## Chapter 1

## Introduction

Notation:
$\mathbb{N}=\{1,2,3, \ldots\}$ is the set of natural numbers.
$\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3, \ldots\}$ is the ring of integers.
$\mathbb{Q}=\left\{\frac{m}{n}: m, n \in \mathbb{Z}, n \neq 0\right\}$ is the field of rational numbers.
$\mathbb{R}$ is the field of real numbers.
$\mathbb{C}=\left\{a+b i: a, b \in \mathbb{R}, i^{2}=-1\right\}$ is the field of complex numbers.
In this chapter, we introduce elliptic curves, modular forms and $L$ functions through examples that motivate the definitions.

### 1.1. Elliptic curves

For the time being, we define an elliptic curve to be any equation of the form

$$
y^{2}=x^{3}+a x^{2}+b x+c
$$

with $a, b, c \in \mathbb{Z}$ and such that the polynomial $x^{3}+a x^{2}+b x+c$ does not have repeated roots. See Section 2.2 for a precise definition.

Example 1.1.1. Are there three consecutive integers whose product is a perfect square?

There are some trivial examples that involve the number zero, for example, 0,1 and 2 , whose product equals $0 \cdot 1 \cdot 2=0=0^{2}$, a square.

Are there any non-trivial examples? If we try to assign variables to our problem, we see that we are trying to find solutions to

$$
\begin{equation*}
y^{2}=x(x+1)(x+2) \tag{1.1}
\end{equation*}
$$

with $x, y \in \mathbb{Z}$ and $y \neq 0$. Equation (1.1) defines an elliptic curve. It turns out that there are no integral solutions other than the trivial ones (see Exercise 1.4.1). Are there rational solutions, i.e., are there solutions with $x, y \in \mathbb{Q}$ ? This is a more delicate question, but the answer is still no (we will prove it in Example 2.7.6). Here is a similar question, with a very different answer:

- Are there three integers that differ by 5, i.e., $x, x+5$ and $x+10$, and whose product is a perfect square?

In this case, we are trying to find solutions to $y^{2}=x(x+5)(x+10)$ with $x, y \in \mathbb{Z}$. As in the previous example, there are trivial solutions (those which involve 0) but in this case, there are non-trivial solutions as well:

$$
\begin{aligned}
(-9) \cdot(-9+5) \cdot(-9+10) & =(-9) \cdot(-4) \cdot 1=36
\end{aligned}=6^{2} .
$$

Moreover, there are also rational solutions, which are far from obvious:

$$
\begin{aligned}
\left(\frac{5}{4}\right) \cdot\left(\frac{5}{4}+5\right) \cdot\left(\frac{5}{4}+10\right) & =\left(\frac{75}{8}\right)^{2} \\
\left(-\frac{50}{9}\right) \cdot\left(-\frac{50}{9}+5\right) \cdot\left(-\frac{50}{9}+10\right) & =\left(\frac{100}{27}\right)^{2}
\end{aligned}
$$

and, in fact, there are infinitely many rational solutions! Here are some of the $x$-coordinates that work:
$x=-9,40, \frac{5}{4}, \frac{-50}{9}, \frac{961}{144}, \frac{7200}{961},-\frac{12005}{1681},-\frac{16810}{2401},-\frac{27910089}{5094049}, \ldots$
In Sections 2.9 and 2.10 we will explain a method to find rational points on elliptic curves and, in Exercise 2.12.23, the reader will calculate all the rational points of $y^{2}=x(x+5)(x+10)$.

Example 1.1.2 (The Congruent Number Problem). We say that $n \geq 1$ is a congruent number if there exists a right triangle whose sides are rational numbers and whose area equals $n$. What natural numbers are congruent?

For instance, the number 6 is congruent, because the right triangle with sides of length $(a, b, c)=(3,4,5)$ has area equal to $\frac{3 \cdot 4}{2}=6$. Similarly, the number 30 is the area of the right triangle with sides $(5,12,13)$; thus, 30 is a congruent number.


Figure 1. A right triangle of area 5 and rational sides.
The number 5 is congruent but there is no right triangle with integer sides and area equal to 5 . However, our definition allowed rational sides, and the triangle with sides $\left(\frac{3}{2}, \frac{20}{3}, \frac{41}{6}\right)$ has area exactly 5. We do not allow, however, triangles with irrational sides even if the area is an integer. For example, the right triangle $(1,2, \sqrt{5})$ has area 1 , but that does not imply that 1 is a congruent number (in fact, 1 is not a congruent number, as we shall see below).

