Foreword

This set of lectures forms a gentle introduction to both the classical theory of the calculus of variations and the more modern developments of optimal control theory from the perspective of an applied mathematician. It focuses on understanding concepts and how to apply them, as opposed to rigorous proofs of existence and uniqueness theorems; and so it serves as a prelude to more advanced texts in much the same way that calculus serves as a prelude to real analysis. The prerequisites are correspondingly modest: the standard calculus sequence, a first course on ordinary differential equations, some facility with a mathematical software package, such as Maple, *Mathematica*® (which I used to draw all of the figures in this book) or MATLAB—nowadays, almost invariably implied by the first two prerequisites—and that intangible quantity, a degree of mathematical maturity. Here at Florida State University, the senior-level course from which this book emerged requires either a first course on partial differential equations—through which most students qualify—or a course on analysis or advanced calculus, and either counts as sufficient evidence of mathematical maturity. These few prerequisites are an adequate basis on which to build a sound working knowledge of the subject. To be sure, there ultimately arise issues that cannot be addressed without the tools of functional analysis; but these are intentionally beyond the scope of this book, though touched on briefly towards the end. Thus, on the one hand, it is by no means necessary for a reader of this book to have been exposed to real analysis; and yet, on the other hand, such prior exposure cannot help but increase the book's accessibility.

Students taking a first course on this topic typically have diverse backgrounds among engineering, mathematics and the natural or social sciences. The range of potential applications is correspondingly broad: the calculus of variations and optimal control theory have been widely used in numerous ways in, e.g., biology [**27, 35, 58**],[1] criminology [**18**], economics [**10, 26**], engineering [**3, 49**], finance [**9**], management science [**12, 57**], and physics [**45, 63**] from a variety of perspectives, so that the needs of students are too extensive to be universally accommodated. Yet one can still identify a solid core of material to serve as a foundation for future graduate studies, regardless of academic discipline, or whether those studies are applied or theoretical. It is this core of material that I seek to expound as lucidly as possible, and in such a way that the book is suitable not only as an undergraduate text, but also for self-study. In other words, this book is primarily a mathematics text, albeit one aimed across disciplines. Nevertheless, I incorporate applications—cancer chemotherapy in Lecture 20, navigational control in Lecture 22 and renewable resource harvesting in Lecture 24—to round out the themes developed in the earlier lectures.

Arnold Arthurs introduced me to the calculus of variations in 1973-74, and these lectures are based on numerous sources consulted at various times over the 35 years that have since elapsed; sometimes with regard to teaching at FSU; sometimes with regard to my own research contributions to the literature on optimal control theory; and only recently with regard to this book. It is hard now to judge the relative extents to which I have relied on various authors. Nevertheless, I have relied most heavily on—in alphabetical order—Akhiezer [**1**], Bryson & Ho [**8**], Clark [**10**], Clegg [**11**], Gelfand & Fomin [**16**], Hadley & Kemp [**19**], Hestenes [**20**], Hocking [**22**], Lee & Markus [**33**], Leitmann [**34**], Pars [**47**], Pinch [**50**] and Pontryagin et al. [**51**]; and other authors are cited in the bibliography. I am grateful to all of them, and to each in a measure proportional to my indebtedness.

[1] Bold numbers in square brackets denote references in the bibliography (p. 245).

To most of the lectures I have added exercises. Quite a few are mine; but most are drawn or adapted from the cited references for their aptitude to reinforce the topic of the lecture. At this level of discourse, various canonical exercises are pervasive in the pedagogical literature and can be found in multiple sources with only minor variation: if these old standards are not *de rigueur*, then they are at least very hard to improve on. I have therefore included a significant number of them, and further problems, if necessary, may be found among the references (as indicated by endnotes). Only solutions or hints for selected exercises appear at the end of the book, but more complete solutions are available from the author.[2]

Finally, a word or two about notation. Just as modelling compels tradeoffs among generality, precision and realism [**36**], so pedagogy compels tradeoffs among generality, rigor and transparency; and central to those tradeoffs is use of notation. At least two issues arise. The first is the subjectivity of signal-to-noise ratio. One person's oasis of terminological correctness may be another person's sea of impenetrable clutter; and in any event, strict adherence to unimpeachably correct notation entails encumbrances that often merely obscure. The second, and related, issue is that diversity is intrinsically valuable [**46**]. In the vast ecosystem of mathematical and scientific literature, polymorphisms of notation survive and prosper because, as in nature, each variant has its advantages and disadvantages, and none is universally adaptive. Aware of both issues, I use a mix of notation that suppresses assumed information when its relevance is not immediate, thus striving at all times to emphasize clarity over rigor.

