Foreword

The idea for writing this book arose as a result of conversations I had with Shannon Reed when she was a student in Math 4980, the problem solving class, at the University of Missouri during the Fall semester of 2004. This was a capstone course for upper division students designed to bring together techniques and ideas from the standard undergraduate curriculum. Another purpose for the course was to teach students problem solving techniques and to prepare them for the national Putnam examination in December. I was excited to discover over the years that these goals, far from being incompatible, work hand in hand to enhance the students' hands on knowledge of mathematics and to provide them with a glimpse into the world of research and discovery.

The idea of a capstone class is absolutely wonderful and it keeps growing on me each year. Too many undergraduate students form an impression of mathematics as being compartmentalized into specific subject areas like analysis, algebra, number theory, topology and others. The idea of unity of mathematics and interactions between fundamental areas are seldom mentioned and almost never taught. The purpose of this book is to attempt to break through this barrier. It is a hopeless task to illustrate even a small sliver of exciting and surprising connections that mathematics is so full of. The author confines himself to the connections that are near and dear to his heart—basic analytic inequalities, probabilistic reasoning, and their connections with geometric combinatorics and number theory.

We begin the book by introducing the reader to the Cauchy-Schwartz and Hölder inequalities. Instead of continuing on to the endless, albeit interesting, world of inequalities, we immediately pursue applications to geometric problems. We hope that the natural appeal and beauty of these connections will help us make the case that, far from being solely an exercise in symbol manipulation, Cauchy-Schwartz and other dry looking estimates reflect fundamental physical realities that can be appreciated on many levels. For example, we show that the Cauchy-Schwartz inequalities can be used to estimate the number of incidences of points and lines, and sizes of projections of discrete point sets. In the course of discussing projections, we quietly sneak in the notion of interpolation, which is so fundamental and unavoidable in research harmonic analysis. When presented for its own sake, this concept can appear dry and specialized. In the context of a concrete problem, however, it is instead at worst a necessary evil needed to resolve the problem at hand. These ideas and their variants occupy the first four chapters of the book.

In chapters 5-8, we move on to the finite field setting, explained in detail without any need for prerequisites, thus simplifying the calculations and eliminating the need for much formalism. This allows us to present much of what is known on the Besicovitch-Kakeya conjecture, one of the most important and central problems of modern harmonic analysis which connects the size of a set with the number of "line segments" of different "slopes" contained within. Chapters 9 and 10 are dedicated to problems and ideas that require basic counting and probabilistic reasoning, which we then connect with some interesting questions in the theory of numbers, thus putting a different perspective on calculations and concepts introduced earlier in the manuscript. Chapters 11 and 12 of the book are dedicated to trigonometric sums, and sums and integrals with applications to problems in geometry and number theory.

We do not aim for the slickest proofs or even the most elegant presentation. The idea is to get the reader to become excited about research mathematics by observing the process in which ideas evolve.

In several chapters we develop the necessary tools in the process of investigation without even alluding to the fact that these are standard results in various areas of mathematics. We aim to get across to the reader that mathematics is not discovered by reproduction and slight modification of techniques found in textbooks, but rather through a painful and often comical process of discovery and rediscovery.

While not all the problems in this book are at the cutting edge of modern mathematics, the techniques are selected precisely for their importance and ubiquity in mathematical research. Connections between different techniques and areas of mathematics are emphasized throughout and constitute one of the most important lessons this book attempts to impart.

The student is expected to work hard while reading this book. This is not bed time reading, nor is it a fantasy novel. You must have a pen and plenty of paper handy, and expect to fill up about ten pages of calculations for every page you read. Mathematics is not a spectator sport, so create in addition to reading and computing. Every time you see a theorem or a calculation, try to formulate a new one. Every time you see a proof, try to find a better one. And most importantly, have fun!

My goal is to make this book interesting and accessible to anyone with the basic knowledge of high school mathematics who is curious about research mathematics. Several chapters require knowledge of calculus of several variables, and this is clearly indicated in the beginning of each of those chapters.

On the other hand, many topics of this book may even be of interest to graduate students in mathematics and professional researchers. While the vast majority of techniques described in this book are well known to professional mathematicians, the perspective and interlacing of topics and ideas may turn out to be unusual and even surprising.

Chapter 1

The Cauchy-Schwarz inequality

In this section we shall follow a procedure often considered nasty, but the one I hope to convince you to appreciate. We shall work backwards, discovering concepts as we go along, instead of stating them ahead of time. No background beyond high school mathematics is required to read this chapter. For further information on the material presented here, the reader is encouraged to take a look at J. Michael Steele's beautiful book, entitled *The Cauchy-Schwarz Master Class* ([**16**]).

