}|^2} = \frac{1}{2\pi}\cdot\frac{1-r^2}{1+2r\cos(\tau-\theta)+r^2}.$$

Note that (2.20) is now the identity

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+2r\cos(\tau-\theta)+r^2}\,d\tau = 1.$$

The following fact states that, as we approach a point in the boundary, the weight of the Poisson kernel concentrates on that point.

2.24. For any $\zeta \in \mathbb{S}$ and $\eta > 0$,

$$\int_{|\xi-\zeta|\ge\eta} P(x,\xi)\,d\sigma(\xi) \to 0$$

as $x \to \zeta$, where the integral is taken over the subset of $\mathbb{S}$ of points $\xi \in \mathbb{S}$ that satisfy $|\xi - \zeta| \ge \eta$.

To verify this limit observe that, if $|\xi-\zeta| \ge \eta$ and $|x-\zeta| < \eta/2$, then

$$|x-\xi| = |\xi - \zeta + \zeta - x| \ge |\xi-\zeta| - |\zeta - x| > \eta - \frac{\eta}{2} = \frac{\eta}{2},$$

so we have

$$P(x,\xi) = \frac{1}{\omega_d}\frac{1-|x|^2}{|x-\xi|^d} \le \frac{1}{\omega_d}\frac{1-|x|^2}{(\eta/2)^d}.$$

Therefore, if $|x-\zeta| < \eta/2$,

$$\int_{|\xi-\zeta|\ge\eta} P(x,\xi)\,d\sigma(\xi) \le \int_{|\xi-\zeta|\ge\eta} \frac{1}{\omega_d}\frac{1-|x|^2}{(\eta/2)^d}\,d\sigma(\xi) \le \left(\frac{2}{\eta}\right)^d (1-|x|^2),$$

and hence

$$\int_{|\xi-\zeta|\ge\eta} P(x,\xi)\,d\sigma(\xi) \to 0$$

as $x \to \zeta$, because $|\zeta| = 1$ and thus $|x| \to 1$.

The facts 2.17, 2.20 and 2.24 make the family

$$\{\xi \mapsto P(x,\xi) : x \in \mathbb{B}\}$$

of functions on $\mathbb{S}$ resemble a family of *good kernels* as $x \to \zeta \in \mathbb{S}$, as defined in [**SS03**]. We will study such families later in this text.

Let $f \in C(\mathbb{S})$. The *Poisson integral of f* is given by

$$\mathcal{P}f(x) = \int_{\mathbb{S}} P(x,\xi) f(\xi) d\sigma(\xi), \tag{2.25}$$

for each $x \in \mathbb{B}$.

The function $u(x) = \mathcal{P}f(x)$ defined by the Poisson integral of f is well defined in $\mathbb{B}$ for any continuous function f on $\mathbb{S}$. This follows because $P(x,\xi)$ is continuous as well in $\mathbb{S}$. In fact, it is not required for f to be continuous on $\mathbb{S}$ for the integral in (2.25) to be defined. It is sufficient for f to be Riemann-integrable on $\mathbb{S}$.

2.26. The Poisson integral $u(x)$ of $f \in C(\mathbb{S})$ is harmonic in $\mathbb{B}$. This is followed by differentiating inside the integral (2.25), and using the fact that $P(x,\xi)$ is harmonic in x. (This is true for a Riemann-integrable function f on $\mathbb{S}$, as well; see Exercise (15).)

In the case $d = 1$, the Poisson integral of $f : \{-1,1\} \to \mathbb{R}$ is the sum

$$\begin{aligned} u(x) &= P(x,-1)f(-1) + P(x,1)f(1) \\ &= \frac{1}{2}(1-x)f(-1) + \frac{1}{2}(1+x)f(1). \end{aligned}$$

This is a linear combination of linear functions, so it is linear and clearly harmonic in $(-1,1)$. Note that $u(x) \to f(\pm 1)$ as $x \to \pm 1$.

The Poisson integral solves the *Dirichlet problem for the ball*: given $f \in C(\mathbb{S})$, find a function u on $\bar{\mathbb{B}}$ such that it is harmonic in the interior and coincides with f on the boundary, that is

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{B} \\ u = f & \text{on } \mathbb{S}. \end{cases} \tag{2.27}$$

We prove Theorem 2.28.

