Preface

Over two hundred years ago, Jean Baptiste Joseph Fourier began to work on the theory of heat and how it flows. His book *Théorie Analytique de la Chaleur* (*The Analytic Theory of Heat*) was published in 1822. In that work, he began the development of one of the most influential bodies of mathematical ideas, encompassing Fourier theory and the field now known as *harmonic analysis* that has grown from it. Since that time, the subject has been exceptionally significant both in its theoretical implications and in its enormous range of applicability throughout mathematics, science, and engineering.

On the theoretical side, the theory of Fourier series was a driving force in the development of mathematical analysis, the study of functions of a real variable. For instance, notions of convergence were created in order to deal with the subtleties of Fourier series. One could also argue that set theory, including the construction of the real numbers and the ideas of cardinality and countability, was developed because of Fourier theory. On the applied side, all the signal processing done today relies on Fourier theory. Everything from the technology of mobile phones to the way images are stored and transmitted over the Internet depends on the theory of Fourier series. Most recently the field of wavelets has arisen, uniting its roots in harmonic analysis with theoretical and applied developments in fields such as medical imaging and seismology.

In this book, we hope to convey the remarkable beauty and applicability of the ideas that have grown from Fourier theory. We present for an advanced undergraduate and beginning graduate student audience the basics of harmonic analysis, from Fourier's heat equation, and the decomposition of functions into sums of cosines and sines (frequency analysis), to dyadic harmonic analysis, and the decomposition of functions into a Haar basis (time localization). In between these two different ways of decomposing functions there is a whole world of time/frequency analysis (wavelets). We touch on aspects of that world, concentrating on the Fourier and Haar cases.

The book is organized as follows. In the first five chapters we lay out the classical theory of Fourier series. In Chapter 1 we introduce Fourier series for periodic functions and discuss physical motivations. In Chapter 2 we present a review of different modes of convergence and appropriate classes of functions. In Chapter 3 we discuss pointwise convergence for Fourier series and the interplay with differentiability. In Chapter 4 we introduce approximations of the identity, the Dirichlet, Fejér, and Poisson kernels, and summability methods for Fourier series. Finally in Chapter 5 we discuss inner-product vector spaces and the completeness of the trigonometric basis in $L^2(\mathbb{T})$.

In Chapter 6, we examine discrete Fourier and Haar analysis on finite-dimensional spaces. We present the Discrete Fourier Transform and its celebrated cousin the Fast Fourier Transform, an algorithm discovered by Gauss in the early 1800s and rediscovered by Cooley and Tukey in the 1960s. We compare them with the Discrete Haar Transform and the Fast Haar Transform algorithm.

In Chapters 7 and 8 we discuss the Fourier transform on the line. In Chapter 7 we introduce the time–frequency dictionary, convolutions, and approximations of the identity in what we call paradise (the Schwartz class). In Chapter 8, we go beyond paradise and discuss the Fourier transform for tempered distributions, construct a time–frequency dictionary in that setting, and discuss the delta distribution as well as the principal value distribution. We survey the mapping properties of the Fourier transform acting on L^p spaces, as well as a few canonical applications of the Fourier transform including the Shannon sampling theorem.

In Chapters 9, 10, and 11 we discuss wavelet bases, with emphasis on the Haar basis. In Chapter 9, we survey the windowed Fourier transform, the Gabor transforms, and the wavelet transform. We develop in detail the Haar basis, the geometry of dyadic intervals, and take some initial steps into the world of dyadic harmonic analysis. In Chapter 10, we discuss the general framework of multiresolution analysis for constructing other wavelets. We describe some canonical applications to image processing and compression and denoising, illustrated in a case study of the wavelet-based FBI fingerprint standard. We state and prove Mallat's Theorem and explain how to search for suitable multiresolution analyses. In Chapter 11, we discuss algorithms and connections to filter banks. We revisit the algorithm for the Haar basis and the multiresolution analysis that they induce. We describe the cascade algorithm and how to implement the wavelet transform given multiresolution analysis, using filter banks to obtain the Fast Wavelet Transform. We describe some properties and design features of known wavelets, as well as the basics of image/signal denoising and compression.

To finish our journey, in Chapter 12 we present the Hilbert transform, the most important operator in harmonic analysis after the Fourier transform. We describe the Hilbert transform in three ways: as a Fourier multiplier, as a singular integral, and as an average of Haar shift operators. We discuss how the Hilbert transform acts on the function spaces L^p, as well as some tools for understanding the L^p spaces. In particular we discuss the Riesz–Thorin Interpolation Theorem and as an application derive some of the most useful inequalities in analysis. Finally we explain the connections of the Hilbert transform with complex analysis and with Fourier analysis.

Each chapter ends with ideas for projects in harmonic analysis that students can work on rather independently, using the material in our book as a springboard. We have found that such projects help students to become deeply engaged in the subject matter, in part by giving them the opportunity to take ownership of a particular topic. We believe the projects will be useful both for individual students using our book for independent study and for students using the book in a formal course.

The prerequisites for our book are advanced calculus and linear algebra. Some knowledge of real analysis would be helpful but is not required. We introduce concepts from Hilbert spaces, Banach spaces, and the theory of distributions as needed. Chapter 2 is an interlude about analysis on intervals. In the Appendix we review vector, normed, and inner-product spaces, as well as some key concepts from analysis on the real line.

