Chapter 1

Warming up to Enumerative Geometry

Enumerative geometry is an old subject that has been revisited extensively over the past 150 years. Enumerative geometry was an active field in the 19^{th} century. Much progress was made, and there was much excitement about what was to come. Indeed, enumerative geometry was the subject of Hilbert's 15^{th} problem. Hilbert's famous problem list was delivered at the beginning of the 20^{th} century and was quite influential in shaping mathematics. A good overview of the influence of the Hilbert problems, including Hilbert's 15^{th}, appears in [**Hilbert**].

Unfortunately, many fundamental enumerative problems eluded the best mathematicians for most of the 20^{th} century. Progress came from a seemingly unlikely source: string theory in physics. In this book we will learn many fundamentals of enumerative geometry and begin to appreciate some of the connections of these ideas to physics.

The basic question of enumerative geometry in its most general form is simply stated as

"How many geometric structures of a given type satisfy a given collection of geometric conditions?"

A trivial example would be the question:

"How many points in the plane lie on each of two given lines?"

However, even this simple example reveals several potential difficulties of a type which we will have to frequently address:

- The answer depends on the choice of lines:
  (1) If the lines are chosen generally, there is exactly one such point, the point of intersection of the two lines.
  (2) The point can "disappear to infinity" if the lines are moved into a parallel position, in a precise way to be explained shortly.
  (3) The answer can be infinite (if the two lines coincide with each other).
- Note as well that the question can be translated into a question of algebra: how many solutions (x, y) are there to the system of equations

$$\begin{aligned} ax + by &= c, \\ dx + ey &= f? \end{aligned}$$

This somewhat cumbersome answer to a very simple question is unpleasant. In enumerative geometry, this is dealt with by changing the question slightly, sacrificing the simplicity of the question in favor of simplicity of the answer, in this case an unequivocal "1".

Let's shift from the context of geometry to the context of algebra, as we just saw we can do in this example.[1] Let's stay in the realm of the elementary and ask the enumerative (algebraic) question: "how many roots does a polynomial of degree n in one variable have?"

For this problem, there is again a range of subcases, revealing a range of potential difficulties. Let's consider a polynomial of degree d

$$f(x) = a_0 x^d + a_1 x^{d-1} + \cdots + a_{d-1} x + a_d,$$

for the moment with real coefficients a_i, and solve $f(x) = 0$. Let's first consider $d = 1$. The solution is $x = -a_1/a_0$, and there is at first glance exactly one solution. There is a difficulty, which we will see is more than a simple issue of semantics: suppose we agree to consider $f(x) = 0 \cdot x + a_1$ as a degenerate degree 1 polynomial; after all, $f(x)$

[1] This ability to move freely from geometry to algebra and back is at the core of the subject called *algebraic geometry*.

can certainly be written in the form $a_0x + a_1$. There are two cases to consider. In the first case, $a_1 \neq 0$, we see that $f(x)$ has no roots. The root can be viewed as having "gone off to ∞" as before: we can write

$$f(x) = \lim_{a_0 \to 0} a_0 x + a_1,$$

and then the root $-a_1/a_0$ goes to ∞ or $-\infty$ as a_0 approaches 0 from one direction or the other. If $a_1 = 0$ as well, then there are infinitely many solutions: any number x trivially satisfies the equation $0 \cdot x = 0$.

The situation already gets much richer if $d = 2$. The quadratic formula gives

$$x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_0}.$$

There are now several possibilities:

(1) $a_0 \neq 0$. There are several well-known subcases. Let $D = a_1^2 - 4a_0a_2$ be the discriminant of $f(x)$.
   (a) $D > 0$. There are two roots.
   (b) $D < 0$. There are no real roots.
   (c) $D = 0$. There is one root.

(2) $a_0 = 0$. There are several possibilities:
   (a) $a_1 = 0$. In this case, there are no solutions (unless additionally $a_2 = 0$, in which case there are infinitely many roots).
   (b) $a_1 \neq 0$. There is now exactly one solution $x = -a_2/a_1$. We can think that the "other root" has "gone off to infinity".

See Exercise 1. The situation will clearly only get worse as d increases.

We can simplify the answer by changing the question. Essentially all of these cases can be unified neatly by a few simple changes in the enumerative problem:

- Use complex coefficients and solutions.
- Count solutions with multiplicity.
- Include infinity.

- Slightly modify our description of polynomials to include infinity.

In any of the cases (1) (a), (b), and (c), there are now exactly two solutions. The situation is familiar from high school algebra. If the coefficients of $f(x)$ are real, then in case (a) there are two real roots, in case (b) there are two complex conjugate roots, and in case (c) there is one real root with multiplicity 2.