Theorem 2.28. *Let $f \in C(\mathbb{S})$ and $u = \mathcal{P}f$ its Poisson integral. Then u is harmonic in $\mathbb{B}$, extends continuously to $\bar{\mathbb{B}}$ and $u|_{\mathbb{S}} = f$.*

Proof. From fact 2.26, we know that u is harmonic in $\mathbb{B}$. It is thus sufficient to prove that, for each $\zeta \in \mathbb{S}$, $u(x) \to f(\zeta)$ as $x \to \zeta$.

Since f is continuous on the compact set $\mathbb{S}$, it is bounded.[3] Let $M > 0$ be such that $|f(\xi)| \le M$ for all $\xi \in \mathbb{S}$.

Given $\varepsilon > 0$, we can choose $\eta > 0$ such that, if $|\xi - \zeta| < \eta$, then

$$|f(\xi) - f(\zeta)| < \frac{\varepsilon}{2}.$$

We write, using identity (2.20),

$$\begin{aligned}
|u(x) - f(\zeta)| &= \Big| \int_{\mathbb{S}} P(x,\xi) f(\xi) d\sigma(\xi) - f(\zeta) \int_{\mathbb{S}} P(x,\xi) d\sigma(\xi) \Big| \\
&\le \int_{\mathbb{S}} P(x,\xi) |f(\xi) - f(\zeta)| d\sigma(\xi) \\
&= \int_{|\xi-\zeta|<\eta} + \int_{|\xi-\zeta|\ge\eta},
\end{aligned}$$

where we have split the last integral over the regions

$$\{\xi \in \mathbb{S} : |\xi - \zeta| < \eta\} \qquad \text{and} \qquad \{\xi \in \mathbb{S} : |\xi - \zeta| \ge \eta\}$$

in the sphere. In the first region, by the choice of η we have that $|f(\xi) - f(\zeta)| < \varepsilon/2$, and thus

$$\int_{|\xi-\zeta|<\eta} P(x,\xi)|f(\xi) - f(\zeta)| d\sigma(\xi) \le \int_{|\xi-\zeta|<\eta} P(x,\xi) \frac{\varepsilon}{2} d\sigma(\xi) \le \frac{\varepsilon}{2},$$

where we have used facts 2.17 and 2.20.

Over the second region, we can no longer control the difference $|f(\xi) - f(\zeta)|$, and we only have the estimate $|f(\xi) - f(\zeta)| \le 2M$ and thus

$$\int_{|\xi-\zeta|\ge\eta} P(x,\xi)|f(\xi) - f(\zeta)| d\sigma(\xi) \le 2M \int_{|\xi-\zeta|\ge\eta} P(x,\xi) d\sigma(\xi).$$

However, by 2.24, there exists $\delta > 0$ such that, if $|x - \zeta| < \delta$, then

$$\int_{|\xi-\zeta|\ge\eta} P(x,\xi) d\sigma(\xi) < \frac{\varepsilon}{4M},$$

and hence we obtain

$$\int_{|\xi-\zeta|\ge\eta} P(x,\xi)|f(\xi) - f(\zeta)| d\sigma < \frac{\varepsilon}{2}.$$

[3] See Section A.2.

Therefore, if $|x-\zeta| < \delta$, $|u(x) - f(\zeta)| < \varepsilon$, and thus we conclude that $u(x) \to f(\zeta)$ as $x \to \zeta$. □

A special case of Theorem 2.28 is *radial convergence*: if $u = \mathcal{P}f$ is the Poisson integral of $f \in C(\mathbb{S})$ and $\zeta \in \mathbb{S}$, then

$$u(r\zeta) \to f(\zeta)$$

as $r \to 1$. Moreover, from the proof of Theorem 2.28, we see that in fact this convergence is uniform in $\mathbb{S}$, because the number η does not depend on ζ as f is in fact uniformly continuous on the compact set $\mathbb{S}$, and hence neither does δ, so $u \to f$ uniformly as we move to the boundary.

We also observe in the proof of Theorem 2.28 that it is not necessary for f to be continuous on all of $\mathbb{S}$ to have $u(x) \to f(\zeta)$, as we only used the continuity of f at ζ. Moreover, since a Riemann-integrable function is continuous at almost every point,[4] we see that $u(x) \to f(\zeta)$, as $x \to \zeta$, for almost every $\zeta \in \mathbb{S}$.

