Introduction

In 1773 Joseph-Louis Lagrange [1736-1813] made the following observation: Assume a homogeneous quadratic polynomial

$$p(x, y) = ax^2 + 2bxy + cy^2$$

in two variables x, y with complex coefficients a, b, c. We replace the variables x, y by

$$x' = x + y \quad \text{and} \quad y' = y,$$

and obtain another homogeneous quadratic polynomial

$$p'(x, y) = p(x + y, y) = a'x^2 + 2b'xy + c'y^2,$$

where

$$a' = a, \ b' = b + a, \ \text{and} \ c' = c + 2b + a.$$

The coefficients of both polynomials satisfy the following equation:

$$ac - b^2 = a'c' - b'^2.$$

This is not an accident. Indeed, let T_λ be an arbitrary change of variables of the form

$$\mathsf{T}_\lambda(x) = x + \lambda y \quad \text{and } \mathsf{T}_\lambda(y) = y,$$

where $\lambda \in \mathbb{C}$ is some complex number. Then the so-called **determinant** of p, namely $\det(p) = ac - b^2$, remains unchanged under the T_λ-action:

$$\det(p) = \det(\mathsf{T}_\lambda(p)),$$

where $\mathsf{T}_\lambda(p)(x,y) = p(x+\lambda y, y)$. We rephrase this result:

A polynomial $p(x,y) = ax^2 + 2bxy + cy^2$ is determined by the three coefficients a, b, c. Thus we interpret the space of homogeneous quadratic polynomials in two variables as a 3-dimensional complex vector space by assigning to a polynomial $p(x,y) = ax^2 + 2bxy + cy^2$ the column vector

$$(a, b, c)^{\mathrm{t}} \in \mathbb{C}^3.$$

The change of variables T_λ becomes a linear map

$$\mathsf{T}_\lambda : \mathbb{C}^3 \longrightarrow \mathbb{C}^3, (a,b,c)^{\mathrm{t}} \mapsto (a, \lambda a + b, \lambda^2 a + 2\lambda b + c)^{\mathrm{t}}$$

given by the matrix

$$\mathsf{T}_\lambda = \begin{bmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ \lambda^2 & 2\lambda & 1 \end{bmatrix}.$$

The family of maps T_λ, where $\lambda \in \mathbb{C}$, forms a group G under matrix multiplication:[1] Since

$$\begin{aligned} \mathsf{T}_\lambda \mathsf{T}_\mu &= \begin{bmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ \lambda^2 & 2\lambda & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \mu & 1 & 0 \\ \mu^2 & 2\mu & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ \lambda+\mu & 1 & 0 \\ (\lambda+\mu)^2 & 2(\lambda+\mu) & 1 \end{bmatrix} \\ &= \mathsf{T}_{\lambda+\mu} \end{aligned}$$

the family is closed under matrix multiplication. Setting $\lambda = 0$ gives us the identity, and the inverse of T_λ is just $\mathsf{T}_{-\lambda}$. Hence, we have a subgroup $G \subset \mathsf{GL}(3,\mathbb{C})$ of the general linear group $\mathsf{GL}(3,\mathbb{C})$ of invertible complex 3×3-matrices acting via matrix multiplication on the 3-dimensional vector space of all homogeneous quadratic polynomials in two variables. Then we consider polynomials in the coefficients, and we end up with the result that the determinant (which is a polynomial of degree 2 in the coefficients) is *invariant* under the action of the group G.

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[1] Indeed, our group G is isomorphic to the additive group of complex numbers. The isomorphism is given by

$$G \longrightarrow \mathbb{C}, \mathsf{T}_\lambda \mapsto \lambda.$$

Next we might ask ourselves whether there are other polynomials in the coefficients a, b, c that are invariant under all linear transformations T_λ. Or, we might ask what properties does the set of all invariant polynomials have? Or, how do we find it? These are some of the classical questions that invariant theorists have studied during the last two centuries and that we will treat in this book.