The congruent number problem is one of the oldest open problems in number theory. For more than a millennium, mathematicians have attempted to provide a characterization of all congruent numbers. The oldest written record of the problem dates back to the early Middle Ages, when it appeared in an Arab manuscript written before 972 (a later 10th century manuscript written by Mohammed Ben Alcohain would go as far as to claim that the principal object of the theory of rational right triangles is to find congruent numbers). It is known that Leonardo Pisano, a.k.a. Fibonacci, was challenged around 1220 by Johannes of Palermo to find a rational right triangle of area
$n=5$, and Fibonacci found the triangle $\left(\frac{3}{2}, \frac{20}{3}, \frac{41}{6}\right)$. We will explain a method to find this triangle below. In 1225 , Fibonacci wrote a more general treatment about the congruent number problem, in which he stated (without proof) that if $n$ is a perfect square, then $n$ cannot be a congruent number. The proof of such a claim had to wait until Pierre de Fermat (1601-1665) settled that the number 1 (and every square number) is not a congruent number (a result that he showed in order to prove the case $n=4$ of Fermat's last theorem).

The connection between the congruent number problem and elliptic curves is as follows:

Proposition 1.1.3. The number $n>0$ is congruent if and only if the curve $y^{2}=x^{3}-n^{2} x$ has a point $(x, y)$ with $x, y \in \mathbb{Q}$ and $y \neq 0$. More precisely, there is a one-to-one correspondence $C_{n} \longleftrightarrow E_{n}$ between the following two sets:

$$
\begin{aligned}
& C_{n}=\left\{(a, b, c): a^{2}+b^{2}=c^{2}, \frac{a b}{2}=n\right\} \\
& E_{n}=\left\{(x, y): y^{2}=x^{3}-n^{2} x, y \neq 0\right\}
\end{aligned}
$$

Mutually inverse correspondences $f: C_{n} \rightarrow E_{n}$ and $g: E_{n} \rightarrow C_{n}$ are given by
$f((a, b, c))=\left(\frac{n b}{c-a}, \frac{2 n^{2}}{c-a}\right), g((x, y))=\left(\frac{x^{2}-n^{2}}{y}, \frac{2 n x}{y}, \frac{x^{2}+n^{2}}{y}\right)$.
The reader can provide a proof (see Exercise 1.4.3). For example, the curve $E: y^{2}=x^{3}-25 x$ has a point $(-4,6)$ that corresponds to the triangle $\left(\frac{3}{2}, \frac{20}{3}, \frac{41}{6}\right)$. But $E$ has other points, such as $\left(\frac{1681}{144}, \frac{62279}{1728}\right)$ that corresponds to the triangle

$$
\left(\frac{1519}{492}, \frac{4920}{1519}, \frac{3344161}{747348}\right)
$$

which also has area equal to 5 . See Figure 2.
Today, there are partial results toward the solution of the congruent number problem, and strong results that rely heavily on famous (and widely accepted) conjectures, but we do not have a full answer yet. For instance, in 1975 (see [Ste75]), Stephens showed that the Birch and Swinnerton-Dyer conjecture (which we will discuss in Section 5.2 ) implies that any positive integer $n \equiv 5,6$ or $7 \bmod 8$ is a


Figure 2. Two rational points on the curve $y^{2}=x^{3}-25 x$.
congruent number. For example, $n=157 \equiv 5 \bmod 8$ must be a congruent number and, indeed, Don Zagier has exhibited a right triangle $(a, b, c)$ whose area equals 157 . The hypotenuse of the simplest such triangle is:

$$
c=\frac{2244035177043369699245575130906674863160948472041}{8912332268928859588025535178967163570016480830}
$$

In Example 5.2.7 we will see an application of the conjecture of Birch and Swinnerton-Dyer to find a rational point $P$ on $y^{2}=x^{3}-157^{2} x$, which corresponds to a right triangle of area 157 via the correspondence in Proposition 1.1.3.