[2] To any instructor or bona fide independent user. For contact details, see **http://www.ams.org/bookpages/stml-50/index.html**.

Lecture 1

The Brachistochrone

Although the roots of the calculus of variations can be traced to much earlier times, the birth date of the subject is widely considered to be June of 1696.[1] That is when John Bernoulli posed the celebrated problem of the *brachistochrone* or curve of quickest descent, i.e., to determine the shape of a smooth wire on which a frictionless bead slides between two fixed points in the shortest possible time.

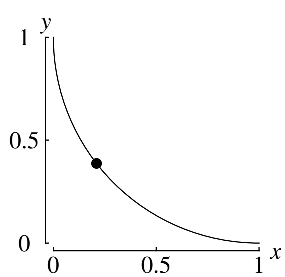

Figure 1.1. A frictionless bead on a wire.

For the sake of definiteness, let us suppose that the points in question have coordinates $(0, 1)$ and $(1, 0)$, and that the bead slides

[1]See, e.g., Bliss [**5**, pp. 12-13 and 174-179] or Hildebrandt & Tromba [**21**, pp. 26-27 and 120-123], although Goldstine [**17**, p. vii] prefers the earlier date of 1662 when Fermat applied his principle of least time to light ray refraction.

along the curve with equation

$$y = y(x). \tag{1.1}$$

Note that it will frequently be convenient to use the same symbol—here y—to denote both a univariate function and the ordinate of its graph, because the correct interpretation will be obvious from context. Needless to say, the endpoints must lie on the curve, and so

$$y(0) = 1, \qquad y(1) = 0. \tag{1.2}$$

Let the bead have velocity

$$\mathbf{v} = \frac{ds}{dt}\boldsymbol{\tau} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = \frac{dx}{dt}\left\{\mathbf{i} + \frac{dy}{dx}\mathbf{j}\right\}, \tag{1.3}$$

where s denotes arc length, t denotes time and $\mathbf{i}$, $\mathbf{j}$ and $\boldsymbol{\tau}$ are unit vectors in the (rightward) horizontal, (upward) vertical and tangential directions, respectively, so that the particle's speed is

$$v = |\mathbf{v}| = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,\frac{dx}{dt} \tag{1.4}$$

implying

$$ds = \sqrt{1 + (y')^2}\,dx, \tag{1.5}$$

where y' denotes $\frac{dy}{dx}$.[2] Let the particle start at $(0,1)$ at time 0 and reach $(1,0)$ at time t_f after travelling distance s_f along the curve. Then its transit time is

$$\int_0^{t_f} dt = \int_0^{s_f} \frac{ds}{v} = \int_0^1 \frac{\sqrt{1 + (y')^2}}{v}\,dx. \tag{1.6}$$

If g is the acceleration due to gravity and the bead has mass m, then its kinetic energy is $\frac{1}{2}mv^2$, its potential energy is mgy and—because there is no friction—the sum of the kinetic and potential energies must be a constant. Because the sum was $\frac{1}{2}m0^2 + mg \cdot 1 = mg$ initially, we have $\frac{1}{2}mv^2 + mgy = mg$ or

$$v = \sqrt{2g(1-y)}, \tag{1.7}$$

[2] As remarked above, the symbol y' may denote either a derivative, i.e., a function, or the value, say $y'(x)$, that this function assigns to an arbitrary element—here x—of its domain. The correct interpretation is obvious from context; e.g., y' in (1.8)-(1.9) denotes the assigned value, for otherwise the integral would not be well defined.

which reduces (1.6) to

$$\frac{1}{\sqrt{2g}}\int_0^1 \frac{\sqrt{1+(y')^2}}{\sqrt{1-y}}\,dx. \tag{1.8}$$

