Chapter 1

Toy Geometries and Main Definitions

In this chapter, we study five toy examples of geometries (symmetries of the equilateral triangle, of the square, the cube, the circle and the sphere) and a model of the geometry of the so-called elliptic plane. These examples prepare us for the main definition (given in Sec. 1.4) of this course: a geometry in the sense of Klein is a set with a transformation group acting on it. Before that, we present some useful general notions related to transformation groups. Further, we study the relationships (called morphisms or equivariant maps) between different geometries, thus introducing the category of all geometries. The notions introduced in this chapter are illustrated by some problems (dealing with toy models of geometries) collected at the end of the chapter.

But before we begin with these topics, we briefly recall some terminology from elementary Euclidean geometry.

1.1. Isometries of the Euclidean plane and space

We assume that the reader is familiar with the basic notions and facts of Euclidean geometry in the plane and in space (these notions and facts are summarized in Chapter 0). One can think of Euclidean geometry as an axiomatic theory (not too rigorously taught in high school) or as a small chapter of linear algebra (the plane $\mathbb{R}^2$ and the space $\mathbb{R}^3$ supplied with the standard metric). It is irrelevant to us which of these two points of view is adopted by the reader, and the aim of this subsection is merely to fix some terminology.

An *isometry* of the Euclidean plane $\mathbb{R}^2$ (or space $\mathbb{R}^3$) is a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ (respectively, $f : \mathbb{R}^3 \to \mathbb{R}^3$) which preserves the distance d between points, i.e.,

$$d(f(P), f(Q)) = d(P, Q)$$

for any pair of points P, Q of the plane, respectively, of space). There are two types of isometries: those which preserve orientation (they are called *motions*, or sometimes *rigid motions*) and those that reverse orientation (*orientation-reversing isometries*).

In the plane, examples of motions are *parallel translations* (determined by a fixed *translation vector*) and *rotations* (determined by a pair (C, α), where C is the *center of rotation* and α is the *oriented angle of rotation*. In space, examples of motions are *parallel translations* and *rotations* (about an axis). Rotations in space are determined by pairs (l, α), where l is the *axis of rotation*, i.e., a straight line with a specified direction on it, and α is the *angle of rotation*; the rotation (l, α) maps any point M in space to the point M' obtained by rotating M in the plane Π perpendicular to l and passing through M by the angle α in a chosen direction; the direction of rotation is of course the same in each plane parallel to Π, it must be either clockwise or counterclockwise if one looks at the plane from "above", i.e., from some point of l obtained from the point $l \cap \Pi$ by moving in the direction of the axis.

Examples of orientation-reversing isometries in the plane are *reflections* (i.e., symmetries with respect to a line). In space, examples of orientation-reversing isometries are given by *mirror symmetries* (i.e., reflections with respect to planes) and *central symmetries* (i.e., reflections with respect to a point).

All other isometries of the Euclidean plane and space are *compositions* of those listed above.

The reader who is uncomfortable with the notions appearing in this subsection is invited to look at the relevant parts of Chapter 0.

1.2. Symmetries of some figures

1.2.1. Symmetries of the equilateral triangle. Consider all the isometries of the equilateral triangle $\triangle = ABC$, i.e., all the distance-preserving mappings of this triangle onto itself. (To be definite, we assume that the letters A, B, C have been assigned to vertices in counterclockwise order.) Denote by s_A, s_B, and s_C the reflections in the bisectors of angles A, B, C of the triangle. Denote by r_0, r_1, r_2 the counterclockwise rotations about its center of gravity by 0, 120, 240 degrees, respectively. Thus r_1 takes the vertex A to B, B to C, and C to A. These six transformations are all called *symmetries* of triangle ABC and the set that they constitute is denoted by Sym($\triangle$). Thus

$$\mathrm{Sym}(\triangle) = \{r_0,\, r_1,\, r_2,\, s_A,\, s_B,\, s_C\}.$$

There are no other isometries of $\triangle$. Indeed, any isometry takes vertices to vertices, each one-to-one correspondence between vertices entirely determines the isometry. (For example, the correspondence $A \to B$, $B \to A$, $C \to C$ determines the reflection s_C.) But there are only six different ways to assign the letters A, B, C to three points, so there cannot be more than 6 isometries of $\triangle$.

In a certain sense, Sym($\triangle$) is the same thing as the family of all permutations of the three letters A, B, C; this remark will be made precise in the next chapter.