The best known result on the congruent number problem is due to J. Tunnell:

Theorem 1.1.4 (Tunnell, 1983, [Tun83]). If $n$ is an odd squarefree positive integer and $n$ is the area of a right triangle with rational sides, then the following cardinalities are equal:

$$
\begin{aligned}
& \#\left\{(x, y, z) \in \mathbb{Z}^{3}: n=2 x^{2}+y^{2}+32 z^{2}\right\} \\
& =\frac{1}{2}\left(\#\left\{(x, y, z) \in \mathbb{Z}^{3}: n=2 x^{2}+y^{2}+8 z^{2}\right\}\right)
\end{aligned}
$$

and, if $n$ is even,

$$
\begin{aligned}
& \#\left\{(x, y, z) \in \mathbb{Z}^{3}: \frac{n}{2}=4 x^{2}+y^{2}+32 z^{2}\right\} \\
& =\frac{1}{2}\left(\#\left\{(x, y, z) \in \mathbb{Z}^{3}: \frac{n}{2}=4 x^{2}+y^{2}+8 z^{2}\right\}\right)
\end{aligned}
$$

Moreover, if the Birch and Swinnerton-Dyer conjecture is true, then, conversely, these equalities imply that $n$ is a congruent number.

For example, for $n=2$ we have $\frac{n}{2}=1=4 x^{2}+y^{2}+32 z^{2}$ if and only if $x=z=0$ and $y= \pm 1$, so the left-hand side of the appropriate equation in Tunnell's theorem is equal to 2 . However, the right-hand side is equal to 1 and the equality does not hold. Hence, 2 is not a congruent number.

For a complete historical overview of the congruent number problem, see [Dic05], Ch. XVI. The book [Kob93] contains a thorough modern treatment of the problem. The reader may also find useful an expository paper [Con08] on the congruent number problem, written by Keith Conrad. Another neat exposition, more computational in nature (using Sage), appears in [Ste08], Section 6.5.3.

Example 1.1.5 (Fermat's last theorem). Let $n \geq 3$. Are there any solutions to $x^{n}+y^{n}=z^{n}$ in integers $x, y, z$ with $x y z \neq 0$ ? The answer is no. In 1637, Pierre de Fermat wrote in the margin of a book (Diophantus' Arithmetica; see Figure 9 in Section 5.5) that he had found a marvellous proof, but the margin was too small to contain it. Since then, many mathematicians tried in vain to demonstrate (or disprove!) this claim. A proof was finally found in 1995 by Andrew Wiles ([Wil95]). We shall discuss the proof in some more detail in Section 5.5. For now, we will outline the basic structure of the argument.

First, it is easy to show that, to prove the theorem, it suffices to show the cases $n=4$ and $n=p \geq 3$, a prime. It is not difficult to show that $x^{4}+y^{4}=z^{4}$ has no non-trivial solutions in $\mathbb{Z}$ (this was first shown by Fermat). Now, suppose that $p \geq 3$ and $a, b, c$ are integers with $a b c \neq 0$ and $a^{p}+b^{p}=c^{p}$. Gerhard Frey conjectured that if such a triple of integers exists, then the elliptic curve

$$
E: y^{2}=x\left(x-a^{p}\right)\left(x+b^{p}\right)
$$



Figure 3. Pierre de Fermat (1601-1665).
would have some unexpected properties that would contradict a wellknown conjecture that Taniyama, Shimura and Weil had formulated in the 1950 's. Their conjecture spelled out a strong connection between elliptic curves and modular forms, which we will describe in Section 5.4. Ken Ribet proved that, indeed, such a curve would contradict the Taniyama-Shimura-Weil (TSW) conjecture. Finally, Andrew Wiles was able to prove the TSW conjecture in a special case that would cover the hypothetical curve $E$. Therefore, $E$ cannot exist and the triple $(a, b, c)$ cannot exist, either.

The Taniyama-Shimura-Weil conjecture (Conjecture 5.4.5), i.e., the modularity theorem 5.4.6, was fully proved by Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor in their article [BCDT01].

### 1.2. Modular forms

Let $\mathbb{C}$ be the complex plane and let $\mathbb{H}$ be the upper half of the complex plane, i.e., $\mathbb{H}=\{a+b i: a, b \in \mathbb{R}, b>0\}$. A modular form is a function
$f: \mathbb{H} \rightarrow \mathbb{C}$ that has several relations among its values (which we will specify in Definitions 4.1.3 and 4.2.1). In particular, the values of the function $f$ satisfy several types of periodicity relations. For example, the modular forms for $\operatorname{SL}(2, \mathbb{Z})$ satisfy, among other properties, the following:

- $f(z)=f(z+1)$ for all $z \in \mathbb{H}$, and
- $f\left(\frac{-1}{z}\right)=z^{k} f(z)$ for all $z \in \mathbb{H}$. The number $k$ is an integer called the weight of the modular form.

