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## Chapter 5

# Gale Diagrams

In the last chapter we saw techniques for visualizing four-dimensional polytopes via their Schlegel diagrams. In this chapter, we will see that we can actually visualize even higher-dimensional polytopes as long as they do not have too many vertices. We do this via a tool called the Gale diagram of the polytope.

Consider  $n$  points  $\mathbf{v}_1, \dots, \mathbf{v}_n$  in  $\mathbb{R}^{d-1}$  whose affine hull has dimension  $d - 1$  and the matrix

$$A := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix} \in \mathbb{R}^{d \times n}.$$

A basic fact of affine linear algebra is that the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are affinely independent (see below) if and only if the vectors

$$(1, \mathbf{v}_1), \dots, (1, \mathbf{v}_n)$$

are linearly independent. If the dimension of  $\text{aff}(\mathbf{v}_1, \dots, \mathbf{v}_n)$  is  $d - 1$ , then there are  $d$  affinely independent vectors in this collection, which in turn implies that the rank of  $A$  is  $d$ . Hence the dimension of the kernel of  $A$  is  $n - d$ . Recall that the kernel of  $A$  is the linear subspace

$$\ker_{\mathbb{R}}(A) := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

Note that  $\mathbf{x} \in \ker_{\mathbb{R}}(A)$  if and only if (1)  $\sum_{i=1}^n \mathbf{v}_i x_i = \mathbf{0}$  and (2)  $\sum_{i=1}^n x_i = 0$ .

- Definition 5.1.**
- (1) Any vector  $\mathbf{x}$  with properties (1) and (2) is called an **affine dependence relation** on  $\mathbf{v}_1, \dots, \mathbf{v}_n$ .
  - (2) If  $\mathbf{x}$  satisfies only (1), then it would be a **linear dependence relation** on  $\mathbf{v}_1, \dots, \mathbf{v}_n$ .
  - (3) If  $\mathbf{x} = \mathbf{0}$  is the only solution to (1) and (2), then  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are said to be **affinely independent**.

Let  $B_1, \dots, B_{n-d} \in \mathbb{R}^n$  be a basis for the vector space  $\ker_{\mathbb{R}}(A)$ . If we organize these vectors as the columns of an  $n \times (n-d)$  matrix

$$B := \begin{pmatrix} B_1 & B_2 & \cdots & B_{n-d} \end{pmatrix},$$

we see that  $AB = \mathbf{0}$ .

**Definition 5.2.** Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\} \subset \mathbb{R}^{n-d}$  be the  $n$  ordered rows of  $B$ . Then  $\mathcal{B}$  is called a **Gale transform** of  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ . The associated **Gale diagram** of  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is the *vector configuration*  $\mathcal{B}$  drawn in  $\mathbb{R}^{n-d}$ .

Later, we will see a more general definition of Gale diagrams. Since the columns of  $B$  can be any basis of  $\ker_{\mathbb{R}}(A)$ , Gale transforms are not unique. However all choices of  $B$  differ by multiplication by a non-singular matrix and we will be happy to choose one basis of  $\ker_{\mathbb{R}}(A)$  and call the resulting  $\mathcal{B}$ , *the* Gale transform of  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ .

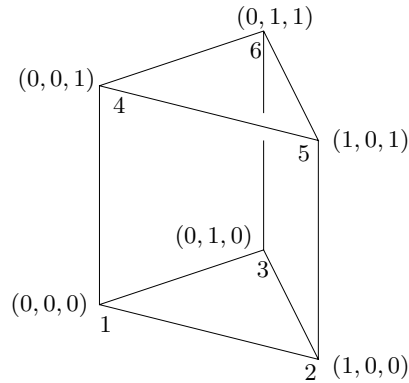
**Example 5.3.** Let  $\{\mathbf{v}_i\}$  be the vertices of the triangular prism shown in Figure 1. Then

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

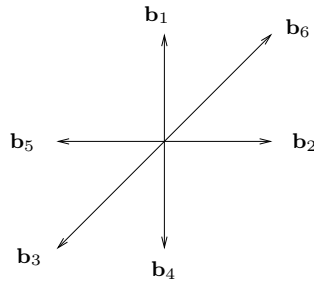
Computing a basis for the kernel of  $A$ , we get

$$B^t = \begin{pmatrix} 0 & 1 & -1 & 0 & -1 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{pmatrix}$$

where  $B^t$  is the transpose of  $B$ . The Gale transform  $\mathcal{B}$  is the vector configuration consisting of the columns of  $B^t$  (or the rows of  $B$ ). In our example,  $\mathcal{B} = \{\mathbf{b}_1 = (0, 1), \mathbf{b}_2 = (1, 0), \mathbf{b}_3 = (-1, -1), \mathbf{b}_4 = (0, -1), \mathbf{b}_5 = (-1, 0), \mathbf{b}_6 = (1, 1)\}$ . The Gale diagram is shown in Figure 2.