We will use the symbol $*$ to denote the *composition* (or *product*) of isometries, in particular, of elements of Sym($\triangle$), and understand expressions such as $r_1 * s_A$ to mean that r_1 is performed first, and then followed by s_A. Obviously, when we compose two elements of Sym($\triangle$), we always obtain an element of Sym($\triangle$).

What element is the composition of two given ones can be easily seen by drawing a picture of the triangle ABC and observing what happens to it when the given isometries are successively performed, but this can also be done without any pictures: it suffices to follow the "trajectory" of the vertices A, B, C. Thus, in the example $r_1 * s_A$, the rotation r_1 takes the vertex A to B, and then B is taken to C by the symmetry s_A; similarly, $B \to C \to B$ and $C \to A \to A$, so that the vertices A, B, C are taken to C, B, A in that order, which means that $r_1 * s_A = s_B$.

The order in which symmetries are composed is important, because the resulting symmetry may change if we inverse the order. Thus, in our example, $s_A * r_1 = s_C \neq s_B$ (as the reader will readily check), so that $r_1 * s_A \neq s_A * r_1$. So for elements of Sym(△), composition is *noncommutative*.

The compositions of all possible pairs of symmetries of △ can be conveniently shown in the following *multiplication table*:

$*$	r_0	r_1	r_2	s_A	s_B	s_C
r_0	r_0	r_1	r_2	s_A	s_B	s_C
r_1	r_1	r_2	r_0	s_B	s_C	s_A
r_2	r_2	r_0	r_1	s_C	s_A	s_B
s_A	s_A	s_C	s_B	r_0	r_1	r_2
s_B	s_B	s_A	s_C	r_2	r_0	r_1
s_C	s_C	s_A	s_B	r_1	r_2	r_0

Here (for instance) the element s_V at the intersection of the fifth column and the third row is $s_B = r_1 * s_A$, the composition of r_1 and s_A in that order (first the transformation r_1 is performed, then s_A).

As we noted above, composition is *noncommutative*, and this is clearly seen from the table (it is not symmetric with respect to its main diagonal).

The composition operation $*$ in Sym(△) is (obviously) *associative*, i.e., $(i*j)*k = i*(j*k)$ for all $i, j, k \in$ Sym(△). The set Sym(□) contains the *identity* transformation r_0 (also denoted id or **1**). Any element i of Sym(□) has an *inverse* i^{-1}, i.e., an element such that $i * i^{-1} = i^{-1} * i =$**1**.

The set Sym(△) supplied with the composition operation $*$ is called the *symmetry group of the equilateral triangle* .

1.2.2. Symmetries of the square. Consider all the isometries of the unit square $\square = ABCD$, i.e., all the distance-preserving mappings of the square to itself.

Let us denote by s_H, s_V, and s_{ac}, s_{bd} the reflections in the horizontal and vertical midlines, and in the diagonals AC, BD, respectively. Denote by r_0, r_1, r_2, r_3 the rotations about the center of the

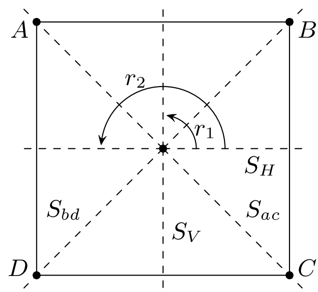

Figure 1.1. Symmetries of the square.

square by 0, 90, 180, 270 degrees, respectively. These eight transformations are all called *symmetries* of the square. We write

$$\mathrm{Sym}(\square) = \{r_0, r_1, r_2, r_3, s_H, s_V, s_{ac}, s_{bd}\}.$$

Just as in the case of the equilateral triangle, the composition of any two symmetries of the square is a symmetry of the square, and a *multiplication table*, indicating the result of all pairwise compositions, can be drawn up:

$*$	r_0	r_1	r_2	r_3	s_H	s_V	s_{ac}	s_{bd}
r_0	r_0	r_1	r_2	r_3	s_H	s_V	s_{ac}	s_{bd}
r_1	r_1	r_2	r_3	r_0	s_{ac}	s_{bd}	s_V	s_H
r_2	r_2	r_3	r_0	r_1	s_V	s_H	s_{bd}	s_{ac}
r_3	r_3	r_0	r_1	r_2	s_{bd}	s_{ac}	s_H	s_V
s_H	s_H	s_{bd}	s_V	s_{ac}	r_0	r_2	r_3	r_1
s_V	s_V	s_{ac}	s_H	s_{bd}	r_2	r_0	r_1	r_3
s_{ac}	s_{ac}	s_H	s_{bd}	s_V	r_1	r_3	r_0	r_2
s_{bd}	s_{bd}	s_V	s_{ac}	s_H	r_3	r_1	r_2	r_0

Here (for instance) the element s_V at the intersection of the sixth column and the fourth row is $s_V = r_2 * s_H$, the composition of r_2 and s_H in that order (first the transformation r_2 is performed, then s_V). Composition is *noncommutative*.

