# Preface

This course is dedicated to the proposition that all geometries are created equal. This was first pointed out by Felix Klein, who declared that *each individual geometry is a set with a transformation group acting on it.* Here we shall study geometries from this point of view not only as individual objects, but also in their social life, i.e., in their relationships (called *morphisms* or *equivariant maps*) within their socium: the *category of geometries.*

Of course, some geometries are more equal than others. Accordingly, we will ignore the most common ones (affine and Euclidean geometries, vector spaces), assuming that they are known to the reader, and concentrate on the most distinguished and beautiful ones. (We assume that our readers are familiar with elementary Euclidean geometry; those who aren't may refer to Chapter 0, which is a précis of the subject, whenever the need arises.)

The reader should not be deceived by the words *groups, morphisms, categories* into thinking that this is a formal algebraic or (heaven forbid) an analytic (coordinate) treatment of geometric topics; it suffices to glance at the numerous figures in the book to realize that we constantly privilege the *visual aspect.*

Category *theory* is not used in this course, but we do use some basic category *language*, which, as the reader will see, is extremely natural in the geometric context. Thus, Cayley's famous phrase: "projective geometry is all geometry" can be given a precise mathematical meaning by using the term *subgeometry* (which means "image by an injective equivariant map"). In the context of this course, it may be rephrased as follows: "The geometries studied in this book (including the three classical ones – hyperbolic, elliptic, and Euclidean) are (almost all) subgeometries of projective geometry."

There is very little in the main body of this book about the axiomatic approach to geometry. This is one of the author's biases: I believe that the classical axiom systems for, say, Euclidean and hyperbolic geometry are hopelessly outdated and no longer belong in contemporary mathematics. Their place is in the *history of mathematics* and in the philosophy of science. Accordingly, here they only appear in one chapter, devoted to the fascinating history of the creation of non-Euclidean geometry, while a detailed treatment of the axiom systems of Euclid and Hilbert is relegated to Appendices A and B.

The use of the plural (*Geometries*) in the title of the book indicates that, to my mind, there is no such *subject* as "geometry", but there are some concrete *mathematical objects* called geometries. In the singular, the word "geometry" should be understood as *a way of thinking about mathematics*, in fact the original one: in Ancient Greece, the word "geometry" was used as a synonym for "mathematics". One can and should think geometrically not only when working with circles and triangles, but also when using commutative diagrams, morphisms, or groups. The famous phrase written above the entrance to Plato's Academy

*Let no one enter who is not a geometer*

should also be displayed on the gates leading to the world of mathematics.

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I will not give a systematic summary of the contents of this course in this Preface, referring the reader to the Table of Contents. Looking at it, the well prepared reader may wonder why some of her/his favorite geometrical topics do not appear in this book among the "distinguished and beautiful" ones promised above. Let me comment on some missing topics, explaining why they are not treated here.

First, there is no *algebraic geometry* in this book. This is because the author believes that this beautiful field of mathematics belongs to algebra, not geometry. Indeed, the mathematicians doing algebraic geometry are typically algebraists, and this is not only true of the great French school (following Grothendieck and his schemes), but also of the more classical Russian school.

Neither is there any *differential geometry*: in its classical low-dimensional aspect it is usually developed in calculus books (where it indeed belongs), in its higher dimensional modern aspect is a part of analysis (under the title of "Calculus on Manifolds") and topology (under the title "Differential Topology").

Other missing topics include *convex geometry* (part of analysis and more specifically optimization theory as "convex analysis"), *symplectic geometry* (part of classical mechanics and dynamical systems), *contact geometry* (part of differential equations), etc.

Of course, contact geometry (say) is formally a geometry in the sense of Klein. In fact, the ideology of transformation groups comes from Sophus Lie as much (if not more than) from Felix Klein, but the context of Lie's beautiful contact geometry is definitely differential equations.

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This book is based on lectures given in the framework of semester courses taught in Russian at the Independent University of Moscow to first-year students in 2003 and 2006, for which I prepared handouts, written in "simple English" and posted on the IUM web site. These lecture notes were published as a 100-page booklet by the Moscow Center of Continuous Mathematical Education in 2006 and used in geometry courses taught to Math in Moscow students of the Independent University.

The brevity of a short semester course (13 lectures) made me restrict the study of such classical geometries as hyperbolic and projective to dimension two, regretfully shelving the three-dimensional case. But then there are many occasions in the course for developing one's *intuition of space*, and indeed the general case is easier to treat from the linear algebra coordinate point of view than from the rather visual synthetic approach characterizing this course. For the reader who wants to go further, I strongly recommend the book by Marcel Berger [2]. I should add that, although my approach to the subject is very different from Berger's, I am heavily indebted to that remarkable book in several specific parts of the exposition. For those who would like to learn more about the axiomatic approach to the classical geometries, there is no better book, to my mind, than N.V. Efimov's *Higher Geometry* [6].

An important, if not the most important, aspect of this course are the problems, which appear at the end of each chapter. It is by solving these problems, much more than by learning the theory, that the reader will become capable of thinking and working geometrically. The sources of the problems are varied. Many were "stolen" from books written by my friend and favorite co-author Victor Prasolov. In many cases, I simply don't know where they originally come from. As handouts for the exercise classes, they were grouped together by Irina Paramonova, who contributed several, as did the other instructors conducting the exercise classes (Vladimir Ivanov and Oleg Karpenkov). I am grateful to all of the people mentioned above, and also to Mikhail Panov, Anton Ponkrashov, and Victor Shuvalov, who produced the computer versions of most of the illustrations, to M.I. Bykova, who corrected many errors in the original handouts, to Victor Prasolov, who found many more in the first draft of this book and to the anonymous referees, whose constructive criticism was very helpful. Finally, I am indebted to Sergei Gelfand, without whose encouragement this book would never have been written.