**Figure 1.** Triangular prism.



**Figure 2.** Gale diagram of the vertices of the triangular prism in Figure 1.

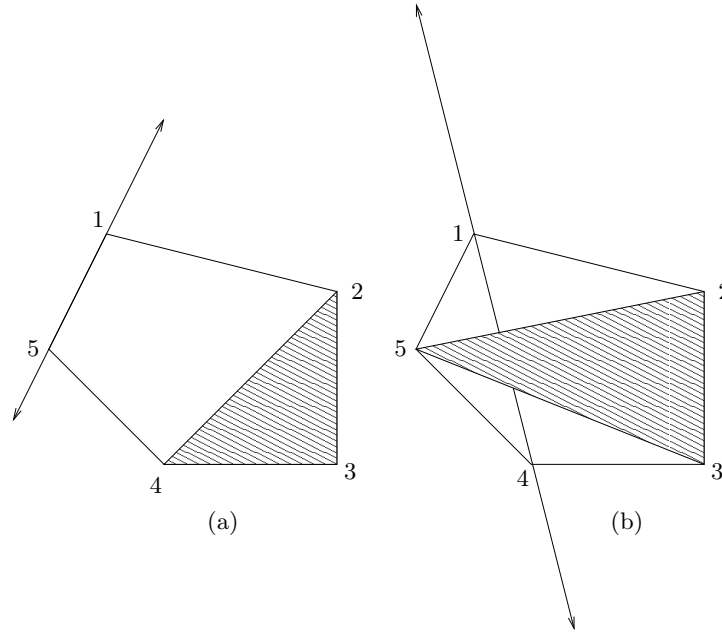
The labeling is very important in the construction of a Gale transform. We label column  $i$  of  $B^t$  as  $\mathbf{b}_i$ .

The main goal of this chapter will be to understand how to read off the face lattice of the  $(d-1)$ -polytope  $P = \text{conv}(\{\mathbf{v}_1, \dots, \mathbf{v}_n\})$  from the Gale diagram of  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ . If  $\{\mathbf{v}_j : j \in J\}$  are all the vertices on a face of  $P$  for some  $J \subseteq [n]$ , it is convenient to simply label this face by  $J$ . Here is a very important characterization of faces.

**Lemma 5.4.** *Let  $P = \text{conv}(\{\mathbf{v}_1, \dots, \mathbf{v}_n\})$ . Then  $J \subseteq [n]$  is a face of  $P$  if and only if*

$$\text{conv}(\{\mathbf{v}_j : j \in [n] \setminus J\}) \cap \text{aff}(\{\mathbf{v}_j : j \in J\}) = \emptyset.$$

Let us illustrate this condition on an example first. In Figure 3(a), note that 15 is a face of the pentagon and that  $\text{conv}(\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\})$  does not intersect the affine hull of the face 15. On the other hand, 14 is not a face of the pentagon and indeed  $\text{conv}(\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5\})$  does intersect the affine hull of the **non-face** 14. See Figure 3(b).



**Figure 3.** Condition in Lemma 5.4.

**Proof.** We may assume that  $P$  is a full-dimensional polytope. If  $J$  is a face of  $P$ , then by definition of a face, both  $J$  and  $\text{aff}(J)$  lie on a supporting hyperplane  $H$  of  $P$ . Choose a supporting hyperplane  $H$  that contains  $J$  but does not contain any higher-dimensional face of  $P$ . One way to do this would be to let the normal vector of  $H$  be the sum of the normal vectors of the affine spans of the facets containing  $J$ . We may assume without loss of generality that  $P$  lies in the halfspace  $H^-$ . Since no  $\mathbf{v}_j$ ,  $j \notin J$ , lies on the face  $\text{conv}(\{\mathbf{v}_j : j \in J\})$  of  $P$ ,  $\text{conv}(\{\mathbf{v}_j : j \in [n] \setminus J\})$  lies in the interior of  $H^-$ , which proves one direction of the lemma. Conversely, if  $\text{conv}(\{\mathbf{v}_j : j \in [n] \setminus J\}) \cap$

$\text{aff}(\{\mathbf{v}_j : j \in J\}) = \emptyset$ , then  $P$  lies in one halfspace defined by the hyperplane  $H$  obtained by extending  $\text{aff}(\{\mathbf{v}_j : j \in J\})$  which is thus a supporting hyperplane of  $P$ . This shows that  $J$  is a face of  $P$ .  $\square$

You might wonder why the lemma was not stated in the seemingly stronger form:  $J \subseteq [n]$  is a face of  $P$  if and only if  $\text{conv}(\{\mathbf{v}_j : j \in [n] \setminus J\}) \cap \text{conv}(\{\mathbf{v}_j : j \in J\}) = \emptyset$ . The above form of the lemma is what is needed to prove the main theorem below.

**Definition 5.5.** Call  $[n] \setminus J$  a **co-face** of  $P$  if  $J$  is a face of  $P$ .

Note that a co-face is not the same as a non-face. In the triangular prism in Figure 1, 123 is both a face and a co-face. (The labeling of the vertices of the prism was fixed by how we ordered them to create the matrix  $A$ .)

In order to understand our main theorem, we need to define formally what we mean by the *interior* and *relative interior* of a polytope. The **interior** of a polytope in  $\mathbb{R}^d$  is the set of all points in the polytope such that we can fit a  $d$ -dimensional ball centered at this point, of infinitesimal (as small as you wish but positive) radius, entirely inside the polytope. A polytope has an interior if and only if it is full-dimensional. For instance the interior of  $C_2$  is the set

$$\text{int}(C_2) = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < 1\}.$$

The line segment  $\text{conv}(\{(1, 0), (0, 1)\}) \subset \mathbb{R}^2$  does not have an interior since there is no point on this segment such that a two-dimensional ball centered at this point will be contained in the line segment. However, this line segment does have an interior if we think of it as a polytope in its affine hull, where it is a full-dimensional polytope. This is known as the **relative interior** of the line segment. In our example,  $\text{relint}(\text{conv}(\{(1, 0), (0, 1)\}))$  equals

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1, x_1 > 0, x_2 > 0\}.$$

We now come to the main theorem of this chapter. The proof of this theorem is taken from [Grü03, page 88].