Obviously, composition is *associative*. The set $\mathrm{Sym}(\square)$ contains the *identity* transformation r_0 (also denoted id or $\mathbf{1}$). Any element i

of Sym(□) has an *inverse* i^{-1}, i.e., an element such that

$$i * i^{-1} = i^{-1} * i = \mathbf{1}.$$

The set Sym(□) supplied with the composition operation is called the *symmetry group of the square*.

1.2.3. Symmetries of the cube.

Let

$$I^3 = \{(x, y, z) \in \mathbb{R}^3 |\ 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$$

be the unit cube. A *symmetry* of the cube is defined as any isometric mapping of I^3 onto itself. The composition of two symmetries (of I^3) is a symmetry. How many are there?

Let us first count the orientation-preserving isometries of the cube (other than the identity), i.e., all its rotations (about an axis) by nonzero angles that take the cube onto itself.

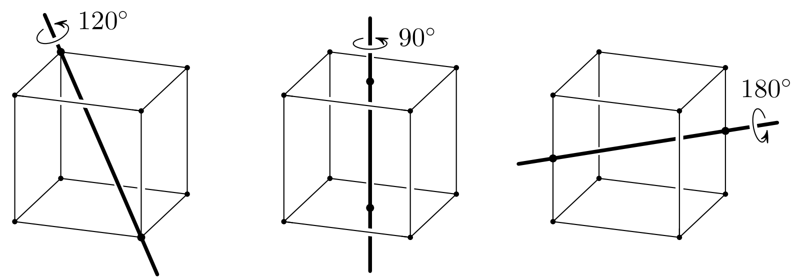

Figure 1.2. Rotations of the cube.

There are three axes of rotation joining the centers of opposite faces, and the rotation angles about each are $\pi/2$, π, $3\pi/2$. There are four axes of rotation joining opposite vertices, the rotation angles for each being $2\pi/3$ and $4\pi/3$. There are six axes of rotation joining midpoints of opposite edges, with only one nonzero rotation for each (by π). This gives us a total of $(3 \times 3) + (4 \times 2) + (6 \times 1) = 23$ orientation-preserving isometries, or 24 if we include the identity.

There are no other orientation-preserving isometries; at this point, we could prove this fact by a tedious elementary geometric counting argument, but we postpone the proof to Chapter 3, where it will be the immediate result of more general and sophisticated algebraic method.

There are also 24 orientation-reversing isometries of the cube. Listing them all is the task prescribed by Problem 1.2 (see the end of the chapter), a task which requires little more than a bit of spatial intuition.

Thus the cube has 48 isometries. All their pairwise compositions constitute a multiplication table, which is a 49 by 49 array of symbols, much too unwieldy to fit on a book page.

The set $\mathrm{Sym}(I^3)$ of all 48 symmetries of the cube supplied with the composition operation is called the *symmetry group of the cube*; it is associative, noncommutative, has an identity, and all its elements have inverses, just as the symmetry groups in the two previous examples.

1.2.4. Symmetries of the circle. Let

$$\bigcirc := \{(x, y) \in \mathbb{R}^2 |\ x^2 + y^2 = 1\}$$

be the unit circle. Denote by $\mathrm{Sym}(\bigcirc)$ the set of all its isometries. The elements of $\mathrm{Sym}(\bigcirc)$ are of two types: the rotations r_φ about the origin by angles φ, $\varphi \in [0, 2\pi)$, and the reflections in lines passing through the origin, s_α, $\alpha \in [0, \pi)$, where α denotes the angle from the x-axis to the line (in the counterclockwise direction). The composition of rotations is given by the (obvious) formula

$$r_\phi * r_\psi = r_{(\phi+\psi)\mathrm{mod}\ 2\pi},$$

where mod 2π means that we subtract 2π from the sum $\phi + \psi$ if the latter is greater than or equal to 2π.