Theorem 2.28 indeed extends the mean value property to any non-central point of $\mathbb{B}$: if u is harmonic in a neighborhood of $\bar{\mathbb{B}}$, then

$$u = \mathcal{P}(u|_{\mathbb{S}}),$$

the Poisson integral of its restriction to $\mathbb{S}$. This follows by the maximal principle, because u and $\mathcal{P}(u|_{\mathbb{S}})$ are harmonic functions in $\mathbb{B}$ that are equal to each other on its boundary (Exercise (9)). We can also consider any other ball by a proper translation and dilation (Exercise (18)).

2.29. The previous observation implies that, if u_n is a sequence of harmonic functions in Ω that converges uniformly to u on any compact subset $K \subset \Omega$, then u is also harmonic in Ω. Indeed, since $u_n \rightrightarrows u$ in any sphere S in Ω, then their Poisson integrals over S converge to the Poisson integral of u over S, which implies that u is equal to its Poisson integral in any ball in Ω. Therefore, u is harmonic. We leave the details as an exercise (Exercise (19)).

2.4. Isolated singularities

Let $\Omega \subset \mathbb{R}^d$ be open and $x_0 \in \Omega$. If u is harmonic in $\Omega \setminus \{x_0\}$, we say that x_0 is an *isolated singularity* of u.

[4]See Section A.3.

Example 2.30. The function $\log|x|$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$, as we saw above. Then 0 is an isolated singularity of $\log|x|$. Similarly, for $d \geq 3$, 0 is an isolated singularity of the function $|x|^{2-d}$ in $\mathbb{R}^d \setminus \{0\}$.

Example 2.31. Consider the function in $\mathbb{R}^2 \setminus \{0\}$ given by

$$u(x, y) = \frac{x}{x^2 + y^2}.$$

Then 0 is an isolated singularity. Note that u is the real part of the holomorphic function $1/z$ in $\mathbb{C} \setminus \{0\}$.

We say that an isolated singularity $x_0 \in \Omega$ of u is *removable* if we can define u at x_0 such that it makes u harmonic in Ω.

In Examples 2.30 and 2.31, the functions are unbounded on every neighborhood of the isolated singularity, so it cannot be removable. It could also be the case for bounded functions near the singularity.

Example 2.32. The function $|x|$ is harmonic in $\mathbb{R} \setminus \{0\}$, so 0 is an isolated singularity, and is not removable because $|x|$ is not equal to a linear function in all of $\mathbb{R}$. Note that, in this case, u is bounded in a neighborhood of 0.

However, such an example is only possible in one dimension, as we shall see below.

For a ball B, we denote by B^* be the *punctured ball*, that is, the ball B minus its center: if $B = B_r(x_0)$, then

$$B_r^*(x_0) = \{x :\in \mathbb{R}^d \ : \ 0 < |x - x_0| < r\}.$$

Theorem 2.33 (Riemann)**.** *Assume $d \geq 2$, and let x_0 be an isolated singularity of the harmonic function u. If u is bounded in some punctured ball around x_0, then x_0 is a removable singularity.*

As seen in Example 2.32, the assumption $d \geq 2$ is necessary.

Proof. By an appropiate translation and dilation, we can assume u is harmonic in $\mathbb{B}^*$ and continuous on $\bar{\mathbb{B}}^*$ (the punctured closed ball). We show that it can be extended to the origin so that it is harmonic in $\mathbb{B}$. We assume that $d > 2$ and leave the case $d = 2$ as an exercise (Exercise (20)).

For $\varepsilon > 0$, we define the function in $\bar{\mathbb{B}}^*$

$$v_\varepsilon(x) = u(x) - \mathcal{P}(u|_{\mathbb{S}})(x) + \varepsilon(|x|^{2-d} - 1).$$

As $|x|^{2-d}$ is harmonic in $\mathbb{R}^d \setminus \{0\}$, v_ε is harmonic in $\mathbb{B}^*$ and, if we set $\mathcal{P}(u|_\mathbb{S})(\xi) = u(\xi)$ if $\xi \in \mathbb{S}$, v_ε is continuous on $\bar{\mathbb{B}}^*$.

