# Preface

This text is intended to provide a student's first encounter with the concepts of measure theory and functional analysis. Its structure and content were greatly influenced by my belief that good pedagogy dictates introducing difficult concepts in their simplest and most concrete forms. For example, the study of abstract metric spaces should come after the study of the metric and topological properties of $\mathbb{R}^n$. Multidimensional calculus should not be introduced in Banach spaces even if the proofs are identical to the proofs for $\mathbb{R}^n$. And a course in linear algebra should precede the study of abstract algebra.

Hence, despite the use of the word "terse" in the title, this text might also have been called "A (Gentle) Introduction to Lebesgue Integration". It is terse in the sense that it treats only a subset of those concepts typically found in a substantive graduate level analysis course. I have emphasized the motivation of these concepts and attempted to treat them in their simplest and most concrete form. In particular, little mention is made of general measures other than Lebesgue until the final chapter. Indeed, we restrict our attention to Lebesgue measure on $\mathbb{R}$ and no treatment of measures on $\mathbb{R}^n$ for $n > 1$ is given. The emphasis is on real-valued functions but complex functions are considered in the chapter on Fourier series and in the final chapter on ergodic transformations. I consider the narrow selection of topics to be an approach at one end of a spectrum whose other end is represented, for example, by the excellent graduate text [**Ru**] by Rudin which introduces Lebesgue measure as a corollary of the Riesz representation theorem. That is a sophisticated and elegant approach, but, in my opinion, not one which is suited to a student's first encounter with Lebesgue integration.

In this text the less elegant, and more technical, classical construction of Lebesgue measure due to Caratheodory is presented, but is relegated to an appendix. The intent is to introduce the Lebesgue integral as a tool. The hope is to present it in a quick and intuitive way, and then go on to investigate the standard convergence theorems and a brief introduction to the Hilbert space of $L^2$ functions on the interval.

This text should provide a good basis for a one semester course at the advanced undergraduate level. It might also be appropriate for the beginning part of a graduate level course if Appendices B and C are covered. It could also serve well as a text for graduate level study in a discipline other than mathematics which has serious mathematical prerequisites.

The text presupposes a background which a student should possess after a standard undergraduate course in real analysis. It is terse in the sense that the density of definition-theorem-proof content is quite high. There is little hand holding and not a great number of examples. Proofs are complete but sometimes tersely written. On the other hand, some effort is made to motivate the definitions and concepts.

Chapter 1 provides a treatment of the "regulated integral" (as found in Dieudonné [**D**]) and of the Riemann integral. These are treated briefly, but with the intent of drawing parallels between their definition and the presentation of the Lebesgue integral in subsequent chapters.

As mentioned above the actual construction of Lebesgue measure and proofs of its key properties are left for an appendix. Instead the text introduces Lebesgue measure as a generalization of the concept of length and motivates its key properties: monotonicity, countable additivity, and translation invariance. This also motivates the concept of $\sigma$-algebra. If a generalization of length has these three key properties, then it needs to be defined on a $\sigma$-algebra for these properties to make sense.

In Chapter 2 the text introduces null sets and shows that any generalization of length satisfying monotonicity and countable additivity must assign zero to them. We then *define* Lebesgue measurable sets to be sets in the $\sigma$-algebra generated by open sets and null sets.

At this point we state a theorem which asserts that Lebesgue measure exists and is unique, i.e., there is a function $\mu$ defined for measurable subsets of a closed interval which satisfies monotonicity, countable additivity, and translation invariance.

The proof of this theorem (Theorem 2.4.2) is included in an appendix where it is also shown that the more common definition of measurable sets (using outer measure) is equivalent to being in the $\sigma$-algebra generated by open sets and null sets.

Chapter 3 discusses bounded Lebesgue measurable functions and their Lebesgue integral. The last section of this chapter, and some of the exercises following it, focus somewhat pedantically on the concept of "almost everywhere." The hope is to develop sufficient facility with the concept that it can be treated more glibly in subsequent chapters.

Chapter 4 considers unbounded functions and some of the standard convergence theorems. In Chapter 5 we introduce the Hilbert space of $L^2$ functions on an interval and show several elementary properties leading up to a definition of Fourier series.

Chapter 6 discusses classical real and complex Fourier series for $L^2$ functions on the interval and shows that the Fourier series of an $L^2$ function converges in $L^2$ to that function. The proof is based on the Stone-Weierstrass theorem which is stated but not proved.

Chapter 7 introduces some concepts from measurable dynamics. The Birkhoff ergodic theorem is stated without proof and results on Fourier series from Chapter 6 are used to prove that an irrational rotation of the circle is ergodic and the squaring map $z \mapsto z^2$ on the complex numbers of modulus 1 is ergodic.

Appendix A summarizes the needed prerequisites providing many proofs and some exercises. There is some emphasis in this section on the concept of countability, to which I would urge students and instructors to devote some time, as countability plays an very crucial role in the study of measure theory.

In Appendix B we construct Lebesgue measure and prove it has the properties cited in Chapter 2. In Appendix C we construct a non-measurable set.

Finally, at the website http://www.ams.org/bookpages/stml-48 we provide solutions to a few of the more challenging exercises. These exercises are marked with a ($\star$) when they occur in the text.

This text grew out of notes I have used in teaching a one quarter course on integration at the advanced undergraduate level. With some selectivity of topics and well prepared students it should be possible to cover all key concepts in a one semester course.