The composition of two reflections s_α and s_β is a rotation by the angle $|\alpha - \beta|$,

$$s_\alpha * s_\beta = r_{2|\alpha-\beta|}.$$

The interested reader will readily verify this formula by drawing a picture and comparing the angles that will appear when the two reflections are composed.

The set of all isometries of the circle supplied with the composition operation is called the *symmetry group of the circle* and is denoted by Sym(◯). The group Sym(◯) has an infinite number of elements. As before, this group is associative, noncommutative, has an identity, and all its elements have inverses.

1.2.5. Symmetries of the sphere. Let

$$\mathbb{S}^2 := \{(x, y) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}$$

be the unit sphere. Denote by Sym($\mathbb{S}^3$) the set of all its isometries and by Rot($\mathbb{S}^3$) the set of all its rotations (by different angles about different axes passing through the center of the sphere). Besides rotations, the transformation group Sym($\mathbb{S}^3$) contains reflections in different planes passing through the center of the sphere, its symmetry with respect to its center, and the composition of these transformations with rotations.

Reflections in planes, unlike rotations, reverse the orientation of the sphere. This means that a little circle oriented clockwise on the sphere (if we are looking at it from the outside) is transformed by any reflection into a counterclockwise oriented circle, and the picture of a left hand drawn on the sphere becomes that of a right hand. Now a reflection in a line passing through the sphere's center does not reverse orientation (unlike reflections with respect to a line in the plane!) because a reflection of the sphere in a line is exactly the same transformation as a rotation about this line by 180°. On the other hand, a reflection of the sphere with respect to its center reverses its orientation (again, this is not the case for reflections of the plane with respect to a point).

Note that the composition of two reflections in planes is a rotation (see Problem 1.11), while the composition of two rotations is another rotation (by what angle and about what axis is the question discussed in Problem 1.12).

The set of all isometries of the sphere supplied with the composition operation is called the *symmetry group of the sphere* and is denoted by Sym($\mathbb{S}^3$). The group Sym($\mathbb{S}^3$) has an infinite number of elements. As before, this group is associative, noncommutative, has an identity, and all its elements have inverses.

1.2.6. A model of elliptic plane geometry. Consider the set $Ant(\mathbb{S}^2)$ of all pairs of antipodal points (i.e., points symmetric with respect to the origin) on the unit sphere $\mathbb{S}^2$); thus elements of $Ant(\mathbb{S}^2)$ are *not* ordinary points, but *pairs of points*. Now consider the family (that we denote O(3)) of all isometries of the space $\mathbb{R}^3$ that do not move the origin[1]. Clearly, any such isometry takes pairs of antipodal points to pairs of antipodal points, thus it maps the set $X = Ant(\mathbb{S}^2)$ to itself.

The family O(3) of transformations of the set $Ant(\mathbb{S}^2)$ is called the *isometry group of the Riemannian elliptic plane*. This is a much more complicated object than the previous "toy geometries". We will come back to its study in Chapter 6.

1.3. Transformation groups

1.3.1. Definitions and notation. Let X be a set (finite or infinite) of arbitrary elements called *points*. By definition, a *transformation group G acting on X* is a (nonempty) set G of bijections of X supplied with the composition operation $*$ and satisfying the following conditions:

(i) G is closed under composition, i.e., for any transformations $g, g' \in G$, the composition $g * g'$ belongs to G;

(ii) G is closed under taking inverses, i.e., for any transformation $g \in G$, its inverse g^{-1} belongs to G.

These conditions immediately imply that G contains the identity transformation. Indeed, take any $g \in G$; by (ii), we have $g^{-1} \in G$; by (i), we have $g^{-1} * g \in G$; but $g^{-1} * g =$id (by definition of inverse element), and so id$\in G$. Note also that composition in G is associative (because the composition of mappings is always associative).

If $x \in X$ and $g \in G$, then by xg we denote the image of the point x under the transformation g. (The more usual notation $g(x)$ is not convenient: we have $x(g * h) = (xg)h$, but $(g * h)(x) = h(g(x))$, with g and h appearing in reverse order in the right-hand side of this equality.)

[1]In linear algebra courses such transformations are called *orthogonal* and O(3) is called the *orthogonal group*.