To handle cases (2), we need to include infinity. We do that by introducing the complex projective line.

Definition 1.1. *The* complex projective line $\mathbf{CP}^1$ *(or just* $\mathbf{P}^1$*) is the set of all ordered pairs of complex numbers* $\{(x,y) \in \mathbf{C}^2 \mid (x,y) \neq (0,0)\}$ *where we identify pairs* (x,y) *and* (x',y') *if one is a scalar multiple of the other:* $(x,y) = \lambda(x',y')$ *for some* $\lambda \in \mathbf{C}^*$*, where* $\mathbf{C}^*$ *is the set of nonzero complex numbers.*

So for example $(1,2)$, $(3,6)$, and $(2+3i, 4+6i)$ all represent the same point of $\mathbf{P}^1$.

This construction is an example of the quotient of a set by an equivalence relation. See Exercise 2.

The idea is that $\mathbf{P}^1$ can be thought of as the union of the set of complex numbers $\mathbf{C}$ and a single point "at infinity". To see this, consider the following subset $U_0 \subset \mathbf{P}^1$:

$$U_0 = \{(x_0, x_1) \in \mathbf{P}^1 \mid x_0 \neq 0\}.$$

Then U_0 is in one-to-one correspondence with $\mathbf{C}$ via the map

$$\phi_0 : U_0 \to \mathbf{C} : \qquad (x_0, x_1) \mapsto \frac{x_1}{x_0}. \tag{1}$$

Note that ϕ_0 is well defined on U_0. First of all, x_0 is not 0 so the division makes sense. Secondly, if (x_0, x_1) represents the same point of $\mathbf{P}^1$ as (x_0', x_1'), then $x_0 = \lambda x_0'$ and $x_1 = \lambda x_1'$ for some nonzero $\lambda \in \mathbf{C}$. Thus $\phi_0((x_0,x_1)) = x_1/x_0 = (\lambda x_1')/(\lambda x_0') = x_1'/x_0' = \phi((x_0',x_1'))$ and ϕ is well defined as claimed. The inverse map is given by

$$\psi_0 : \mathbf{C} \to U_0, \qquad z \mapsto (1, z).$$

The complement of U_0 is the set of all points of $\mathbf{P}^1$ of the form $(0, x_1)$. But since $(0, x_1) = x_1(0,1)$, all of these points coincide with

$(0,1)$ as a point of $\mathbf{P}^1$. So $\mathbf{P}^1$ is obtained from a copy of $\mathbf{C}$ by adding a single point.

This point can be thought of as the point at infinity. To see this, consider a complex number t, and identify it with a point of U_0 using ψ_0; i.e., we identify it with $\psi_0(t) = (1,t)$. Now let $t \to \infty$. The beautiful feature is that the limit now exists in $\mathbf{P}^1$! To see this, rewrite $(1,t)$ as $(1/t,1)$ using scalar multiplication by $1/t$. This clearly approaches $(0,1)$ as $t \to \infty$, so $(0,1)$ really should be thought of as the point at infinity!

We have been deliberately vague about the precise meaning of limits in $\mathbf{P}^1$. This is a notion from topology, which we will deal with later in Chapter 4. The property that limits exist in a topological space is a consequence of the *compactness* of the space, and the process of enlarging $\mathbf{C}$ to the compact space $\mathbf{P}^1$ is our first example of the important process of *compactification*. This makes the solutions to enumerative problems well-defined, by preventing solutions from going off to infinity. A precise definition of compactness will be given in Chapter 4.

We now have to modify our description of complex polynomials by associating to them polynomials $F(x_0,x_1)$ on $\mathbf{P}^1$. Before turning to their definition, note that the equation $F(x_0,x_1) = 0$ need not make sense as a well-defined equation on $\mathbf{P}^1$, since it is conceivable that a point could have different representatives (x_0,x_1) and (x_0',x_1') such that $F(x_0,x_1) = 0$ while $F(x_0',x_1') \neq 0$. We avoid this problem by requiring that $F(x_0,x_1)$ be a *homogeneous polynomial*; i.e., all terms in F have the same total degree, which is called the degree of F. So

$$F(x_0,x_1) = \sum_{i=0}^{d} a_i x_0^i x_1^{d-i} \tag{2}$$

is the general form of a homogeneous polynomial of degree d. If $\lambda \in \mathbf{C}^*$, then we compute $F(\lambda x_0, \lambda x_1) = \lambda^d F(x_0,x_1)$, so $F(x_0,x_1) = 0$ if and only if $F(\lambda x_0,\lambda x_1) = 0$, and the equation $F(x_0,x_1) = 0$ is a well-defined condition on a point $(x_0,x_1) \in \mathbf{P}^1$.