**Theorem 5.6.** Let  $P = \text{conv}(\{\mathbf{v}_1, \dots, \mathbf{v}_n\})$ ,  $\mathbf{v}_i \in \mathbb{R}^{d-1}$ , and let  $\mathcal{B}$  be the Gale transform of  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ . Then  $J$  is a face of  $P$  if and only if either  $J = [n]$  or  $\mathbf{0} \in \text{relint}(\text{conv}(\{\mathbf{b}_k : k \notin J\}))$ .

**Proof.** Note that  $J = [n]$  if and only if  $J$  is the whole polytope  $P$  which is an improper face of  $P$ . So we have to show that  $J \subsetneq [n]$  is a face of  $P$  if and only if  $\mathbf{0} \in \text{relint}(\text{conv}(\{\mathbf{b}_k : k \notin J\}))$ . Let  $\dim(P) = d - 1$ .

If  $J \subsetneq [n]$  is not a face of  $P$ , then by Lemma 5.4,

$$\text{aff}(\{\mathbf{v}_k : k \in J\}) \cap \text{conv}(\{\mathbf{v}_k : k \notin J\}) \neq \emptyset.$$

Let  $\mathbf{z}$  be in this intersection. Then  $\mathbf{z} = \sum_{k \in J} p_k \mathbf{v}_k = \sum_{k \notin J} q_k \mathbf{v}_k$  with

$$\sum_{k \in J} p_k = 1, \quad \sum_{k \notin J} q_k = 1, \quad \text{and } q_k \geq 0 \text{ for all } k \notin J$$

or, equivalently,

$$\sum_{k \in [n]} r_k \mathbf{v}_k = \mathbf{0}, \quad \sum_{k \in [n]} r_k = 0 \text{ and } \sum_{k \notin J} r_k = 1, \quad r_k \geq 0 \text{ for all } k \notin J$$

by taking  $r_k = q_k$  when  $k \notin J$  and  $r_k = -p_k$  when  $k \in J$ .

The first two conditions imply that  $\mathbf{r} = (r_1, \dots, r_n)$  lies in  $\ker_{\mathbb{R}}(A)$  where

$$A := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}.$$

Let  $B$  be the matrix from which the Gale transform  $\mathcal{B}$  was taken. Since the columns of  $B$  form a basis for  $\ker_{\mathbb{R}}(A)$ , there exists  $\mathbf{t} \in \mathbb{R}^{n-d}$  such that

$$\mathbf{r} = B\mathbf{t} \text{ or, equivalently, } r_k = \mathbf{b}_k \cdot \mathbf{t} \text{ for all } k = 1, \dots, n.$$

Since  $r_k \geq 0$  for all  $k \notin J$ , we get that  $r_k = \mathbf{b}_k \cdot \mathbf{t} \geq 0$  for all  $k \notin J$ , which means that all the  $\mathbf{b}_k$ 's with  $k \notin J$  lie in the halfspace defined by  $\mathbf{t} \cdot \mathbf{x} \geq 0$  in  $\mathbb{R}^{n-d}$ . Also since  $\sum_{k \notin J} r_k = 1$ , it cannot be that  $r_k = \mathbf{b}_k \cdot \mathbf{t} = 0$  for all  $k \notin J$  or, in other words, not all the  $\mathbf{b}_k$ 's with  $k \notin J$  lie in the hyperplane defined by  $\mathbf{t} \cdot \mathbf{x} = 0$ . Thus  $\mathbf{0}$  is not in the relative interior of  $\text{conv}(\{\mathbf{b}_k : k \notin J\})$ . Reversing all the arguments, you get the other direction of the theorem.  $\square$

**Example 5.7.** Let's use Theorem 5.6 to read off the face lattice of the triangular prism from the Gale diagram in Figure 1. First, note that for each  $i = 1, \dots, 6$ ,  $\mathbf{0} \in \text{relint}(\text{conv}(\mathbf{b}_k : k \neq i))$ . This implies that all the singletons  $1, 2, 3, 4, 5, 6$  are faces of  $P$ , as indeed they are. Now let's find the edges of  $P$ . These will be all pairs

$ij$  such that  $\mathbf{0} \in \text{relint}(\text{conv}(\{\mathbf{b}_k : k \neq i, j\}))$ . For instance 14 is an edge of  $P$  since  $\mathbf{0} \in \text{relint}(\text{conv}(\{\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_5, \mathbf{b}_6\}))$ . However, 16 is not an edge of  $P$  since  $\mathbf{0} \notin \text{relint}(\text{conv}(\{\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5\}))$ . Can you find all the other edges? The face 123 is witnessed by the fact that  $\mathbf{0} \in \text{relint}(\text{conv}(\{\mathbf{b}_4, \mathbf{b}_5, \mathbf{b}_6\}))$ , but 245 is not a face since  $\mathbf{0} \notin \text{relint}(\text{conv}(\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_6\}))$ .

