Preface

This book is a result of the MASS course "Real and p-adic analysis" that I gave in the MASS program in the fall of 2000. The notes were first published in *MASS Selecta* [**12**], and a Russian translation of a revised version appeared in [**7**]. The present text is further revised and expanded.

The choice of the topic was motivated by the internal beauty of the subject of p-adic analysis, an unusual one in the undergraduate curriculum, and abundant opportunities to compare it with its much more familiar real counterpart.

There are several pedagogical advantages of this approach. Both real and p-adic numbers are obtained from the rationals by a procedure called completion, which can be applied to any metric space, by using different distances on the rationals: the usual Euclidean distance for the reals and a new p-adic distance for each prime p, for the p-adics. The p-adic distance satisfies the "strong triangle inequality" that causes surprising properties of p-adic numbers and leads to interesting deviations from the classical real analysis much like how the renunciation of the fifth postulate of Euclid's *Elements*, the axiom of parallels, leads to non-Euclidean geometry. Similarities, on the other hand, arise when the fact does not depend on the "strong triangle inequality", and in these cases the same proof works in the real and

p-adic cases. Analysis of the differences and similarities helps the students to better understand the proofs in both contexts.

The material covered in this book appears in several classical and recent sources [**2, 4, 8, 9, 10, 11, 14, 16**] but either remains on an elementary level with more emphasis on number theory than on analysis or quickly leads to matters way too sophisticated for the undergraduate students. My only contribution was to choose the appropriate material and to present it in the proper context, simplifying the proofs in some cases.

I included several topics from real analysis and elementary topology which are not usually covered in undergraduate courses (totally disconnected spaces and Cantor sets, points of discontinuity of maps and the Baire Category Theorem, surjectiveness of isometries of compact metric spaces). They enhanced the students' understanding of real analysis and intertwined the real and p-adic contexts of the course. In [**15**], a standard reference for real analysis, many of these topics appear only in exercises. Basic algebraic notions are discussed only briefly, and the reader is referred to any standard text in abstract algebra, e.g. [**6**].

The course entailed a large number of exercises (with emphasis on proofs) which appear in this book. Particularly, some parts of proofs in the main body of the text are relegated to exercises. While it may create some difficulties for uninterrupted reading, it helps students' deeper understanding of the subject and makes this book particularly appropriate for self-study. A few harder exercises are marked with a $*$. Answers, hints, and solutions for most of the exercises are included at the end of the book.

Besides solving the homework exercises, the students in the MASS program were asked to give presentations on additional topics. These presentations, some of which were quite advanced, included the following: "The Signum function"; "Euclidean models of the p-adic integers"; "Interpolation series and the p-adic Weierstrass theorem"; "Finite extensions of p-adic numbers and p-adic circles"; "Isometries on the p-adic integers"; "p-adic solenoid"; "The X-adic norm of power series"; "Equiareal triangulations"; "A graphical model of the Peano curve via the Cantor set"; "Revised harmonic series". The first three of these topics are now included in the book.

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