We will describe modular forms in detail in Chapter 4. Let us see some examples that motivate our interest in these functions.

Example 1.2.1 (Representations of integers as sums of squares). Is the number $n>0$ a sum of two (integer) squares? In other words, are there $a, b \in \mathbb{Z}$ such that $n=a^{2}+b^{2}$ ? And if so, in how many different ways can you represent $n$ as a sum of two squares?

For instance, the number $n=3$ cannot be represented as a sum of two squares but the number $n=5$ has 8 distinct representations:

$$
5=( \pm 1)^{2}+( \pm 2)^{2}=( \pm 2)^{2}+( \pm 1)^{2} .
$$

Notice that here we consider $(-1)^{2}+2^{2}, 1^{2}+2^{2}$ and $2^{2}+1$ as distinct representations of 5 . A general formula for the number of representations of an integer $n$ as a sum of 2 squares, due to Lagrange, Gauss and Jacobi, is given by

$$
\begin{equation*}
S_{2}(n)=2\left(1+\left(\frac{-1}{n}\right)\right) \sum_{d \mid n}\left(\frac{-1}{d}\right), \tag{1.2}
\end{equation*}
$$

where $\left(\frac{m}{n}\right)$ is the Jacobi symbol and $\sum_{d \mid n}$ is a sum over all positive divisors of $n$ (including 1 and $n$ ). Here we just need the easiest values $\left(\frac{-1}{n}\right)=(-1)^{(n-1) / 2}$ of the Jacobi symbol. Let us see that the formula works:

$$
\begin{aligned}
& S_{2}(3)=2\left(1+\left(\frac{-1}{3}\right)\right) \sum_{d \mid 3}\left(\frac{-1}{d}\right)=2(1+(-1))(1+(-1))=0 \\
& S_{2}(5)=2\left(1+\left(\frac{-1}{5}\right)\right) \sum_{d \mid 5}\left(\frac{-1}{d}\right)=2(1+1)(1+1)=8
\end{aligned}
$$

and $S_{2}(9)=4$. Indeed, the number nine has 4 different representations: $9=( \pm 3)^{2}+0^{2}=0^{2}+( \pm 3)^{2}$. Let us explore other similar questions.

Let $n>0$ and $k \geq 2$. Is the number $n>0$ a sum of $k$ (integer) squares? In other words, are there $a_{1}, \ldots, a_{k} \in \mathbb{Z}$ such that $n=$ $a_{1}^{2}+\cdots+a_{k}^{2}$ ? And if so, in how many different ways can you represent $n$ as a sum of $k$ squares? Lagrange showed that every natural number can be represented as a sum of $k \geq 4$ squares, but how many different representations are there?

Let $S_{k}(n)$ be the number of representations of $n$ as a sum of $k$ squares. Determining exact formulas for $S_{k}(n)$ is a classical problem in number theory. There are exact formulas known in a number of cases (e.g. Eq. 1.2). The formulas for $k=4,6$ and 8 are due to Jacobi and Siegel. We write $n=2^{\nu} g$, with $\nu \geq 0$ and odd $g>0$ :

$$
\begin{aligned}
& S_{4}(n)=8 \sum_{d \mid n, 4 \nmid d} d, \\
& S_{6}(n)=\left(\left(\frac{-1}{g}\right) 2^{2 \nu+4}-4\right) \sum_{d \mid g}\left(\frac{-1}{d}\right) d^{2}, \\
& S_{8}(n)=16 \cdot \begin{cases}\sum_{d \mid n} d^{3} & \text { if } n \text { is odd, } \\
\sum_{d \mid n} d^{3}-2 \sum_{d \mid g} d^{3} & \text { if } n \text { is even. }\end{cases}
\end{aligned}
$$

For example, $S_{4}(4)=8(1+2)=24$ and, indeed

$$
\begin{aligned}
4 & =( \pm 1)^{2}+( \pm 1)^{2}+( \pm 1)^{2}+( \pm 1)^{2}=( \pm 2)^{2}+0+0+0 \\
& =0+( \pm 2)^{2}+0+0=0+0+( \pm 2)^{2}+0=0+0+0+( \pm 2)^{2} .
\end{aligned}
$$