We observe that, for $\xi \in \mathbb{S}$, $v_\varepsilon(\xi) = 0$. Moreover, as $x \to 0$, we have $v_\varepsilon(x) \to \infty$ because u is bounded near 0. Thus, by the maximum principle, $v_\varepsilon(x) > 0$ for all $x \in \mathbb{B}^*$, because otherwise u would take a negative minimum in $\mathbb{B}^*$, and that is not possible by the maximum principle. Since $\varepsilon > 0$ is arbitrary, we obtain that $u(x) \geq \mathcal{P}(u|_\mathbb{S})(x)$ for all $x \in \mathbb{B}^*$.

If we repeat the argument for $-u$, we obtain $u(x) \leq \mathcal{P}(u|_\mathbb{S})(x)$ for all $x \in \mathbb{B}^*$. Thus $u(x) = \mathcal{P}(u|_\mathbb{S})(x)$ in $\mathbb{B}^*$.

Therefore, the Poisson integral $\mathcal{P}(u|_\mathbb{S})(x)$ is the harmonic extension of u to all of $\mathbb{B}$. □

Note that, in the proof of Theorem 2.33, we are extending u to $x = 0$ by its average over $\mathbb{S}$. It is actually not necessary to assume that u is bounded near x_0 to conclude that x_0 is a removable singularity. See Exercise (21).

Exercises

(1) A *translation* in $\mathbb{R}^d$ is a map $T : \mathbb{R}^d \to \mathbb{R}^d$ of the form $T(x) = x + h$, for some $h \in \mathbb{R}^d$.
   (a) If T is a translation, $\Delta(u \circ T) = (\Delta u) \circ T$.
   (b) If u is harmonic in $\mathbb{R}^d$ and T is a translation, then $u \circ T$ is also harmonic in $\mathbb{R}^d$.

(2) An *orthogonal transformation* in $\mathbb{R}^d$ is a map $P : \mathbb{R}^d \to \mathbb{R}^d$ of the form $P(x) = Ax$, for some orthogonal $n \times n$ matrix A, that is, A satisfies that $AA^t = I_n$, where A^t is the transpose of A and I_n is the $n \times n$ identity matrix.
   (a) If P is an orthogonal transformation, then
$$\Delta(u \circ P) = (\Delta u) \circ P.$$
   (b) If u is harmonic in $\mathbb{R}^d$ and P is orthogonal, then $u \circ P$ is also harmonic in $\mathbb{R}^d$.

(3) Prove fact 2.7.

(4) Let $u(x) = ax + b$. Then

$$u(x_0) = \frac{1}{2r} \int_{x_0-r}^{x_0+r} u(x)dx.$$

(5) Prove that the function v in the proof of Theorem 2.4 is harmonic.

(6) Suppose u is harmonic in a neighborhood of $\bar{\Omega}$, where Ω is a C^1 domain. Then

$$\int_{\partial\Omega} \partial_\nu u \, d\sigma = 0.$$

(7) Let f be Riemann-integrable on the rectangle R, and continuous at the interior point $x_0 \in R$. As $\varepsilon \to 0$,

(a) $\displaystyle\frac{1}{|B_\varepsilon(x_0)|} \int_{B_\varepsilon(x_0)} f \to f(x_0)$; and

(b) $\displaystyle\frac{1}{|S_\varepsilon(x_0)|} \int_{S_\varepsilon(x_0)} f \to f(x_0)$.

(8) If $\Omega \subset \mathbb{R}^d$ is a bounded domain and u is harmonic in Ω and continuous on $\bar{\Omega}$, then u takes its maximum and its minimum on $\partial\Omega$.

(9) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, u and v harmonic in Ω and continuous on $\bar{\Omega}$. If $u = v$ on $\partial\Omega$, then $u = v$ in Ω.

(10) The following exercises provide the details of the proof of the fundamental theorem of algebra.

(a) If f is a holomorphic function without zeroes in its domain Ω, then $1/f$ is holomorphic in Ω.

(b) If $p(z)$ is a polynomial over $\mathbb{C}$, then either $p(z)$ is constant or $|p(z)| \to \infty$ as $|z| \to \infty$.