Clearly, changing the curve on which the bead slides down will change the value of the above integral, which is therefore a function of y: it is a function of a function, or a *functional* for short. Whenever we wish to emphasize that a functional J depends on y, we will denote it by $J[y]$, as in

$$J[y] \;=\; \int_0^1 \sqrt{\frac{1+\left(y'\right)^2}{1-y}}\,dx. \tag{1.9}$$

At other times, however, we may prefer to emphasize that the functional depends on the curve $y = y(x)$, i.e., on the graph of y, which we denote by Γ; in that case, we will denote the functional by $J[\Gamma]$. At other times still, we may have no particular emphasis in mind, in which case, we will write the functional as plain old J. For example, if Γ is a straight line, then

$$y(x) \;=\; 1-x, \tag{1.10}$$

and (1.9) yields

$$J \;=\; \int_0^1 \frac{\{1+(-1)^2\}^{\frac{1}{2}}}{\sqrt{x}}\,dx \;=\; 2\sqrt{2}\int_0^1 \frac{d}{dx}\{x^{\frac{1}{2}}\}\,dx \;=\; 2\sqrt{2} \tag{1.11}$$

or approximately 2.82843; whereas if Γ is a quarter of the circle of radius 1 with center $(1,1)$, then

$$y(x) \;=\; 1-\sqrt{2x-x^2} \tag{1.12}$$

and

$$J \;=\; \int_0^1 \frac{1}{(2x-x^2)^{\frac{3}{4}}}\,dx \;\approx\; 2.62206 \tag{1.13}$$

on using numerical methods.[3]

[3] E.g., the *Mathematica* command `NIntegrate[(2x-x^2)^(-3/4),{x,0,1}]`.

Here two remarks are in order. First, multiplication by a constant of a quantity to be optimized has no effect on the optimizer.[4] So, from (1.8) and (1.9), the brachistochrone problem is equivalent to that of finding y to minimize $J[y]$. Second, from (1.11) and (1.13), the bead travels faster down a circular arc than down a straight line: whatever the optimal curve is, it is not a straight line. But is there a curve that yields an even lower transit time than the circle?

One way to explore this question is to consider a one-parameter family of *trial curves* satisfying (1.2), e.g., the family defined by

$$y \;=\; y_\epsilon(x) \;=\; 1 - x^\epsilon \tag{1.14}$$

for $\epsilon > 0$. Note the contrast with (1.1). Now each different *trial function* y_ϵ is distinguished by its value of ϵ; y is used only to denote the ordinate of its graph, as illustrated by Figure 1.2(a). When (1.14) is substituted into (1.9), J becomes a function of ϵ: we obtain

$$\begin{aligned} J(\epsilon) \;=\; J[y_\epsilon] \;&=\; \int_0^1 \frac{\{1+\{y_\epsilon'(x)\}^2\}^{\frac{1}{2}}}{\sqrt{1-y_\epsilon(x)}}\,dx \\ &=\; \int_0^1 x^{-\frac{\epsilon}{2}}\{1+\epsilon^2 x^{2\epsilon-2}\}^{\frac{1}{2}}\,dx \end{aligned} \tag{1.15}$$

after simplification. This integral cannot be evaluated analytically (except when $\epsilon = 1$), but is readily evaluated by numerical means with the help of a software package such as Maple, *Mathematica*® or MATLAB. Because, from Figure 1.2(a), the curve is too steep initially when ϵ is very small, is too close to the line when ϵ is close to 1 and bends the wrong way for $\epsilon > 1$, let us consider only values between, say, $\epsilon = 0.2$ and $\epsilon = 0.8$. A table of such values is

ϵ	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$J(\epsilon)$	2.690	2.634	2.602	2.587	2.589	2.608	2.647

and the graph of J over this domain is plotted in Figure 1.2(b). We see that $J(\epsilon)$ achieves a minimum at $\epsilon = \epsilon^* \approx 0.539726$ with

[4] For example, the polynomials $x(2x-1)$ and $3x(2x-1)$ both have minimizer $x = \frac{1}{4}$, although in the first case the minimum is $-\frac{1}{8}$ and in the second case the minimum is $-\frac{3}{8}$.

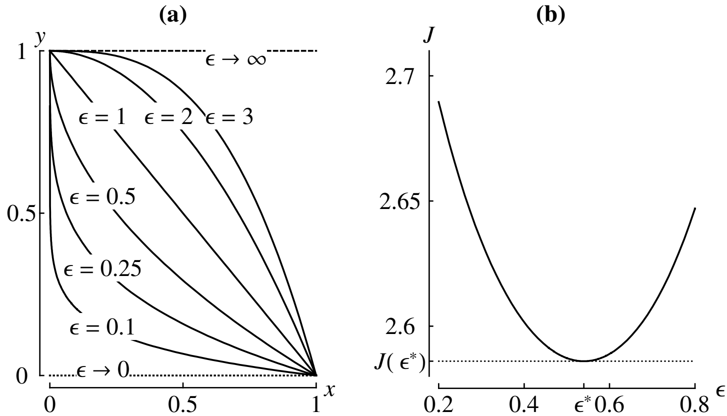

Figure 1.2. **(a)** A class of trial functions. **(b)** $J = J(\epsilon)$ on $[0.2, 0.8]$.