1.3.2. Examples. The five toy geometries considered in the previous section all give examples of transformation groups. The five transformation groups Sym act (by isometries) on the equilateral triangle, the square, the cube, the circle, and the sphere, respectively. In the last example (1.2.5), the orthogonal group O(3) acts on pairs of antipodal points on the sphere, these pairs being regarded as "points" of the "elliptic plane".

More examples are given by the transformation group consisting of all the bijections $\mathrm{Bij}(X)$ of any set X. By definition of transformation groups, $\mathrm{Bij}(X)$ is the largest (by inclusion) transformation group acting on the given set X. At the other extreme, any set X has a transformation group consisting of a single element, the identity transformation.

When the set X is finite and consists of n objects, the group $\mathrm{Bij}(X)$ of all its bijections is called the *permutation group* on n objects and is denoted by S_n. This group is one of the most fundamental notions of mathematics, and plays a key role in abstract algebra, linear algebra, and, as we shall see already in the next chapter, in geometry.

1.3.3. Orbits, stabilizers, class formula. Let $(X : G)$ be some transformation group acting on a set X and let $x \in X$. Then the *orbit* of x is defined as

$$\mathrm{Orb}(x) := \{xg | g \in G\} \subset X,$$

and the *stabilizer* of x is

$$\mathrm{St}(x) := \{g \in G | xg = x\} \subset G.$$

For example, if $X = \mathbb{R}^2$ and G is the rotation group of the plane about the origin, then the set of orbits consists of the origin and all concentric circles centered at the origin; the stabilizer of the origin is the whole group G, and the stabilizers of all the other points of $\mathbb{R}^2$ are trivial (i.e., they consist of one element – the identity $\mathrm{id} \in G$).

Suppose $(X : G)$ is an action of a finite transformation group G on a finite set X. Then the number of points of G is (obviously) given by

$$|G| = |\mathrm{Orb}(x)| \times |\mathrm{St}(x)| \tag{1.1}$$

for any $x \in X$. Now let $A \subset X$ be a set that intersects each orbit at exactly one point. Then the number of points of X is given by the formula

$$|X| = \sum_{x\in A} \frac{|G|}{|\mathrm{St}(x)|}, \tag{1.2}$$

called the *class formula*. This formula, just as the previous one, follows immediately from definitions.

1.3.4. Fundamental domains. If X is a subset of $\mathbb{R}^n$ (e.g. $\mathbb{R}^n$ itself) and G is a transformation group acting on X, then a subset $F \subset X$ is called a *fundamental domain* of the action of G on X if

- F is an open set in X;
- $F \bigcap Fg = \varnothing$ for any $g \in G$ (except $g = \mathrm{id}$);
- $X = \bigcup_{g\in G} \mathrm{Clos}(Fg)$, where $\mathrm{Clos}(.)$ denotes the closure of a set.

For example, in the case of the square, a fundamental domain of the action of $\mathrm{Sym}(\square)$ is the interior of the triangle AOM, where O is the center of the square and M is the midpoint of side AB; of course $\mathrm{Sym}(\square)$ has many other fundamental domains. Thus fundamental domains are not necessarily unique. Moreover, fundamental domains don't always exist: for instance, $\mathrm{Sym}(S^1)$ (and other "continuous" geometries) do not have any fundamental domains.

1.3.5. Morphisms. According to one of the main principles of the category approach to mathematics, as soon as an important class of objects is defined, one must define their *morphisms*, i.e., the natural class of relationships between them. Following this principle, we say any mapping of transformation groups $\alpha : G \to H$ is a *homomorphism* if α respects the product (composition) structure, i.e.,

$$\alpha(g_1 * g_2) = \alpha(g_1) * \alpha(g_2) \quad \text{for all} \quad g_1, g_2 \in G. \tag{1.3}$$

Let us look at a few examples of homomorphisms:

(i) the mapping μ :Sym(□) $\to$ Sym(I^3) obtained by placing the square on top of the cube and extending its isometries to the whole cube in the natural way (e.g. assigning the rotation by 90° about the vertical axis passing through the centers of the horizontal faces of the cube to the 90° rotation of the square);

(ii) the mapping ν :Sym(△)→Sym(◯) assigning to each rotation of the triangle the rotation of the circle by the same angle and, to the reflections s_A, s_B, s_C, the reflections $s_0, s_{2\pi/3}, s_{4\pi/3}$ of the circle;

(iii) the mapping π :Rot(I^3) $\to$ Rot(□) induced by the projection of the cube on its bottom horizontal face Φ, i.e., assigning the identity element to all isometries of the cube that do not map Φ to Φ, and assigning, to all the other isometries of the cube, their restriction to Φ;

(iv) the mapping $\iota : S_3$ →Sym(△) assigning to each permutation of the symbols A, B, C the isometry that performs that permutation of the vertices A, B, C of the triangle.