A quick perspective before we begin the nuts and bolts of the mathematical discussion. Inequalities are a dime a dozen. The statement that $2 \leq 3$ is a true inequality, but it is meaningless. An interesting inequality is one that comes close to not being true, but does not quite cross that precarious threshold. This is not much different than any area of learning. Saying that the United States has a bigger per capita income than Chad is a true but meaningless statement. On the other hand, the fact that the People's Republic of Congo has land area equal to nearly two thirds of the land area of the United States is much more precise, interesting and surprising. To put it bluntly, to say something interesting, one must walk on the very edge of the cliff of falsehood, yet never fall off. It is time to begin the perilous journey.

Let a and b denote two real numbers. Then

$$(a-b)^2 \geq 0.$$

This statement is so vacuous, you are probably wondering why I am telling you this. Nevertheless, expand the left hand to see that

$$a^2 - 2ab + b^2 \geq 0,$$

which implies that

$$ab \leq \frac{a^2+b^2}{2}. \tag{1.1}$$

Now consider

$$A_N = \sum_{k=1}^{N} a_k = a_1 + \cdots + a_N,$$

$$B_N = \sum_{k=1}^{N} b_k = b_1 + \cdots + b_N,$$

where $a_1, \ldots, a_N$, and $b_1, \ldots, b_N$ are real numbers. Let

$$X_N = \left(\sum_{k=1}^{N} a_k^2\right)^{\frac{1}{2}},$$

$$Y_N = \left(\sum_{k=1}^{N} b_k^2\right)^{\frac{1}{2}}.$$

Our goal is to take advantage of (1.1). Let's take a look at

$$\begin{aligned}\sum_{k=1}^{N} a_k b_k &= X_N Y_N \sum_{k=1}^{N} \frac{a_k}{X_N} \cdot \frac{b_k}{Y_N} \\ &\leq X_N Y_N \sum_{k=1}^{N} \left[\frac{1}{2}\left(\frac{a_k}{X_N}\right)^2 + \frac{1}{2}\left(\frac{b_k}{Y_N}\right)^2\right].\end{aligned} \tag{1.2}$$

Exercise 1.1. Explain why if C is a constant, then

$$\sum_{k=1}^{N} C a_k = C \sum_{k=1}^{N} a_k.$$

Exercise 1.2. Explain why

$$\sum_{k=1}^{N}(a_k + b_k) = \sum_{k=1}^{N} a_k + \sum_{k=1}^{N} b_k.$$

We now use Exercise 1.1 and 1.2 to rewrite the right hand side of (1.2) in the form

$$\begin{aligned} & X_N Y_N \frac{1}{2}\frac{1}{X_N^2}\sum_{k=1} a_k^2 + X_N Y_N \frac{1}{2}\frac{1}{Y_N^2}\sum_{k=1}^{N} b_k^2 \\ & = X_N Y_N \frac{1}{2}\frac{1}{X_N^2} X_N^2 + X_N Y_N \frac{1}{2}\frac{1}{Y_N^2} Y_N^2 \\ & = \frac{1}{2} X_N Y_N + \frac{1}{2} X_N Y_N = X_N Y_N. \end{aligned} \tag{1.3}$$

Using (1.3) and putting everything together, we obtain the Cauchy-Schwarz inequality:

Theorem 1.4. *Let a_k, b_k be real numbers. Then*

$$\sum_{k=1}^{N} a_k b_k \leq \left(\sum_{k=1}^{N} a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{N} b_k^2\right)^{\frac{1}{2}}. \tag{1.5}$$

Exercise 1.3. Prove that that equality in (1.5) occurs if and only if $a_k = b_k$ for all k. Hint: How did this all begin? Surely the equality in the inequality $(a-b)^2 \geq 0$ happens if and only if $a = b$...

We now use a variant of the same procedure to deduce the following generalization of the Cauchy-Schwarz inequality known as Hölder's inequality.

Theorem 1.6. *Let $1 < p < \infty$ and define the "dual" exponent p' by the equation*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Then

$$\sum_{k=1}^{N} a_k b_k \leq \left(\sum_{k=1}^{N} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{N} |b_k|^{p'}\right)^{\frac{1}{p'}}. \tag{1.7}$$

Following the proof of Cauchy-Schwartz above it is not difficult to see that it suffices to prove that

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}. \tag{1.8}$$

Exercise 1.4. Check the details of this and demonstrate line by line that Hölder's inequaity indeed follows from (1.8)!

To prove (1.8) we need to use first year calculus. If you are not familiar with calculus, take (1.8) for granted and move for now. Do not worry...

If you are familiar with basic calculus, set $a^p = e^x$ and $b^{p'} = e^y$. Let $\frac{1}{p} = t$ and observe that $0 \leq t \leq 1$. We are then reduced to showing that for any real valued x, y and $t \in [0,1]$,

$$e^{(1-t)x+ty} \leq (1-t)e^x + te^y.$$

Let

$$F(t) = (1-t)e^x + te^y - e^{(1-t)x+ty}$$

and observe that $F(0) = F(1) = 0$. Also observe that

$$F''(t) = -(x-y)^2 e^{(1-t)x+ty} \leq 0.$$

What do we have? A twice differentiable function F is equal to 0 at $t = 0$ and $t = 1$. Moreover, F is concave up because $F''(t) \leq 0$. We are forced to conclude that F is always non-negative and (1.8) follows.