$J(\epsilon^*) \approx 2.58598$. Comparing with (1.13), we find that $y = y_{\epsilon^*}(x)$ yields a lower transit time than the circular arc.

But that doesn't make $y = y_{\epsilon^*}(x)$ the solution of the brachistochrone problem, because the true minimizing function may not belong to the family defined by (1.14). If $y = y^*(x)$ is the true minimizing curve, then all we have shown is that

$$J[y^*] \;\leq\; J(\epsilon^*) \approx 2.58598. \tag{1.16}$$

In other words, we have found an upper bound for the true minimum. It turns out, in fact, that the true minimizer is a *cycloid* defined parametrically by

$$x \;=\; \frac{\theta + \sin(\theta)\cos(\theta) + \frac{1}{2}\pi}{\cos^2(\theta_1)}, \quad y \;=\; 1 - \left\{\frac{\cos(\theta)}{\cos(\theta_1)}\right\}^2 \tag{1.17}$$

for $-\frac{1}{2}\pi \leq \theta \leq \theta_1$, where $\theta_1 \approx -0.364791$ is the larger of the only two roots of the equation

$$\theta_1 + \sin(\theta_1)\cos(\theta_1) + \tfrac{1}{2}\pi \;=\; \cos^2(\theta_1) \tag{1.18}$$

and $J[y^*] \approx 2.5819045$; see Lecture 4, especially (4.26)-(4.27). We compare y^* with y_{ϵ^*} in Figure 1.3. Both curves are initially vertical; however, the cycloid is steeper (has a more negative slope) than the

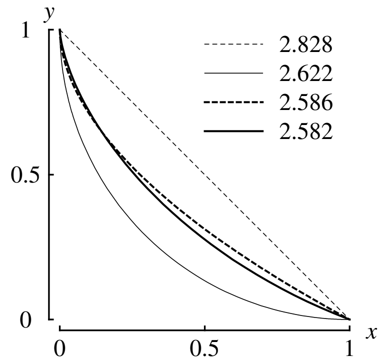

Figure 1.3. Values achieved (top right) for $J[y]$ by a straight line, a quarter-circle, the best trial function and a cycloid.

graph of the best trial function for values of x between about 0.02 and 0.55, and it slopes more gently elsewhere.

But how could we have known that the cycloid is the curve that minimizes transit time—in other words, that the cycloid is the brachistochrone? At this stage, we couldn't have: we need the calculus of variations, which was first developed to solve this problem. We will start to develop it ourselves in Lecture 2.

Exercises 1

1. Rotating a curve between $(0,1)$ and $(1,2)$ about the x-axis generates a surface of revolution. Obtain an upper bound on the minimum value S^* of its surface area by using the trial-function method (and a software package for numerical integration).
2. Obtain an upper bound on the minimum value J^* of

$$J[y] \;=\; \int_0^1 y^2 {y'}^2 \, dx$$

subject to $y(0) = 0$ and $y(1) = 1$ by using the trial functions $y = y_\epsilon(x) = x^\epsilon$ with $\epsilon > \frac{1}{4}$.