So there are $16+2+2+2+2=24$ possible representations of the number 4 as a sum of 4 squares. Notice that $S_{4}(2)=S_{4}(4)$. In how many ways can 4 be represented as a sum of 6 squares? We write $4=2^{2} \cdot 1$, so $\nu=2$ and $g=1$, and thus,

$$
S_{6}(4)=\left(\left(\frac{-1}{1}\right) 2^{2 \cdot 2+4}-4\right)\left(\left(\frac{-1}{1}\right) \cdot 1^{2}\right)=\left(2^{8}-4\right) \cdot 1=252 .
$$

The formulas for $S_{k}(n)$ given above are derived using the theory of modular forms, as follows. We define a formal power series $\Theta(q)$ by

$$
\Theta(q)=\sum_{j=-\infty}^{\infty} q^{j^{2}}
$$

and, for $k \geq 2$, consider the power series expansion of the $k$ th power of $\Theta$ :

$$
\begin{aligned}
(\Theta(q))^{k} & =\left(\sum_{j=-\infty}^{\infty} q^{j^{2}}\right)^{k} \\
& =\left(\sum_{a_{1}=-\infty}^{\infty} q^{a_{1}^{2}}\right) \cdots\left(\sum_{a_{k}=-\infty}^{\infty} q^{a_{k}^{2}}\right)=\sum_{n \geq 0} c_{n} q^{n} .
\end{aligned}
$$

What is the $n$th coefficient, $c_{n}$, of $\Theta^{k}$ ? If the readers stare at the previous equation for a while, they will find that $c_{n}$ is given by

$$
c_{n}=\#\left\{\left(a_{1}, \ldots, a_{k}\right) \in \mathbb{Z}^{k}: a_{1}^{2}+\cdots+a_{k}^{2}=n\right\} .
$$

Therefore, $c_{n}=S_{k}(n)$ and $(\Theta(q))^{k}=\sum_{n \geq 0} S_{k}(n) q^{n}$. In other words, $\Theta^{k}$ is a generating function for $S_{k}(n)$. But, how do we find closed formulas for $S_{k}(n)$ ? This is where the theory of modular forms becomes particularly useful, for it provides an alternative description of the coefficients of $\Theta^{k}$.

It turns out that, for even $k \geq 2$, the function $\Theta^{k}$ is a modular form of weight $\frac{k}{2}$ (more precisely, it is a modular form for the group $\left.\Gamma_{1}(4)\right)$, and the space of all modular forms of weight $\frac{k}{2}$, denoted by $M_{\frac{k}{2}}\left(\Gamma_{1}(4)\right.$ ), is finite dimensional (we will carefully define all these terms later). For instance, let $k=4$. Then $M_{2}\left(\Gamma_{1}(4)\right)$, the space of modular forms of weight $\frac{4}{2}=2$ for $\Gamma_{1}(4)$, is a 2 -dimensional $\mathbb{C}$-vector space and a basis is given by modular forms with $q$-expansions:

$$
\begin{aligned}
f(q) & =1+24 q^{2}+24 q^{4}+96 q^{6}+24 q^{8}+144 q^{10}+96 q^{12}+\cdots \\
g(q) & =q+4 q^{3}+6 q^{5}+8 q^{7}+13 q^{9}+12 q^{11}+14 q^{13}+\cdots .
\end{aligned}
$$

Therefore, $\Theta^{4}(q)=\lambda f(q)+\mu g(q)$ for some constants $\lambda, \mu \in \mathbb{C}$. We may compare $q$-expansions to find the values of $\lambda$ and $\mu$ :

$$
\begin{aligned}
\Theta^{4}(q)=\sum_{n \geq 0} S_{4}(n) q^{n} & =1+8 q+24 q^{2}+32 q^{3}+24 q^{4}+\cdots \\
\lambda f(q)+\mu g(q) & =\lambda+\mu q+24 \lambda q^{2}+4 \mu q^{3}+\cdots
\end{aligned}
$$

Therefore, it is clear that $\lambda=1$ and $\mu=8$, so $\Theta^{4}=f+8 g$. Since the expansions of $f$ and $g$ are easy to calculate (for example, using Sage; see Appendix A.2), we can easily calculate the coefficients of the $q$-expansion of $\Theta$ and, therefore, values of $S_{4}(n)$.