Invariant theory is a blend of many different areas of mathematics. We need to understand the groups involved and their geometry. We need notions and methods from combinatorics and commutative algebra in order to make questions like *Have we found all invariants?* precise. In the example above we find that since the determinant of the polynomial p is invariant, so is its square or any other power. This observation leads to the statement that the set of invariant polynomials forms an algebra over the ground field; see Proposition 3.14. Note that the linear polynomial $\hat{p} = a$ is also invariant. Now, again *Is that all?* There are several ways to interpret this question. One would be *Does the ring of invariants form a finitely generated algebra?* Another would be *How many invariants are there in any given degree?* It is relatively easy to see that the homogeneous invariants in a fixed degree form a finite-dimensional vector space over the ground field; see Exercise 3 in Chapter 3. It is much harder to answer the former question. We will prove in Chapter 6, using the First Fundamental Theorem of Invariant Theory for Σ_n, that any ring of invariants of a finite group is a finitely generated algebra. We will reprove this result in Chapter 10 using algebraic methods. Apart from geometry, combinatorics, and algebra, invariant theory draws from fields like representation theory, group cohomology, and algebraic topology. These areas also profit from invariant theoretic results. In addition, invariant theory has many applications to problems in other mathematical areas and areas outside mathematics like physics and engineering. We will illustrate this with several "interchapters" on applications to physics, engineering, numerical analysis, combinatorics, coding theory, and graph theory. Some of these applications, as well as many examples and results throughout the text, are based on work by students (graduate as well as undergraduate) showing that invariant theory is a vibrant ever-expanding area of mathematics.

This text is written for advanced undergraduate and first-year graduate students who wish to have a concise introduction to the theory of polynomial invariants of finite groups. We will deal only with ground fields of characteristic zero. However, much of what we present remains valid in finite characteristic.

We assume knowledge of abstract and linear algebra from standard undergraduate courses. However, Part 1 of the book (Chapters 1 and 2) is a collection of fundamental notions of groups and rings that we will use throughout the text. Most of the material there should be known; except for perhaps group representations (see Section 1.3) and some material on graded rings in Chapter 2. We want to emphasize that apart from the group algebra (appearing in Chapter 8 all our rings will be graded, commutative rings with unity. In Part 2 we introduce the basic objects of study. We define group actions, in particular linear group actions on polynomial rings and their invariants, in Chapter 3. Chapter 4 deals with permutation representations and their invariants. In particular we will find a complete description of all invariant polynomials. In Part 3 of the book we introduce various methods to construct invariant polynomials. Then we show that every ring of polynomial invariants is finitely generated as an algebra over the ground field. Along the way to that result we will prove the First Fundamental Theorem of Invariant Theory for the symmetric group. Furthermore, we present important families of examples like pseudoreflection groups and vector invariants. Part 4 lays the algebraic foundations of the field: the classical Noether theorems. In order to do so we will need to introduce some advanced abstract algebra, in particular the notion of modules and their basic properties. Indeed, historically much of this material has been developed in order to answer invariant theoretic questions. The goal of Part 5 is to prove the Shephard-Todd-Chevalley Theorem, which characterizes the invariants of pseudoreflection groups. For that we introduce some advanced counting methods in Chapters 11 and 12 that are interesting and useful in their own right (see the application to combinatorics). We close the book with an appendix on rational invariants, i.e., on classical Galois theory.

I have added many examples and exercises. In particular, I have added biographical questions like Exercise 21 in Chapter 1:

Find out to whom the term Pauli Matrices refers!

You can find answers in your library in books on the history of mathematics. Or you could go online: At the URL

http://www-groups.dcs.st-andrews.ac.uk/%7Ehistory/

you will find a data base of biographies of famous mathematicians and at

http://darkwing.uoregon.edu/~wmnmath/

you will find a data base of biographies of female mathematicians.[2]

A one-semester course for students with basic algebraic skills could consist of the first three parts: At the end the students would be able to identify the basic objects of studies and their basic properties. They would know how to phrase the basic invariant theoretic questions. Furthermore, they would be able to prove the First Fundamental Theorem of Invariant Theory for the Symmetric Group; they could prove that rings of invariants are finitely generated algebras; they could prove Noether's bound; they could construct invariants; and they will have encountered the most important families of invariants. In addition they will have seen applications to "real life" problems. Students who have a bit more background in abstract and commutative algebra might be able to finish Part 4 too. Otherwise, Parts 4 and 5 make a fine second term, or could serve as supplemental reading for a first course in commutative algebra.

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[2]The symmetric difference of the two data bases does not only contain biographies of male mathematicians.

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Finally, I will put a list of errata on my website
http://www.math.ttu.edu/~mneusel/titelseite.html

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