Introduction

Probably, the one most famous book in the whole history of mathematics is Euclid's "Elements". In Europe it was used as a standard textbook of geometry in all schools for about 2000 years.

One of the first theorems in the "Elements" is the following Proposition I.5, of which we quote only the first half.

Theorem 1 (Euclid). *In isosceles triangles the angles at the base are equal to one another.*

Proof. Every high school student knows the standard modern proof of this proposition. It is very short.

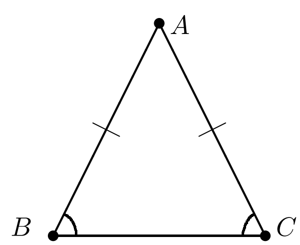


Figure 1. An isosceles triangle

STANDARD PROOF. Let ABC be the given isosceles triangle (Figure 1). Since $AB = AC$, there exists a plane movement (reflection) that takes A to A, B to C and C to B. Under this movement, $\angle ABC$ goes into $\angle ACB$; therefore, these two angles are equal. $\square$

It seems that there is nothing interesting about this theorem. However, wait a little and look at Euclid's original proof (Figure 2).

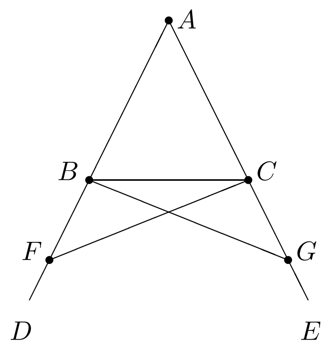


Figure 2. Euclid's proof

Euclid's original proof. On the prolongations AD and AE of the sides AB and AC, choose points F and G such that $AF = AG$. Then $\triangle ABG = \triangle ACF$; hence $\angle ABG = \angle ACF$. Also $\triangle CBG = \triangle BCF$; hence $\angle CBG = \angle BCF$. Therefore $\angle ABC = \angle ABG - \angle CBG = \angle ACF - \angle BCF = \angle ACB$. □

In mediaeval England, Proposition I.5 was known under the name of *pons asinorum* (asses' bridge). In fact, the part of Figure 2 formed by the points F, B, C, G and the segments that join them really resembles a bridge. Poor students who could not master Euclid's proof were compared to asses that could not surmount this bridge.

From a modern viewpoint Euclid's argument looks cumbersome and weird. Indeed, why did he ever need these auxiliary triangles ABG and ACF? Why was he not happy just with the triangle ABC itself? The reason is that Euclid just could not use *movements* in geometry: this was forbidden by his philosophy, stating that "mathematical objects are alien to motion".

This example shows that the use of movements can elucidate geometrical facts and greatly facilitate their proof. But movements are

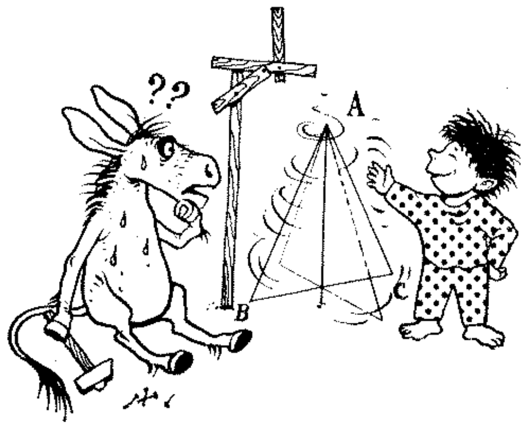


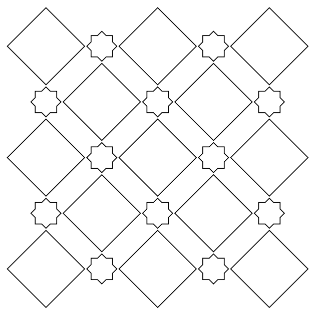
Figure 3. Asses's Bridge

important not only when they are studied separately. It is very interesting to study the *social behaviour* of movements, i.e. the structure of *sets of interrelated movements* (or more general transformations). In this area, the most important notion is that of a *transformation group*.

The theory of groups, as a mathematical theory, appeared not so long ago, only in the nineteenth century. However, examples of objects that are directly related to transformation groups had been created back in ancient civilizations, both oriental and occidental. This refers to the art of ornament, called "the oldest aspect of higher mathematics expressed in an implicit form" by the famous twentieth century mathematician Hermann Weyl.

Figure 4 shows two examples of ornaments found on the walls of the mediaeval Alhambra Palace in Spain.

Both patterns are highly symmetric in the sense that they are preserved by many plane movements. In fact, the symmetry properties of Figure 4a are very close to those of Figure 4b: each ornament has an infinite number of translations, rotations by 90° and 180°, reflections and glide reflections. However, they are not identical. The difference between them is in the way these movements are related to



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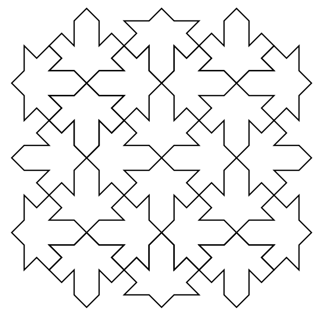

b

Figure 4. Two ornaments from Alhambra

each other for each of the two patterns. The exact meaning of these words can only be explained in terms of group theory, which says that the symmetry groups of Figures 4a and 4b are not isomorphic (this is the contents of Exercise 129, at the end of Chaper 5).

The problem of determining and classifying *all* the possible types of wall pattern symmetry was solved in the late nineteenth century independently by the Russian scientist E. S. Fedorov and the German scientist A. Schoenflies. It turned out that there are exactly 17 different types of plane crystallographic groups (see the table at the end of Chapter 5).

Of course, the significance of group theory goes far beyond the classification of plane ornaments. In fact, it is one of the key notions in the whole of mathematics, widely used in algebra, geometry, topology, calculus, mechanics, etc.

This book provides an elementary introduction into the theory of groups. We begin with some examples from elementary Euclidean geometry, where plane movements play an important role and the ideas of group theory naturally arise. Then we explicitly introduce the notion of a transformation group and the more general notion of an abstract group, and discuss the algebraic aspects of group theory and its applications in number theory. After that we pass to group actions, orbits, invariants, and some classification problems, and finally go as far as the application of continuous groups to the solution of differential equations. Our primary aim is to show how the notion of group works in different areas of mathematics, thus demonstrating that mathematics is a unified science.

The book is intended for people with the beginning of a basic college mathematical education, including the knowledge of elementary algebra, geometry and calculus.

You will find many problems given with detailed solutions, and many exercises, supplied with hints and answers at the end of the book. It goes without saying that the reader who wants to really understand what's going on must try to solve as many problems as possible.