The exact formulas given above for $S_{k}(n)$, however, follow from some deeper facts. Here is a sketch of the ideas involved (the reader may skip these details for now and return here after reading Chapter 4): given $\Theta^{4}=\sum c_{n} q^{n}$ and $F(q)=\sum\left(\sum_{d \mid n} d\right) q^{n}$, one can find an eigenvector $G(q)=\sum b_{n} q^{n}$ for a collection of linear maps $T_{n}$ (the socalled Hecke operators, $\left.T_{n}: M_{2}\left(\Gamma_{1}(4)\right) \rightarrow M_{2}\left(\Gamma_{1}(4)\right)\right)$ among spaces of modular forms, i.e., $T_{n}(G)=\lambda_{n} G$ for $n>1$, and the eigenvalues $\lambda_{n}=b_{n} / b_{1}=\sum_{d \mid n} d$. Moreover, the eigenvector $G$ can be written explicitly as a combination of $\Theta^{4}$ and $F$. Finally, one can show that the coefficients $c_{n}$ must be given by the formula $c_{n}=8 \sum_{d \mid n, 4 \nmid d} d$ (see [Kob93], III, §5, for more details).

## 1.3. $L$-functions

An $L$-function is a function $L(s)$, usually given as an infinite series of the form

$$
L(s)=\sum_{n=1}^{\infty} a_{n} n^{-s}=\sum_{n=1}^{\infty} \frac{a_{n}}{n^{s}}=a_{1}+\frac{a_{2}}{2^{s}}+\frac{a_{3}}{3^{s}}+\cdots
$$

with some coefficients $a_{n} \in \mathbb{C}$. Typically, the function $L(s)$ converges for all complex numbers $s$ in some half-plane (i.e., those $s$ with real part larger than some constant), and in many cases $L(s)$ has an analytic or meromorphic continuation to the whole complex plane. Mathematicians are interested in $L$-functions because they are objects from analysis that, sometimes, capture very interesting algebraic information.

Example 1.3.1 (The Riemann zeta function). The Riemann zeta function, usually denoted by $\zeta(s)$, is perhaps the most famous $L$ function:

$$
\zeta(s)=\sum_{n=1}^{\infty} \frac{1}{n^{s}}=1+\frac{1}{2^{s}}+\frac{1}{3^{s}}+\cdots
$$

The reader may already know some values of $\zeta$. For example $\zeta(2)=$ $\sum \frac{1}{n^{2}}$ is convergent by the $p$-series test, and its value is $\pi^{2} / 6$ (this value can be computed using Fourier analysis and Parseval's equality). The connection between $\zeta(s)$ and number theory comes from the fact that $\zeta(s)$ has an Euler product:

$$
\begin{aligned}
\zeta(s) & =\sum_{n=1}^{\infty} \frac{1}{n^{s}}=\prod_{p \text { prime }} \frac{1}{1-p^{-s}} \\
& =\left(\frac{1}{1-2^{-s}}\right) \cdot\left(\frac{1}{1-3^{-s}}\right) \cdot\left(\frac{1}{1-5^{-s}}\right) \cdots
\end{aligned}
$$

This Euler product is not difficult to establish (Exercise 1.4.8) and has the very interesting consequence that any information on the distribution of the zeros of $\zeta(s)$ can be translated into information about the distribution of prime numbers among the natural numbers.

Example 1.3.2 (Dirichlet $L$-function). Let $a, N \in \mathbb{N}$ be relatively prime integers. Are there infinitely many primes $p$ of the form $a+k N$ (i.e., $p \equiv a \bmod N$ ) for $k \geq 0$ ? The answer is yes and this fact, known as Dirichlet's theorem on primes in arithmetic progressions, was first proved by Dirichlet using a particular kind of $L$-function that we know today as a Dirichlet $L$-function.

Let $N>0$. A Dirichlet character (modulo $N$ ) is a function $\chi:(\mathbb{Z} / N \mathbb{Z})^{\times} \rightarrow \mathbb{C}^{\times}$that is a homomorphism of groups, i.e., $\chi(n m)=$ $\chi(n) \chi(m)$ for all $n, m \in(\mathbb{Z} / N \mathbb{Z})^{\times}$. Notice that $\chi(n) \in \mathbb{C}$ and $\chi(n)^{\varphi(N)}=1$ for all $\operatorname{gcd}(n, N)=1$. Therefore, $\chi(n)$ must be a root of unity. We extend $\chi$ to $\mathbb{Z}$ as follows. Let $a \in \mathbb{Z}$. If $\operatorname{gcd}(a, N)=1$, then $\chi(a)=\chi(a \bmod N)$. Otherwise, if $\operatorname{gcd}(a, N) \neq 1$, then $\chi(a)=0$.