**Exercise 5.8.** Compute the face lattice of the cyclic polytope in  $\mathbb{R}^4$  with seven vertices. The Gale transform consists of the columns of the matrix

$$\begin{pmatrix} -1 & 5 & -10 & 10 & -5 & 1 & 0 \\ -5 & 24 & -45 & 40 & -15 & 0 & 1 \end{pmatrix}.$$

(**Hint:** For a simplicial polytope, it suffices to know the facets to write down the whole face lattice.)

Theorem 5.6 can be used to read off the face lattice of any polytope. But it is most useful when the Gale diagram is in a low-dimensional space such as  $\mathbb{R}$  or  $\mathbb{R}^2$ . Three-dimensional Gale diagrams are already quite challenging. However, there is a nice trick to reduce the dimension of the Gale diagram by one. These Gale diagrams are known as *affine Gale diagrams*. See [Zie95] for a formal definition. We give the idea below.

We can think of a Gale diagram in  $\mathbb{R}^{n-d}$  as  $n$  vectors that poke out through a  $(n-d-1)$ -sphere. If we look at this sphere from outside, we only see one hemisphere, which we will call the northern hemisphere. We can mark all the vectors that poke out through this hemisphere with a dot and label them as before. The rest of the vectors poke out through the southern hemisphere and we will mark their antipodal vectors on the northern hemisphere with an open circle and change labels to the old labels with bars on top. You should always choose the equator so that no vector pokes out through the equator. This can always be done since there are only finitely many vectors in the Gale diagram. Let's first try this on the Gale diagram from Figure 1.

We first put a circle (1-sphere) around the Gale diagram with the dotted line chosen to be the equator. See Figure 4. Let's declare the right hemisphere to be the northern hemisphere. The Gale vectors 1, 6, 2 intersect this hemisphere. We mark those points with black

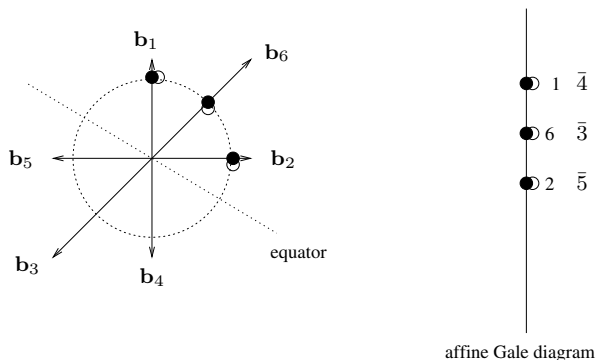


Figure 4. The affine Gale diagram of the triangular prism.

dots. The antipodals of the other vectors also intersect the northern hemisphere at the same points. We mark those intersections with open circles and label them  $\bar{4}, \bar{3}, \bar{5}$ . On the right we see the affine Gale diagram, which lives in  $\mathbb{R}$ . Can we read off the face lattice from this affine Gale diagram? To do this, we need to say what condition on a collection of black and white dots is equivalent to the origin being in the relative interior of the Gale vectors with the same indices. For instance to check whether 1346 is a face of  $P$ , we remove the dots with labels 1,  $\bar{3}$ ,  $\bar{4}$  and 6. This leaves the black dot 2 and the white dot  $\bar{5}$  which are at the same position. This means that  $\mathbf{b}_2$  and  $\mathbf{b}_5$  are opposite to each other and  $\mathbf{0}$  is in the relative interior of their convex hull. Thus 1346 is a face.

**Exercise 5.9.** What conditions on a collection of black and white dots in the affine Gale diagram guarantees that the origin is in the relative interior of the corresponding Gale vectors?

**Exercise 5.10.** Compute the face lattice of the cyclic polytope in  $\mathbb{R}^3$  with seven vertices.

In this case,

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 9 & 16 & 25 & 36 & 49 \\ 1 & 8 & 27 & 64 & 125 & 216 & 343 \end{pmatrix}.$$



Using a computer package that does linear algebra, we compute a basis for  $\ker_{\mathbb{R}}(A)$  to get

$$B^t = \begin{pmatrix} 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 4 & -15 & 20 & -10 & 0 & 1 & 0 \\ 10 & -36 & 45 & -20 & 0 & 0 & 1 \end{pmatrix}.$$

Let's try to draw the affine Gale diagram for this example. We can start by positioning the last three vectors at the corners of an equilateral triangle that will be in the center of the hemisphere we can see. In our case then, we are looking at the sphere along the vector  $(1, 1, 1)$  toward the origin. Can you finish and write down the face lattice? (**Hint:** Read the rest of this page for a methodical procedure.)

As the above exercise shows, it is hard to draw affine Gale diagrams precisely, with the description we have of it so far. We need a more methodical procedure for drawing them, which we now describe.