Exercise 1.5. Prove the Cauchy-Schwartz inequality in the following way. Let $a = (a_1, \dots, a_N)$ and $b = (b_1, \dots, b_N)$. Define

$$< a, b >= a_1 b_1 + \cdots + a_N b_N,$$

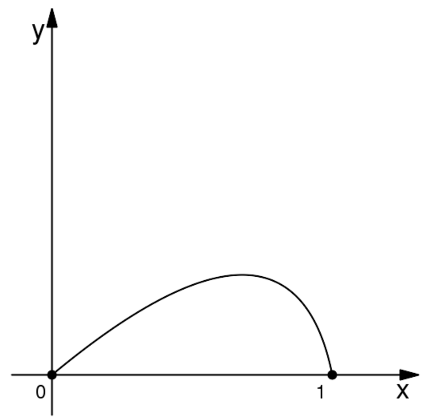
and define

$$||a||^2 = a_1^2 + a_2^2 + \cdots + a_N^2.$$

Cauchy-Schwarts takes the form

$$< a, b > \leq ||a|| \cdot ||b||.$$

Consider $< a - tb, a - tb >$, expand it out, write it as a quadratic polynomial in t, minimize it in t, and complete the proof.



Exercise 1.6. Prove the following related inequality. Let $x_1, \ldots, x_d$ denote positive real numbers. Then

$$(x_1 \cdot x_2 \cdot \cdots \cdot x_n)^{\frac{1}{n}} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}. \tag{1.9}$$

Hint: There are many ways to do this, but I suggest exploring the following idea. Prove this inequality for $n = 2$ and then extend it, using induction, to $n = 2^k$, $k = 1, 2 \ldots$. Then prove that if the result holds for $n + 1$, then it must hold for n, thus filling the gaps between the powers of two.

Another approach: Write $a_j = e^{\log(a_j)}$ and use the convexity of the exponential function, i.e, the fact that

$$e^{t_1 x_1 + t_2 x_2 + \cdots + t_n x_n} \leq t_1 e^{x_1} + t_2 e^{x_2} + \cdots + t_n e^{x_n}$$

with $t_j \geq 0$ and

$$t_1 + t_2 + \cdots + t_n = 1.$$

Exercise 1.7. Let $x_1, \ldots, x_n$ and $a_1, \ldots, a_n$ be positive real numbers. Then

$$x_1^{a_1} \cdot x_2^{a_2} \cdot \cdots \cdot x_n^{a_n} \leq \frac{(x_1 + \cdots + x_n)^{a_1 + \cdots + a_n}}{a_1 \cdot a_2 \cdot \cdots \cdot a_n} a_1^{a_1} \cdot a_2^{a_2} \cdot \cdots \cdot a_n^{a_n}.$$

Is it possible to choose a_js in a way that reduces this inequality to either (1.7) or (1.9)? If the answer is yes, demonstrate it. If the answer is no, explain why not? Hint: Under what conditions does the equality hold in all the inequalities involved in this exercise.

Exercise 1.8. Generalize (1.8) to see that

$$a_1 a_2 \dots a_n \le \frac{a_1^{p_1}}{p_1} + \cdots + \frac{a_n^{p_n}}{p_n},$$

where each $p_j > 1$ and

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1.$$

While this follows fairly easily from the method of the previous exercise, solve this proof using Lagrange multipliers. More precisely, let

$$f(a_1, \dots, a_n) = \frac{a_1^{p_1}}{p_1} + \cdots + \frac{a_n^{p_n}}{p_n},$$

and let

$$g(a_1, \dots, a_n) = a_1 a_2 \dots a_n.$$

Now minimize $f(a_1, \dots, a_n)$ under the constraint $g(a_1, \dots, a_n) = c$, where c is an arbitrary positive number.

1. Notes, remarks and difficult questions

Many high school students have seen the Cauchy-Schwartz inequality, at least in the case $N = 2$, without ever realizing it. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two vectors in the plane. We know from high school mathematics that

$$x \cdot y = x_1 y_1 + x_2 y_2 = |x| \cdot |y| \cdot \cos(\theta),$$

where

$$|x| = \sqrt{x_1^2 + x_2^2},$$

and θ is the angle between x and y. Since $|\cos(\theta)| \le 1$, we see that

$$|x \cdot y| \le |x| \cdot |y|,$$

which is the Cauchy-Schwartz inequality in the case $N = 2$.

Can you use the fact that two vectors determine a plane, even in d-dimensions, to extend the above reasoning to $\mathbb{R}^d$, thus giving yet another proof of the Cauchy-Schwartz inequality?

Provided that you were able to answer the question in the previous paragraph, we now have three proofs of the Cauchy-Schwartz inequality. Do the first two inequalities rely on the assumption that the left hand side of the Cauchy-Schwartz inequality, $\sum_{k=1}^{N} a_k b_k$, is a finite sum? What about the third inequality?