A Dirichlet $L$-function is a function of the form

$$
L(s, \chi)=\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s}}
$$



Figure 4. Johann Peter Gustav Lejeune Dirichlet (18051859) and Georg Friedrich Bernhard Riemann (1826-1866).
where $\chi$ is a given Dirichlet character. For example, one can take $\chi_{0}$ to be the trivial Dirichlet character, i.e., $\chi_{0}(n)=1$ for all $n \geq 1$. Then $L\left(s, \chi_{0}\right)$ is the Riemann zeta function $\zeta(s)$. Dirichlet $L$-functions also have Euler products:

$$
L(s, \chi)=\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s}}=\prod_{p} \frac{1}{1-\chi(p) p^{-s}} .
$$

The idea of the proof of Dirichlet's theorem generalizes the following proof, due to Euler, of the infinitude of the primes. Consider $\zeta(s)=\sum_{n=1}^{\infty} \frac{1}{n^{s}}=\prod_{p} \frac{1}{1-p^{-s}}$ and suppose there are only finitely many primes. Then the product over all primes is finite, and therefore its value at $s=1$ would be finite (a rational number, in fact). However, $\zeta(1)=\sum_{n=1}^{\infty} 1 / n$ is the harmonic series, which diverges! Therefore, there must be infinitely many prime numbers.

Dirichlet adapted this argument by looking instead at a different function:

$$
\Psi_{a, N}(s)=\sum_{p \equiv a \bmod N} \frac{1}{p^{s}} .
$$

He showed that (a) for every non-trivial Dirichlet character $\chi$ modulo $N$, we have $L(1, \chi) \neq 0$ or $\infty$, and (b) this implies that $\Psi_{a, N}(1)$ diverges to $\infty$. Part (b) follows from the equality

$$
\begin{aligned}
& \log (\zeta(s))+\sum_{\substack{\chi \not \bmod N \\
\chi \neq 1}} \chi(a)^{-1} \log (L(s, \chi)) \\
& =\phi(N)\left(\sum_{p \equiv a \bmod N} \frac{1}{p^{s}}\right)+g(s)
\end{aligned}
$$

where $g(s)$ is a function with $g(1)$ finite, and $\phi$ is the Euler $\phi$-function. Therefore, there cannot be a finite number of primes of the form $p \equiv a \bmod N$.

Example 1.3.3 (Representations of integers as sums of squares). Is the number $n>0$ a sum of three integer squares? In Subsection 1.2, we saw formulas for the number of representations of an integer as a sum of $k=2,4,6$ and 8 integer squares, but we avoided the same question for odd $k$. The known formulas for $S_{3}(n), S_{5}(n)$ and $S_{7}(n)$ involve values of Dirichlet $L$-functions.

Let us first define the Dirichlet character that we shall use here. The reader should be familiar with the Legendre symbol $\left(\frac{n}{p}\right)$, which is equal to 0 if $p \mid n$, equal to 1 if $n$ is a square $\bmod p$, and equal to -1 if $n$ is not a square $\bmod p$. Let $m>0$ be a natural number with prime factorization $m=\prod_{i} p_{i}$ (the primes are not necessarily distinct). First we define

$$
\left(\frac{n}{2}\right)= \begin{cases}0 & \text { if } n \text { is even } \\ 1 & \text { if } n \equiv \pm 1 \bmod 8 \\ -1 & \text { if } n \equiv \pm 3 \bmod 8\end{cases}
$$

Now we are ready to define the Kronecker symbol of $n$ over $m>0$ by

$$
\left(\frac{n}{m}\right)=\prod_{i}\left(\frac{n}{p_{i}}\right)
$$

For any $n>0$, the symbol $(\underline{-n})$ induces a Dirichlet character $\chi_{n}$ defined by $\chi_{n}(a)=\left(\frac{-n}{a}\right)$, and we can define the associated $L$-function
by

$$
L\left(s, \chi_{n}\right)=\sum_{a=1}^{\infty} \frac{\chi_{n}(a)}{a^{s}}
$$

We are ready to write down the formula for $S_{3}(n)$, due to Gauss, Dirichlet and Shimura (there are also formulas for $S_{5}(n)$, due to Eisenstein, Smith, Minkowski and Shimura, and a formula for $S_{7}(n)$, also due to Shimura). For simplicity, let us assume that $n$ is odd and square free (for the utmost generality, please check [Shi02]):

$$
S_{3}(n)= \begin{cases}0 & \text { if } n \equiv 7 \bmod 8 \\ \frac{24 \sqrt{n}}{\pi} L\left(1, \chi_{n}\right) & \text { otherwise }\end{cases}
$$

The reader is encouraged to investigate this problem further by attempting Exercises 1.4.6 and 1.4.7.