We view the book as an introduction to serious analysis and computational harmonic analysis through the lens of Fourier and wavelet analysis.

Examples, exercises, and figures appear throughout the text. The notation $A := B$ and $B =: A$ both mean that A is defined to be the quantity B. We use the symbol $\Box$ to mark the end of a proof and the symbol $\Diamond$ to mark the end of an example, exercise, aside, remark, or definition.

Suggestions for instructors

The first author used drafts of our book twice as the text for a one-semester course on Fourier analysis and wavelets at the University of New Mexico, aimed at upper-level undergraduate and graduate students. She covered most of the material in Chapters 1 and 3–11 and used a selection of the student projects, omitting Chapter 12 for lack of time. The concepts and ideas in Chapter 2 were discussed as the need arose while lecturing on the other chapters, and students were encouraged to revisit that chapter as needed.

One can design other one-semester courses based on this book. The instructor could make such a course more theoretical (following the L^p stream, excluding Chapters 6, 10, and 11) or more computational (excluding the L^p stream and Chapter 12 and including Chapter 6 and parts of Chapters 10 and 11). In both situations Chapter 2 is a resource, not meant as lecture material. For a course exclusively on Fourier analysis, Chapters 1–8 have more than enough material. For an audience already familiar with Fourier series, one could start in Chapter 6 with a brief review and then do the Discrete Fourier and Haar Transforms, for which only linear algebra is needed, and move on to Fourier integrals, Haar analysis, and wavelets. Finally, one could treat the Hilbert transform or instead supplement the course with more applications, perhaps inspired by the projects. We believe that the emphasis on the Haar basis and on dyadic harmonic analysis make the book distinctive, and we would include that material.

The twenty-four projects vary in difficulty and sophistication. We have written them to be flexible and open-ended, and we encourage instructors to modify them and to create their own. Some of our projects are structured sequentially, with each part building on earlier parts, while in other projects the individual parts are independent. Some projects are simple in form but quite ambitious, asking students to absorb and report on recent research papers. Our projects are suitable for individuals or teams of students.

It works well to ask students both to give an oral presentation on the project and to write a report and/or a literature survey. The intended audience is another student at the same stage of studies but without knowledge of the specific project content. In this way, students develop skills in various types of mathematical communication, and students with differing strengths get a chance to shine. Instructors can reserve the last two or three weeks of lectures for student talks if there are few enough students to make this practicable. We find this to be a worthwhile use of time.

It is fruitful to set up some project milestones throughout the course in the form of a series of target dates for such tasks as preparing outlines of the oral presentation and of the report, drafting summaries of the first few items in a literature survey, rehearsing a presentation, completing a draft of a report, and so on. Early planning and communication here will save much stress later. In the second week of classes, we like to have an initial ten-minute conversation with each student, discussing a few preliminary sentences they have written on their early ideas for the content and structure of the project they plan to do. Such a meeting enables timely intervention if the proposed scope of the project is not realistic, for instance. A little later in the semester, students will benefit from a brief discussion with the instructor on the content of their projects and their next steps, once they have sunk their teeth into the ideas.

Students may find it helpful to use the mathematical typesetting package LaTeX for their reports and software such as Beamer to create slides for their oral presentations. Working on a project provides good motivation for learning such professional tools. Here is a natural opportunity for instructors to give formal or informal training in the use of such tools and in mathematical writing and speaking. We recommend Higham's book [**Hig**] and the references it contains as an excellent place to start. Instructors contemplating the task of designing and planning semester-long student projects will find much food for thought in Bean's book [**Bea**].

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Some parts of Chapters 10 and 11 are drawn from material in the first author's book [**MP**]. This material originated in lecture notes by the first author for minicourses delivered by her (in Argentina in 2002 and in Mexico in 2006) and by Martin Mohlenkamp (in Honduras in 2004; he kindly filled in for her when her advanced pregnancy made travel difficult).

Early drafts of the book were written while the second author was a member of the Department of Mathematics, Harvey Mudd College, Claremont, California.

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In writing this book, we have drawn on many sources, too numerous to list in full here. An important debt is to E. M. Stein and R. Shakarchi's book [**SS03**], which we used in our Princeton lectures as the main text for classical Fourier theory. We were particularly inspired by T. Körner's book [**Kör**].

Students who took the wavelets courses at the University of New Mexico proofread the text, suggesting numerous improvements. We have incorporated several of their ideas for projects. Harvey Mudd College students Neville Khambatta and Shane Markstrum and Claremont Graduate University student Ashish Bhan typed LaTeX notes and created Matlab figures for the second author's wavelets course. In the final stages, Matthew Dahlgren and David Weirich, graduate students at the University of New Mexico, did a careful and thoughtful reading, giving us detailed comments from the point of view of an informed reader working through the book without a lecture course. They also contributed to the subject index. Lisa Schultz, an Honours student at the University of South Australia, completed both indexes and helped with the final touches. Over the years colleagues and students have given suggestions on draft versions, including Wilfredo Urbina, University of New Mexico graduate students Adam Ringler and Elizabeth Kappilof, and Brown University graduate student Constance Liaw. We thank Jorge Aarão for reading and commenting on our almost-final manuscript.

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María Cristina Pereyra, University of New Mexico
Lesley A. Ward, University of South Australia
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