(c) If $p(z)$ is polynomial over $\mathbb{C}$ with no roots, then $1/p(z)$ is an entire bounded function.

(11) If f is an entire function and its real part is nonnegative, then f is constant.

(12) If u is a radial harmonic function in $\mathbb{B}$, then it is constant.

(13) $x \mapsto P(x,\xi)$ is harmonic in $\mathbb{B}$, for each $\xi \in \mathbb{S}$. (*Hint:* Write $P(x,\xi) = \omega_d^{-1}(1 - |x|^2)|x-\xi|^{-d}$ and use the identity $\Delta(uv) = (\Delta u)v + 2\nabla u \cdot \nabla v + u\Delta v$.)

(14) *Symmetry Lemma*: If $x \in \mathbb{B}$ and $\xi \in \mathbb{S}$, then

$$\left| |x|\xi - \frac{x}{|x|} \right| = |x - \xi|.$$

See Figure 2.4.

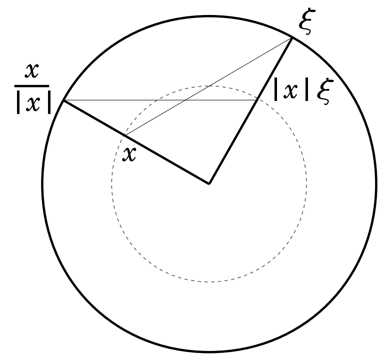

Figure 2.4. If $x \in \mathbb{B}$ and $\xi \in \mathbb{S}$, the distance between the points $x/|x|$ and $|x|\xi$ is the same as the distance between x and ξ, as stated by the symmetry lemma.

(15) Let f be Riemann-integrable on $\mathbb{S}$. Then its Poisson integral u is harmonic in $\mathbb{B}$.

(16) *Hopf lemma*: If u is a nonconstant harmonic function in $\mathbb{B}$, is continuous on $\bar{\mathbb{B}}$, and attains its maximum at $\zeta \in \mathbb{S}$, then there exists $c > 0$ such that

$$u(\zeta) - u(r\zeta) > c(1 - r),$$

for any $0 < r < 1$.

(17) *Harnack inequality*: If u is a harmonic function in $\mathbb{B}$, is continuous on $\bar{\mathbb{B}}$, and is positive, then

$$\frac{1 - |x|}{(1 + |x|)^{d-1}} u(0) \leq u(x) \leq \frac{1 + |x|}{(1 - |x|)^{d-1}} u(0)$$

for all $x \in \mathbb{B}$.

(18) If u is harmonic in Ω and $\bar{B}_r(x_0) \subset \Omega$, then the values of u in $B_r(x_0)$ are determined by its values on $S_r(x_0)$.

(19) Let u_n be a sequence of harmonic functions in Ω such that $u_n \rightrightarrows u$ on any compact $K \subset \Omega$. Then u is harmonic in Ω.

(20) Prove Theorem 2.33 for $d = 2$.

(21) Let u be harmonic in a domain in $\mathbb{R}^d$ with an isolated singularity at x_0. If $d = 2$ and

$$\lim_{x \to x_0} u(x) \log |x - x_0| = 0,$$

or $d > 2$ and

$$\lim_{x \to x_0} u(x) |x - x_0|^{d-2} = 0,$$

then x_0 is a removable singularity.

Notes

The results of this chapter are basic classical results, proven in every text in harmonic functions and partial differential equations. A classical reference for the theory of harmonic functions is [**Kel67**]. Theorem 2.4 is a result by Gauss [**Gau40**]. The proof presented here is the most popular and can be found, for instance, in [**ABR01**] or in [**Fol95**]. It can also be proven by differentiating the integral over a sphere with respect to its radius, as in [**Eva10**] or in [**MS13**]. The proof of Theorem 2.9 is also in [**Fol95**]. The proof of Theorem 2.14 is an elaboration of the proof by Nelson [**Nel61**]. Siméon Denis Poisson developed explicit expressions to solutions to the Laplace equation in terms of integrals over the sphere in [**Poi20**], and thus the Poisson kernel and integral are named after him. Theorem 2.33 is a result by Riemann [**Rie51**]. The proof presented here can be found in [**ABR01**].