Having enlarged $\mathbf{C}$ to $\mathbf{P}^1$, we correspondingly need to "extend" an arbitrary polynomial $f(x)$ to a homogeneous polynomial $F(x_0,x_1)$.

Letting d be the degree of $f(x)$, we define its *homogenization* as

$$F(x_0, x_1) = x_0^d f\left(\frac{x_1}{x_0}\right). \tag{3}$$

Note that the homogenization satisfies

$$f(x) = F(\psi_0(x)) = F(1, x). \tag{4}$$

But notice that (4) does not uniquely specify $F(x_0, x_1)$ given $f(x)$, since multiplication of F by any power of x_0 will not alter the validity of (4). In other words, we have homogeneous polynomials F of different degrees which satisfy (4). So we can, if we want to, replace the occurrence of d in (3) by an integer $e \geq d$ and put $F(x_0, x_1) = x_0^e f(x_1/x_0)$, which is easily checked to be the unique homogeneous solution to (4) of degree e. However, when we speak of the homogenization of $f(x)$, our meaning is (3), i.e., the choice $e = d$. The process of going from $F(x_0, x_1)$ to $f(x)$ by (4) is referred to as *dehomogenization*.[2]

Now let's go back to the case of a polynomial $f(x)$ of degree 1. The homogenization of $f(x) = a_0 x + a_1$ as a polynomial of degree 1 is $F(x_0, x_1) = a_0 x_1 + a_1 x_0$. Note how the case $a_0 = 0$ makes no difference and is treated on equal footing with the case $a_0 \neq 0$: as long as $f(x)$ is not the zero polynomial (equivalently, F is not the zero polynomial), there is exactly one root in $\mathbf{P}^1$, namely $(x_0, x_1) = (-a_0, a_1)$.

The homogenization of a general degree 2 polynomial is given by $F(x_0, x_1) = a_0 x_1^2 + a_1 x_0 x_1 + a_2 x_0^2$. Again, as long as F is not the zero polynomial, there are exactly two roots in $\mathbf{P}^1$ (including multiplicity). For example, if $a_0 = 0$, then the roots are $(0, 1)$ and $(-a_1, a_2)$. Note that this is exactly what we found earlier: identifying $\mathbf{C}$ with U_0 via (1), then the point $(0, 1)$ is thought of as being at infinity, and the point $(-a_1, a_2)$ is identified with the point $\phi_0((-a_1, a_2)) = -a_2/a_1$ found earlier. See Exercise 4.

We can now state an easy generalization:

Theorem 1.2. *Any nonzero homogeneous polynomial $F(x_0, x_1)$ of degree d has exactly d roots in $\mathbf{P}^1$ including multiplicity.*

[2] We could more precisely have referred to this as dehomogenization with respect to x_0 to emphasize that x_0 is the variable that gets set to 1.

Proof. By induction on d. The case $d = 1$ is already proven. In the general case, factor out all powers of x_0, writing $F(x_0, x_1) = x_0^r G(x_0, x_1)$ where G is homogeneous of degree $d-r$ and is not divisible by x_0. Then the point $(0, 1)$ at infinity is a root of F with multiplicity r. If $r > 0$, then G has $d - r$ roots by induction. These roots are also roots of F, and when they are combined with the r roots at $(0, 1)$, we have found all d roots and we are done in this case.

So we may assume that F is not divisible by x_0. This implies that it has a nonzero term involving x_1^d, so that its dehomogenization $f(x)$ also has degree d. Now the fundamental theorem of algebra implies that $f(x)$ has a complex root $x = a$. Write $f(x) = (x - a)h(x)$ where h has degree $d - 1$. Then $F(x_0, x_1) = (x_1 - ax_0)H(x_0, x_1)$, where H, the homogenization of h, has degree $d - 1$. By induction, H has $d - 1$ roots. These are all roots of F as well, and F has the additional root $(1, a)$, giving d roots in all. □

An essential role was played by the fundamental theorem of algebra, which is why we had to use complex coefficients. The proof shows that a similar result would have held if we had used coefficients in an arbitrary algebraically closed field.

Later, in Chapter 11, we will rederive Theorem 1.2 using elementary instances of deep ideas from physics including supersymmetry and topological quantum field theories.

The notion of $\mathbf{P}^1$ generalizes.