Let  $\mathcal{B} \subset \mathbb{R}^{n-d}$  be the Gale transform. Choose a vector  $\mathbf{y} \in \mathbb{R}^{n-d}$  such that  $\mathbf{y} \cdot \mathbf{b} \neq 0$  for any  $\mathbf{b} \in \mathcal{B}$ . We now compute  $\mathbf{b}' := \frac{\mathbf{b}}{\mathbf{b} \cdot \mathbf{y}}$  for each  $\mathbf{b} \in \mathcal{B}$ . Then the points  $\mathbf{b}'$  lie on the hyperplane  $H := \{\mathbf{x} \in \mathbb{R}^{n-d} : \mathbf{y} \cdot \mathbf{x} = 1\}$ . If  $\mathbf{b}_i \cdot \mathbf{y} > 0$ , then label  $\mathbf{b}'$  with  $i$  and mark it with a black dot. If  $\mathbf{b}_i \cdot \mathbf{y} < 0$ , then label  $\mathbf{b}'$  with  $\bar{i}$  and mark it with a white dot. Since  $H$  is isomorphic to  $\mathbb{R}^{n-d-1}$ , we simply have to find an explicit isomorphism that will help us draw our new points on  $H \subset \mathbb{R}^{n-d}$  in  $\mathbb{R}^{n-d-1}$ . Projection of the points onto the first  $n-d-1$  coordinates turns out to be such an isomorphism in the examples you will see in these chapters.

**Exercise 5.11.** Compute the face lattice of the four-dimensional cross-polytope  $C^\Delta(4)$  by drawing its affine Gale diagram.

**Exercise 5.12.** Now replace the vertex  $\mathbf{e}_1 \in \mathbb{R}^4$  in  $C^\Delta(4)$  with  $\alpha \cdot \mathbf{e}_1$ . For different values of  $\alpha \in \mathbb{R}$ , how will this new convex polytope change? How is this change reflected in the affine Gale diagram?

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## Chapter 6

# Bizarre Polytopes

In this chapter we will see that Gale diagrams are powerful tools for studying polytopes beyond their ability to encode the faces of a polytope. Let us first investigate some properties of Gale diagrams. The most fundamental question you can ask is if any vector configuration can be the Gale diagram of some polytope. The material in this chapter is taken from [Zie95, Chapter 6].

As in Chapter 5, let  $V := \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset \mathbb{R}^{d-1}$  and let

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix} \in \mathbb{R}^{d \times n}.$$

Assume that  $\text{rank}(A) = d$ , and choose a matrix  $B \in \mathbb{R}^{n \times (n-d)}$  whose columns form a basis of  $\ker_{\mathbb{R}}(A)$ . Recall that the Gale transform  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\} \subset \mathbb{R}^{n-d}$  consists of the rows of  $B$ .

- Definition 6.1.** (1) A vector configuration  $\{\mathbf{w}_1, \dots, \mathbf{w}_p\} \subset \mathbb{R}^q$  is said to be **acyclic** if there exists a vector  $\alpha \in \mathbb{R}^q$  such that  $\alpha \cdot \mathbf{w}_i > 0$  for all  $i = 1, \dots, p$ . Geometrically this means that all the vectors  $\mathbf{w}_i$  lie in the interior of a halfspace defined by a hyperplane in  $\mathbb{R}^q$  containing the origin.
- (2) A vector configuration  $\{\mathbf{w}_1, \dots, \mathbf{w}_p\} \subset \mathbb{R}^q$  is said to be **totally cyclic** if there exists a vector  $\beta > \mathbf{0}$  in  $\mathbb{R}^p$  such that  $\beta_1 \mathbf{w}_1 + \dots + \beta_p \mathbf{w}_p = \mathbf{0}$ . Geometrically this means that the

$\mathbf{w}_i$  are arranged all the way around the origin and are not entirely on one side of any hyperplane through the origin.

**Lemma 6.2.** *The columns of  $A$  form an acyclic configuration in  $\mathbb{R}^d$  since they all lie in the open halfspace  $\{\mathbf{x} \in \mathbb{R}^d : x_1 > 0\}$ , while the Gale transform  $\mathcal{B}$  is a totally cyclic configuration in  $\mathbb{R}^{n-d}$  since  $\mathbf{b}_1 + \dots + \mathbf{b}_n = \mathbf{0}$ . (Note that the first row of  $A$ , which is a row of ones, dots to zero with the rows of  $B$ .)*

Suppose we start with a totally cyclic vector configuration  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\} \subset \mathbb{R}^{n-d}$  and a vector  $\beta > \mathbf{0}$  such that  $\sum \beta_i \mathbf{b}_i = \mathbf{0}$ . By rescaling the elements of  $\mathcal{B}$ , we may assume that  $\beta = (1, 1, \dots, 1)$ . If  $B \in \mathbb{R}^{n \times (n-d)}$  is the matrix whose rows are the elements of  $\mathcal{B}$ , then we can also assume that  $\text{rank}(B) = n - d$ . This means that  $\ker_{\mathbb{R}}(B^t) = \{\mathbf{x} \in \mathbb{R}^n : B^t \mathbf{x} = \mathbf{0}\}$  is a linear subspace of rank  $n - (n - d) = d$ . Let  $A \in \mathbb{R}^{d \times n}$  be a matrix whose rows form a basis for  $\ker_{\mathbb{R}}(B^t)$ . The columns of  $A$  form the vector configuration  $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{R}^d$ . Then  $\mathcal{A}$  is said to be a **Gale dual** of  $\mathcal{B}$  and  $\mathcal{B}$  a Gale dual of  $\mathcal{A}$ . By our assumption that  $\beta = (1, \dots, 1)$ , we may assume that the first row of  $A$  is a row of all ones or, in other words,  $\mathbf{a}_i = \begin{pmatrix} 1 \\ \mathbf{v}_i \end{pmatrix}$  for all  $i = 1, \dots, n$ . Now the question is, what conditions on  $\mathcal{B}$  will ensure that  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is the vertex set of a  $(d-1)$ -polytope? To state our answer formally, we introduce the notion of *circuits* and *co-circuits* of vector configurations.