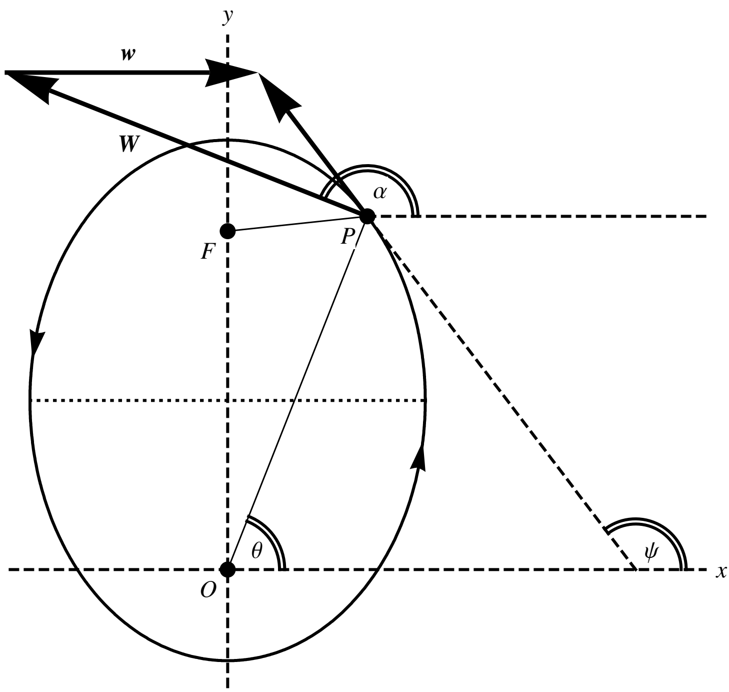

Figure 22.3. The solution to Chaplygin's problem. The optimal trajectory is an ellipse of eccentricity $e = \frac{w}{W}$ with foci at F and at O, which is also the origin of coordinates. Here P denotes the plane's position; θ is the polar angle; $OP = r$; $\mathbf{W}$ and $\alpha = \theta + \frac{1}{2}\pi$ are the plane's velocity and heading relative to the air; $\mathbf{w}$ is the wind velocity; and

$$\psi \;=\; \pi + \tan^{-1}\Big(\frac{\cos(\theta)}{e - \sin(\theta)}\Big)$$

is the plane's true heading, obtained by using $\tan(\psi) = \frac{dy}{dx} = \dot{y}/\dot{x}$ in conjunction with (22.31) and (22.42). The defining property of this ellipse is that $OP + PF$ is constant and equal to the length of its major axis.

known quantity; and the optimal enclosed area will be

$$\frac{1}{2}\int_0^T r^2\dot{\theta}\,dt \;=\; \frac{1}{2}\int_0^{2\pi} r^2\,d\theta \;=\; \frac{1}{4\pi}W^2T^2\Big(1 - \{\tfrac{w}{W}\}^2\Big)^{\frac{3}{2}} \tag{22.49}$$

(Exercise 22.7).

Exercises 22

1. Verify (22.14).
2. Verify that the boat's initial heading relative to dry land is precisely half of that relative to water when $W = U$ in Figure 22.1(a). What if $W = 3U$?
3. Verify (22.26) and (22.30).
4. **(a)** What is wrong with the solution according to pp. 185-187 of Zermelo's problem for open water when $W = 0.6U$?
   (b) What is wrong with the solution according to pp. 188-189 of Zermelo's problem for a river when $W = 0.8U$?
5. Verify that (22.46) is the polar equation of the ellipse in Figure 22.3, whose defining property is that $OP + PF = 2\mu$.
6. Use the result that, on $(0, \pi)$ or on $(\pi, 2\pi)$,
   $$\Psi(\theta) \;=\; \tfrac{e\cos(\theta)}{(1-e^2)\{e\sin(\theta)-1\}} - \tfrac{2}{(1-e^2)^{3/2}}\tan^{-1}\left(\tfrac{e-\tan(\frac{1}{2}\theta)}{\sqrt{1-e^2}}\right)$$
   is an anti-derivative of $\{1 - e\sin(\theta)\}^{-2}$ to verify (22.48).
   Hint: Why is $\Psi(\theta)$ not an anti-derivative of $\{1 - e\sin(\theta)\}^{-2}$ on $(0, 2\pi)$? How must Ψ be modified?
7. Verify (22.49).
8. Suppose that a boat with constant speed W must cross the open water in Figure 22.1(a) through the current defined by (22.7) as quickly as possible from the point $(l, -h)$ to any point on the shoreline. What are the optimal heading and trajectory? How long does the crossing take?
9. Suppose that a boat with constant speed W must cross the river in Figure 22.1(b) through the current defined by (22.24) as quickly as possible from the point $(l, -h)$ to any point on the opposite bank. What is the optimal solution?
10. A circular island of radius 1 has its center at the origin and is surrounded by open water with a current defined by
    $$\mathbf{q} \;=\; \tfrac{1}{6}U(x^2 + y^2 - 1)\{y\mathbf{i} - x\mathbf{j}\}.$$
    Where would a boat reach the island from the point $(2, 0)$, where $|\mathbf{q}| = U$, at constant speed W in the least amount of time?

Endnote. For further time-optimal navigational control problems, see Bryson & Ho [**8**, pp. 82-86].