Definition 1.3. *The* n-dimensional complex projective space $\mathbf{CP}^n$ *(or just* $\mathbf{P}^n$*) is the set of all ordered* $(n+1)$*-tuples of complex numbers*

$$\{\mathbf{x} = (x_0, \ldots, x_n) \in \mathbf{C}^{n+1} \mid \mathbf{x} \neq (0, \ldots, 0)\}$$

where we identify pairs which are scalar multiples of each other: $\mathbf{x} = \lambda \mathbf{x}'$ *for some* $\lambda \in \mathbf{C}^*$.

Complex projective space plays a fundamental role in complex algebraic geometry.

The process of projectivization can be generalized to any complex vector space V, including infinite-dimensional spaces.

Definition 1.4. *The* projectivization $\mathbf{P}(V)$ *of* V *is the quotient of* $V - \mathbf{0}$ *by the equivalence relation* $x \sim x'$ *if* $x = \lambda x'$ *for some* $\lambda \in \mathbf{C}^*$.

In the statement of Definition 1.4, the zero element of V has been denoted by $\mathbf{0}$.

See Exercise 3. Another description of $\mathbf{P}(V)$ is given in Exercise 5.

Let's return to our first enumerative question about two lines in the plane. We replace the plane by $\mathbf{C}^2$, which we then compactify by enlarging it to $\mathbf{P}^2$. More generally, $\mathbf{P}^n$ has a subset

$$U_0 = \{(x_0, \ldots, x_n) \in \mathbf{P}^n \mid x_0 \neq 0\}$$

which is in one-to-one correspondence with $\mathbf{C}^n$ via the map

$$\phi_0 : U_0 \to \mathbf{C}^n : \qquad (x_0, \ldots, x_n) \mapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0} \right).$$

The inverse map is given by

$$\psi_0 : \mathbf{C}^n \to U_0, \qquad (x_1, \ldots, x_n) \mapsto (1, x_1 \ldots, x_n).$$

So $\mathbf{P}^n$ is an enlargement of $\mathbf{C}^n$ (in fact, it's a compactification). The complement of U_0 in $\mathbf{P}^n$ is naturally in one-to-one correspondence with $\mathbf{P}^{n-1}$ (Exercise 6).

Similarly, we have subsets U_i defined by $x_i \neq 0$ for each $0 \leq i \leq n$. Each of these U_i is in one-to-one correspondence with $\mathbf{C}^n$, as will be seen explicitly in Example 4.20.

We have a notion of homogeneous polynomials on $\mathbf{P}^n$: these are the polynomials for which all terms have the same total degree. If $F(x_0, \ldots, x_n)$ is homogeneous, then its zero locus

$$Z(F) = \{\mathbf{x} \in \mathbf{P}^n \mid F(x_0, \ldots, x_n) = 0\}$$

is well defined and is called the *hypersurface defined by* F. See Exercise 7. We refer to F as a *defining equation* of $Z(F)$.

Given a hypersurface $Z \subset \mathbf{P}^n$, its defining equation is far from unique; e.g. $Z(F) = Z(\lambda F)$ for any $\lambda \in \mathbf{C}^*$, and $Z(F) = Z(F^n)$ for any positive integer n. The *degree* of Z is the minimal degree of a defining equation for Z. Taking the minimal degree effectively eliminates the possibility of introducing extraneous powers in F or any of its factors. Sometimes we will want to consider multiplicities and then we have to put extra powers back in.

In particular we can state an easy converse of Theorem 1.2 as saying that any set $Z \subset \mathbf{P}^1$ consisting of d points (including multiplicity) is the zero locus of a degree d homogeneous polynomial $f(x_0, x_1)$, unique up to a scalar multiple. If we have points $(a_i, b_i) \in \mathbf{P}^1$ with multiplicities r_i and $\sum_i r_i = d$, we can take f to be the degree d polynomial

$$f(x_0, x_1) = \prod_i (b_i x_0 - a_i x_1)^{r_i}.$$

Just as for $\mathbf{P}^1$, we have a notion of the dehomogenization of $F(x_0, \ldots, x_n)$:

$$f(x_1, \ldots, x_n) = F(\psi_0(x_1, \ldots, x_n)) = F(1, x_1, \ldots, x_n) \tag{5}$$

and a notion of the homogenization of $f(x_1, \ldots, x_n)$:

$$F(x_0, \ldots, x_n) = x_0^d f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right), \tag{6}$$

where d is the degree of $f(x_1, \ldots, x_n)$, the maximum total degree of any term in f.

Compare with [**Hulek**, Sections 2.1, 2.2], a text containing an introduction to algebraic geometry with minimal prerequisites. [**Reid**] and [**Fulton3**] are other algebraic geometry texts readily accessible to undergraduates. For graduate level algebraic geometry, see, e.g., [**Miranda**] for complex algebraic geometry in dimension 1, [**GH**] for complex algebraic geometry more generally, or see either [**Harris**] or [**Hartshorne**] for algebraic geometry over arbitrary fields.

