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# Preface

The standard model for the diffusion of heat uses the idea that heat spreads randomly in all directions at some rate. The *heat equation* is a deterministic (non-random), partial differential equation derived from this intuition by averaging over the very large number of particles. This equation can and has been traditionally studied as a deterministic equation. While much can be said from this perspective, one also loses much of the intuition that can be obtained by considering the individual random particles.

The idea in these notes is to introduce the heat equation and the closely related notion of harmonic functions from a probabilistic perspective. Our starting point is the random walk which in continuous time and space becomes Brownian motion. We then derive equations to understand the random walk. This follows the modern approach where one tries to combine probabilistic and deterministic methods to analyze diffusion.

Besides the random/deterministic dichotomy, another difference in approach comes from choosing between discrete and continuous models. The first chapter of this book starts with discrete random walk and then uses it to define harmonic functions and the heat equations on the integer lattice. Here one sees that linear functions arise, and the deterministic questions yield problems in linear algebra. In

particular, solutions of the heat equation can be found using diagonalization of symmetric matrices.

The next chapter goes to continuous time and continuous space. We start with the Brownian motion which is the limit of random walk. This is a fascinating subject in itself and it takes a little work to show that it exists. We have separated the treatment into Sections 2.1 and 2.6. The idea is that the latter section does not need to be read in order to appreciate the rest of the chapter. The traditional heat equation and Laplace equation are found by considering the Brownian particles. Along the way, it is shown that the matrix diagonalization of the previous chapter turns into a discussion of Fourier series.

The third chapter introduces a fundamental idea in probability, martingales, that is closely related to harmonic functions. The viewpoint here is probabilistic. The final chapter is an introduction to fractal dimension. The goal, which is a bit ambitious, is to determine the fractal dimension of the random Cantor set arising in Chapter 3.

This book is derived from lectures given in the Research Experiences for Undergraduates (REU) program at the University of Chicago. The REU is a summer program taken in part or in full by about eighty mathematics majors at the university. The students take a number of mini-courses and do a research paper under the supervision of graduate students. Many of the undergraduates also serve as teaching assistants for one of two other summer programs, one for bright junior high and high school students and another designed for elementary and high school teachers. The first two chapters in this book come from mini-courses in 2007 and 2008, and the last two chapters from a 2009 course.

The intended audience for these lectures was advanced undergraduate mathematics majors who may be considering graduate work in mathematics or a related area. The idea was to present probability and analysis in a more advanced way than found in undergraduate courses. I assume the students have had the equivalent of an advanced calculus (rigorous one variable calculus) course and some exposure to linear algebra. I do not assume that the students have had a course in probability, but I present the basics quickly. I do not assume measure theory, but I introduce many of the important ideas along the way,

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such as: Borel-Cantelli lemma, monotone and dominated convergence theorems, Borel measure, conditional expectation, etc. I also try to firm up the students' grasp of the advanced calculus throughout the book. For example, analysis of simple random walk leads to Stirling's formula whose proof uses Taylor's theorem with remainder.

It is hoped that this book will be interesting to undergraduates, especially those considering graduate studies, as well as to graduate students and faculty whose specialty is not probability or analysis. This book could be used for advanced seminars or for independent reading. There are a number of exercises at the end of each section. They vary in difficulty and some of them are at the challenging level that corresponds to summer projects for undergraduates at the REU.

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