**Definition 6.3.** (1) The **sign** of a vector  $\mathbf{u} \in \mathbb{R}^n$  is the vector  $\text{sign}(\mathbf{u}) \in \{+, 0, -\}^n$  defined as

$$\text{sign}(\mathbf{u})_i := \begin{cases} + & \text{if } u_i > 0 \\ - & \text{if } u_i < 0 \\ 0 & \text{if } u_i = 0. \end{cases}$$

(2) The **support** of a vector  $\mathbf{u} \in \mathbb{R}^n$  is the set

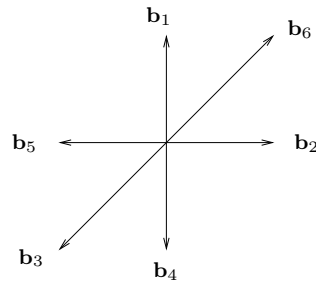
$$\text{supp}(\mathbf{u}) := \{i : u_i \neq 0\} \subseteq [n].$$

Note that the supports of a collection of vectors can be partially ordered by set inclusion.

**Definition 6.4.** Let  $\mathcal{W} = \{\mathbf{w}_1, \dots, \mathbf{w}_p\} \subset \mathbb{R}^q$  be a vector configuration.

- (1) A **circuit** of  $\mathcal{W}$  is any non-zero vector  $\mathbf{u} \in \mathbb{R}^p$  of minimal support such that  $\mathbf{w}_1 u_1 + \dots + \mathbf{w}_p u_p = \mathbf{0}$ . The vector  $\text{sign}(\mathbf{u})$  is called a **signed circuit** of  $\mathcal{W}$ .
- (2) A **co-circuit** of  $\mathcal{W}$  is any non-zero vector of minimal support of the form  $(\mathbf{v} \cdot \mathbf{w}_1, \dots, \mathbf{v} \cdot \mathbf{w}_n)$  where  $\mathbf{v} \in \mathbb{R}^q$ . The sign vector of a co-circuit is called a **signed co-circuit**.

**Example 6.5.** Consider the vector configuration shown in Figure 1 that is the Gale transform of the triangular prism from Chapter 5. If we take  $\mathbf{v} = (1, 0)$  in Definition 6.4(2), then we get the co-circuit  $(0, 1, -1, 0, -1, 1)$  and the signed co-circuit  $(0, +, -, 0, -, +)$ . On the other hand, the vector  $(1, 0, 0, 1, 0, 0)$  is a circuit of the configuration, and hence  $(+, 0, 0, +, 0, 0)$  is a signed circuit of the configuration.



**Figure 1.** Gale diagram of the vertices of the triangular prism from Chapter 5.

The signed circuits (or, equivalently, signed co-circuits) of a vector configuration completely determine the combinatorics of the configuration. In fact there is a very rich theory of circuits and co-circuits that we will not get into here. It is also a fact that if  $\mathcal{A}$  and  $\mathcal{B}$  are Gale duals, then the circuits of  $\mathcal{A}$  are exactly the co-circuits of  $\mathcal{B}$  and vice versa. For instance, in our triangular prism example,  $\mathcal{A}$  can be taken to be the columns of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The co-circuit  $(0, 1, -1, 0, -1, 1)$  of  $\mathcal{B}$  does indeed form a circuit of  $\mathcal{A}$ : Check first that this vector lies in the kernel of  $A$ . To see that it is a circuit, i.e., a dependency on the columns of  $A$  of minimal support, you have to check that all subsets of columns 2, 3, 5, 6 are in fact linearly independent.

Circuits and co-circuits come in symmetric pairs: the negative of a circuit is again a circuit and similarly for co-circuits. It suffices to record one member of each pair.

**Theorem 6.6.** ([Zie95, Theorem 6.19]) *Let  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\} \subset \mathbb{R}^{n-d}$  be a totally cyclic vector configuration with  $\sum \mathbf{b}_i = \mathbf{0}$  and the matrix  $B$  having rank  $n-d$  as before. Then  $\mathcal{B}$  is a Gale transform of a  $(d-1)$ -polytope with  $n$  vertices if and only if every co-circuit of  $\mathcal{B}$  has at least two positive coordinates.*

**Proof.** Recall the matrix  $A$  constructed from  $\mathcal{B}$  as before. We have to show that  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is the vertex set of the  $(d-1)$ -polytope  $P = \text{conv}(\{\mathbf{v}_1, \dots, \mathbf{v}_n\})$  if and only if every co-circuit of  $\mathcal{B}$  has at least two positive coordinates. Since  $\mathcal{B}$  is totally cyclic, every co-circuit of  $\mathcal{B}$  has at least one positive entry and one negative entry. Some co-circuit of  $\mathcal{B}$  has exactly one positive entry — say in position  $j$  — if and only if  $\mathbf{0} \notin \text{relint}(\text{conv}(\mathbf{b}_i : i \neq j))$  which, by Theorem 5.6, is if and only if  $\mathbf{v}_j$  is not a vertex of  $P$ . This proves the theorem.  $\square$

**Remark 6.7.** If every  $\mathbf{v}_i$  is a vertex of  $\text{conv}(\{\mathbf{v}_1, \dots, \mathbf{v}_n\})$ , then we say that the  $\mathbf{v}_i$  are in **convex position**. Theorem 6.6 provides conditions on a vector configuration  $\mathcal{B}$  with the stated assumptions that precisely guarantee when  $\mathcal{B}$  is the Gale dual of a configuration  $\mathcal{A}$  whose columns are all in convex position.