### 1.4. Exercises

Exercise 1.4.1. Use the divisibility properties of integers to show that the only solutions to $y^{2}=x(x+1)(x+2)$ with $x, y \in \mathbb{Z}$ are $(0,0),(-1,0)$ and $(-2,0)$. (Hint: If $a$ and $b$ are relatively prime and $a b$ is a square, then $a$ is a square and $b$ is a square.)

Exercise 1.4.2. Find all the Pythagorean triples $(a, b, c)$, i.e., $a, b, c \in$ $\mathbb{Z}$ and $a^{2}+b^{2}=c^{2}$, such that $b^{2}+c^{2}=d^{2}$ for some $d \in \mathbb{Z}$. In other words, find all the integers $a, b, c, d$ such that $(a, b, c)$ and $(b, c, d)$ are both Pythagorean triples. (Hint: You may assume that $y^{2}=$ $x(x+1)(x+2)$ has no rational points other than $(0,0),(-1,0)$ and $(-2,0)$.)

Exercise 1.4.3. Prove Proposition 1.1.3; i.e., show that $f((a, b, c))$ is a point in $E_{n}$, that $g((x, y))$ is a triangle in $C_{n}$ and that $f(g((x, y)))=$ $(x, y)$ and $g(f((a, b, c)))=(a, b, c)$.

Exercise 1.4.4. Calculate $S_{4}(n)$, for $n=1,3,5,6$, by hand, using Jacobi's formula and also by finding all possible ways of writing $n$ as a sum of 4 squares.

Exercise 1.4.5. The goal of this problem is to find the $q$-expansion of $\Theta^{6}(q)$ :
(1) Find by hand the values of $S_{6}(n)$, for $n=0,1,2$; i.e., find all possible ways to write $n=0,1,2$ as a sum of 6 squares.
(2) Using Sage, calculate the dimension of $M_{\frac{k}{2}}\left(\Gamma_{1}(4)\right)$ (see Appendix A.2) and a basis of modular forms for $k=6$.
(3) Write $\Theta^{6}$ as a linear combination of the basis elements found in part 2.
(4) Use part 3 to write the $q$-expansion of $\Theta^{6}$ up to $O\left(q^{20}\right)$.
(5) Use the expansion of $\Theta^{6}$ to verify that $S_{6}(4)=252$. Also, calculate $S_{6}(19)$ using Jacobi's formula and verify that it coincides with the coefficient of $\Theta^{6}$ in front of the $q^{19}$ term.

Exercise 1.4.6. Show that any integer $n \equiv 7 \bmod 8$ cannot be represented as a sum of three integer squares.

Exercise 1.4.7. Find the number of representations of $n=3$ as a sum of 3 squares. Then compare your result with the value of the formula given in Example 1.3.3; i.e., use a computer to approximate

$$
S_{3}(3)=\frac{24 \sqrt{3}}{\pi} L\left(1, \chi_{3}\right)=\frac{24 \sqrt{3}}{\pi} \sum_{a=1}^{\infty} \frac{\left(\frac{-3}{a}\right)}{a}
$$

by adding the first 10,000 terms of $L\left(1, \chi_{3}\right)$. Do the same for $n=5$ and $n=11$. Does the formula seem to work for $n=2$ ? (Note: the command kronecker ( $-\mathrm{n}, \mathrm{m}$ ) calculates the Kronecker symbol $\left(\frac{-n}{m}\right)$ in Sage.)

Exercise 1.4.8. Prove that the Riemann zeta function $\zeta(s)=\sum_{n=1}^{\infty} \frac{1}{n^{s}}$ has an Euler product; i.e., prove the following formal equality of series

$$
\sum_{n=1}^{\infty} \frac{1}{n^{s}}=\prod_{p \text { prime }} \frac{1}{1-p^{-s}}
$$

(Hint: There are two possible approaches:
Hint (a). Expand the right-hand side using the Fundamental Theorem of Arithmetic and the algebraic equality $\frac{1}{1+x}=\sum_{k=0}^{\infty} x^{k}$. [This approach helps build an intuition about what is going on, but may be hard to write into a rigorous proof]
Hint (b). Calculate $\left(1-1 / 2^{s}\right) \zeta(s)$ and $\left(1-1 / 3^{s}\right)\left(1-1 / 2^{s}\right) \zeta(s)$, etc.)

