Preface

These lectures were prepared for the advanced undergraduate course in *Geometric Combinatorics* at the Park City Mathematics Institute in July 2004. Many thanks to the organizers of the undergraduate program, Bill Barker and Roger Howe, for inviting me to teach this course. I also wish to thank Ezra Miller, Vic Reiner and Bernd Sturmfels, who coordinated the graduate research program at PCMI, for their support. Edwin O'Shea conducted all the tutorials at the course and wrote several of the exercises seen in these lectures. Edwin was a huge help in the preparation of these lectures from beginning to end.

The main goal of these lectures was to develop the theory of convex polytopes from a geometric viewpoint to lead up to recent developments centered around secondary and state polytopes arising from point configurations. The geometric viewpoint naturally relies on linear optimization over polytopes. Chapters 2 and 3 develop the basics of polytope theory. In Chapters 4 and 5 we see the tools of Schlegel and Gale diagrams for visualizing polytopes and understanding their facial structure. Gale diagrams have been used to unearth several bizarre phenomena in polytopes, such as the existence of polytopes whose vertices cannot have rational coordinates and others whose facets cannot be prescribed. These examples are described in Chapter 6. In Chapters 7–9 we construct the secondary polytope of a graded point configuration. The faces of this polytope index the regular subdivisions of the configuration. Secondary polytopes appeared in the literature in the early 1990's and play a crucial role in combinatorics, discrete optimization and algebraic geometry. The secondary polytope of a point configuration is naturally refined by the state polytope of the toric ideal of the configuration. In Chapters 10–14 we establish this relationship. The state polytope of a toric ideal arises from the theory of Gröbner bases, which is developed in Chapters 10–12. Chapter 13 establishes the connection between the Gröbner bases of a toric ideal and the regular triangulations of the point configuration defining the ideal. Finally, in Chapter 14 we construct the state polytope of a toric ideal and relate it to the corresponding secondary polytope.

These lectures are meant to be self-contained and do not require any background beyond basic linear algebra. The concepts needed from abstract algebra are developed in Chapters 1, 10, 11 and 12.

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