We can also characterize affine Gale diagrams by reinterpreting Theorem 6.6.

**Corollary 6.8.** ([Zie95, Corollary 6.20]) *A point configuration  $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\} \subset \mathbb{R}^{n-d-1}$ , each of them declared to be either black or white, that affinely spans  $\mathbb{R}^{n-d-1}$ , is the affine Gale diagram of a  $(d-1)$ -polytope with  $n$  vertices if and only if the following condition is satisfied: for every oriented hyperplane  $H$  in  $\mathbb{R}^{n-d-1}$  spanned by some points of  $\mathcal{C}$ , the number of black dots on the positive side of  $H$*

plus the number of white dots on the negative side of  $H$  is at least two.

**Exercise 6.9.** Check that Corollary 6.8 is a straight translation of Theorem 6.6 to affine Gale diagrams.

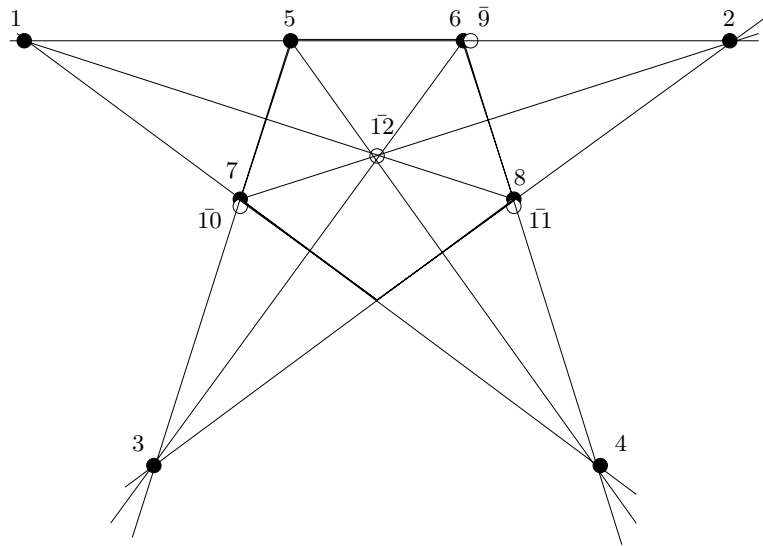
**Exercise 6.10.** Check that the condition of Corollary 6.8 is true for the affine Gale diagram of the triangular prism from Chapter 5.

We are now ready to get to the fun. We could try to use Gale diagrams to classify  $(d - 1)$ -polytopes with  $n$  vertices. Any  $(d - 1)$ -polytope with  $d$  vertices is a simplex. The Gale diagram in this case is in zero-dimensional space  $\mathbb{R}^0$  and all the  $\mathbf{b}_i = 0 \in \mathbb{R}^0$ . If  $P$  is a  $(d - 1)$ -polytope with  $d + 1$  vertices, then its Gale diagram is a totally cyclic vector configuration in  $\mathbb{R}$  and its affine Gale diagram is a cloud of black and white points in  $\mathbb{R}^0$ . It is known that there are  $\lfloor (d - 1)^2/4 \rfloor$  combinatorial types of  $(d - 1)$ -polytopes with  $d + 1$  vertices. Of these,  $\lfloor (d - 1)/2 \rfloor$  are simplicial polytopes and the others are multiple pyramids over simplicial polytopes of this type. This is a non-obvious but classical result. Further results are known. See [Grü03, Chapter 6] for details. Our goal in the rest of the chapter will be to show that Gale diagrams exhibit the existence of some really bizarre polytopes.

**Theorem 6.11.** ([Grü03, page 94]) *There exists a non-rational eight-dimensional polytope with twelve vertices.*

**Proof.** Using Corollary 6.8, check that the point configuration shown in Figure 2 is the affine Gale diagram of an 8-polytope  $P$  with twelve vertices. It turns out that this point configuration cannot be realized by rational coordinates without violating the prescribed combinatorics. By “prescribed combinatorics” we mean that the same points should be collinear, or on a plane, etc. as in Figure 2. First note that fixing the combinatorics implies that there will always be a pentagon in the middle of the configuration. It is harder to see that this pentagon has to be regular (do you see it?). Furthermore, a regular pentagon cannot be embedded in the plane with rational coordinates as its coordinates will involve  $\sqrt{5}$ , which is not rational.

Let  $Q$  be any polytope that is combinatorially equivalent to  $P$ . Then the affine Gale diagram of  $Q$  also has the same combinatorics, i.e., same collinearities, circuits, coincidences, etc. Thus  $Q$  cannot be realized with rational coordinates either. In particular, neither  $P$  nor any polytope combinatorially equivalent to it can be realized with rational coordinates.