By a *line* in $\mathbf{P}^2$, we mean the hypersurface defined by a homogeneous polynomial of degree 1 in (x_0, x_1, x_2) (degree 1 hypersurfaces in $\mathbf{P}^n$ are more generally called *hyperplanes*). In other words, the polynomial $ax + by = c$ in $\mathbf{C}^2$ can be homogenized to get $ax_1 + bx_2 = cx_0$, and there is a similar homogenization for $dx + ey = f$ (note that (x, y) has been replaced by (x_1, x_2) here).

Now lines that were parallel in the plane meet in $\mathbf{P}^2$! Consider for instance the lines $x + y = 1$ and $x + y = 2$. Their homogenizations are $x_1 + x_2 = x_0$ and $x_1 + x_2 = 2x_0$. Subtracting these equations, we get $x_0 = 0$, indicating that any solutions are at infinity (which we already knew since the lines were parallel in the finite plane). Then $x_2 = -x_1$ from either equation. This solution $(0, x_1, -x_1)$ is just a single point $(0, 1, -1) \in \mathbf{P}^2$.

So if we replace our first question by:

"How many points in $\mathbf{P}^2$ lie in each of two given lines?",

the answer is much simpler than the corresponding question for the plane: the result is that the answer is always 1, as long as the two lines are distinct (Exercise 8). Note that the same result would have held with real coefficients. The point is that linear equations can be solved over the reals. We only need the complex numbers (an example of an algebraically closed field) when we go to higher degree.

We still have the problem of an infinity of solutions if the lines coincide. For that matter, we had the same problem with the infinity of roots of the zero polynomial. We will come back to this later in this book when we discuss excess intersection theory.

Exercises

1. Use limits to verify the assertion that a root of a quadratic equation goes off to infinity as its leading coefficient a_0 approaches 0 while a_1 remains finite. Verify that two solutions go off to infinity as a_1 and a_0 both approach 0 (assuming a_2 stays nonzero).

2. Consider a relation $\sim$ between elements of a set S, written $a \sim b$ for elements $a, b \in S$. We say that $\sim$ is an *equivalence relation* if $\sim$ has the following three properties:
   (a) Reflexivity: $a \sim a$ for all $a \in S$.
   (b) Symmetry: if $a \sim b$, then $b \sim a$.
   (c) Transitivity: if $a \sim b$ and $b \sim c$, then $a \sim c$.

   If $\sim$ is an equivalence relation and $a \in S$, we define the *equivalence class* of a to be

$$[a] = \{b \in S \mid a \sim b\}.$$

Prove that any two equivalence classes $[a]$ and $[b]$ are either equal or disjoint and that S is the union of all of the equivalence classes.

The set of equivalence classes is sometimes denoted by $S/\sim$ and is called the *quotient of S by the equivalence relation* $\sim$. We think of $S/\sim$ as the set S with elements $a, b \in S$ identified if $a \sim b$.

3. Let V be a vector space over a field k and consider the relation on $V - \mathbf{0}$ defined by scalar multiplication: $v \sim v'$ if and only if $v = \lambda v'$ for some $\lambda \in k - \{0\}$. Prove that $\sim$ is an equivalence relation.
4. Show that every nonzero homogeneous polynomial $F(x_0, x_1)$ of degree 2 has exactly two roots in $\mathbf{P}^1$ including multiplicity.
5. Let V be a vector space. Find a natural one-to-one correspondence between $\mathbf{P}(V)$ and the set of all 1-dimensional linear subspaces of V.
6. Show that $\mathbf{P}^n - U_0$ is in natural one-to-one correspondence with $\mathbf{P}^{n-1}$. This subset is sometimes called the *hyperplane at infinity*.
7. Suppose that $F(x_0, x_1, \ldots, x_n)$ is a nonzero homogeneous polynomial of degree d.
   (a) Prove that $F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n)$ for any complex number λ.
   (b) Prove that the zero locus $Z(F) \subset \mathbf{P}^n$ is well defined; i.e., $F(\mathbf{x}) = 0$ if and only if $F(\lambda \mathbf{x}) = 0$ for any $\lambda \in \mathbf{C}^*$.
   (c) Show that if F is a polynomial and $Z(F)$ is well defined in the sense of (b) above, then F must be homogeneous of some degree.
8. Prove that any two distinct lines in $\mathbf{P}^2$ have exactly one point of intersection.