**Figure 2.** Affine Gale diagram for Theorem 6.11.

□

The above example is due to Perles. No non-rational polytope with less than twelve vertices is known. However, Richter-Gebert has constructed 4-polytopes (with about thirty vertices) which are non-rational. This stands in contrast to the fact that all polytopes of dimension at most three can be realized with rational coordinates. Also, all  $(d-1)$ -polytopes with at most  $d+2$  vertices can be realized with rational coordinates. Can you see how to construct infinitely many polytopes of dimension  $d \geq 8$  and at least  $d+4$  vertices that do not have rational realizations, beginning with the above example?

We now turn to a different feature of polytopes that can be uncovered via their Gale diagrams. It is known that for all polytopes of dimension  $d \leq 3$  or with at most  $d + 3$  vertices, one can prescribe the shape of a facet. This means that if a particular facet is known to be an octahedron, say, then we can start with any embedding of an octahedron as this facet and then complete the construction of the polytope according to the combinatorics prescribed. This contrasts the following theorem whose proof is from [Zie95, Theorem 6.22].

**Theorem 6.12.** ([Stu88]) *There is a 4-polytope  $P$  with seven facets for which the shape of a facet cannot be prescribed.*

**Proof.** Let  $P^\Delta$  be the bi-pyramid over a square pyramid. Let the  $A$  matrix for this be

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

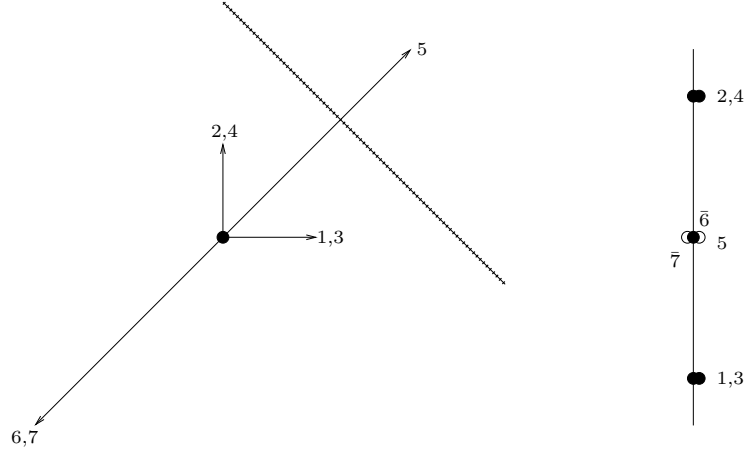
To see that the convex hull of the columns of  $A$  is a bi-pyramid over a square pyramid, first note that the convex hull of the first four columns of  $A$  is a square, and the convex hull of the first five columns of  $A$  is a square pyramid. Next note that the average of the last two columns of  $A$  is  $(1, 0, 0, 1/2, 0)$ , which is the midpoint of the line segment perpendicular to the base of the pyramid, dropped from the apex of the pyramid. The columns of

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 2 & -2 & -2 \\ 0 & 1 & 0 & 1 & 2 & -2 & -2 \end{pmatrix}$$

form the Gale transform  $\mathcal{B}$  of  $P^\Delta$ . The Gale diagram is shown in Figure 3.

Now we examine an operation on polytopes that we have not seen so far. The **vertex figure** of a polytope  $Q$  at a vertex  $v$  is the intersection of  $Q$  with a hyperplane  $H$  that “chops off” vertex  $v$  very near vertex  $v$  — i.e.,  $v$  lies on one side of  $H$  and all other vertices of  $Q$  lie on the other side of  $H$ . The resulting polytope is denoted as  $Q/v$ .





**Figure 3.** Gale and affine Gale diagram for Theorem 6.12.

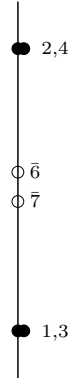
Let's consider the vertex figure of our bi-pyramid  $P^\Delta$  at the vertex 5. The Gale diagram of  $P^\Delta/5$  is obtained from the Gale diagram of  $P^\Delta$  by deleting the point 5 from the diagram. (This is Exercise 6.13.) The resulting Gale diagram is that of a regular octahedron, for instance the one with

$$A' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Check that for  $P^\Delta$ ,  $5\bar{6}$  and  $5\bar{7}$  are co-facets. However, this requires that the points  $\bar{6}$  and  $\bar{7}$  coincide in the affine Gale diagram of  $P^\Delta/5$  or, equivalently, that the Gale vectors 6 and 7 are in the linear span of the Gale vector 5 in the opposite direction from 5. Therefore, if we start with a non-regular octahedron such as the following one with

$$A' = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1/6 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

which has the affine Gale diagram shown in Figure 4, then it is not



**Figure 4.** Affine Gale diagram of a non-regular octahedron.

the vertex figure of a 4-polytope that is combinatorially isomorphic to  $P^\Delta$ .

Recall that the face lattices of  $P$  and  $P^\Delta$  are anti-isomorphic, which means that the vertex 5 of  $P^\Delta$  corresponds to a facet of  $P$ . This facet has the same combinatorics as the vertex figure  $P^\Delta/5$ . Thus by showing that a vertex figure of  $P^\Delta$  cannot be prescribed, we have shown that a facet of  $P$  cannot be prescribed.  $\square$

**Exercise 6.13.** Argue that the Gale diagram of the vertex figure  $P^\Delta/5$  is obtained from the Gale diagram of  $P^\Delta$  by deleting the point 5 from the diagram. This result is true in general.