The proof of the fact that these mappings are indeed homomorphisms, i.e., relation (1.3) holds, is a straightforward verification left to the reader.

A homomorphism α of transformation groups is said to be a *monomorphism* if the mapping α is injective (i.e., takes different elements to different ones). Examples of monomorphisms are the homomorphisms μ and ν above. A homomorphism α of transformation groups is said to be an *epimorphism* if α is surjective (i.e., is an onto map). An example is the mapping π above. A homomorphism α of transformation groups is said to be an *isomorphism* if it is both a monomorphism and an epimorphism, i.e., if the mapping α is bijective.

Two transformation groups G and H acting on two sets X and Y (the case $X = Y$ is not forbidden) are called *isomorphic* if there exists an isomorphism $\phi : G \to H$. If two isomorphic groups are finite, then they necessarily have the same number of elements (but the number of points in the sets on which they act can differ, as for example in the case of the isomorphic groups Sym(△) and S_3). Note in this context that Sym(□) and S_4 are *not* isomorphic, because the first of these groups consists of 8 elements, while the second has $4! = 24$.

1.3.6. Order. The *order* of a transformation group G is, by definition, the number of its elements; we denote it by $|G|$. Thus

$$|\operatorname{Sym}(\triangle)| = 6, \quad |\operatorname{Sym}(\square)| = 8, \quad |\operatorname{Sym}(\bigcirc)| = \infty.$$

The *order* of an element g of a transformation group G is, by definition, the least positive integer k such that the element $g*g*\cdots*g$ (k factors) is the identity; this integer is denoted by $\operatorname{ord}(g)$; if there is no such integer, then g is said to be of *infinite order*. For example, the rotation by 30° in Sym(◯) has order 12, while the rotation by $\sqrt{2}\pi$ is of infinite order. (The last fact follows from the irrationality of $\sqrt{2}$.)

1.3.7. Subgroups. Many important classes of objects have naturally defined "subobjects" (e.g. spaces and subspaces, manifolds and submanifolds, algebras and subalgebras). Transformation groups are no exception: if G is a transformation group and H is a subset of G, then H is called a *subgroup* of G if H itself is a group with respect to the composition operation $*$, i.e., if it satisfies the two conditions:

(i) H is closed under composition, i.e., $g, g' \in H \Longrightarrow g * g' \in H$;

(ii) H is closed under taking inverses, i.e., $g \in G \Longrightarrow g^{-1} \in G$.

According to this definition, any transformation group G has at least two subgroups: G itself and its one-element subgroup, i.e., the group $\{\mathrm{id}\}$ consisting of the identity element. We will call these two subgroups *trivial*, and all the others, *nontrivial*.

For example, the subset of all rotations of the group Sym(□) is a (nontrivial) subgroup of Sym(□) (of order 4), the set consisting of the identity element and a reflection s_α is a subgroup of order 2 in Sym(◯), while the set of all rotations of Sym(◯) is a subgroup of infinite order.

If g is an element of order k in a transformation group G, then the set of k elements$\{g,\, g*g, \dots,\, g*g*\cdots*g = \mathrm{id}\}$ is a subgroup of G of order k; it is called the *cyclic subgroup of G generated by g*.

This terminology is also used when g is an element of infinite order, but then the subgroup $\{\mathrm{id}, g, g * g, \ldots, g * g * \cdots * g, \ldots\}$ is also of infinite order.

1.4. The category of geometries

In this section, we present the main definition of this course (that of a geometry) and define some related basic concepts.

1.4.1. Geometries in the sense of Klein. A pair $(X : G)$, where X is a set and G is a transformation group acting on X will be called a *geometry in the sense of Klein*. The six examples in Sec. 1.2 define the geometry of the equilateral triangle, the geometry of the square, the geometry of the cube, the geometry of the circle, the geometry of the sphere, and the geometry of Riemann's elliptic plane. Another example is the set $\mathrm{Bij}(X)$ of all bijections of any set X.

1.4.2. The Erlangen program. The idea that geometries are sets of objects with transformation groups acting on them was first stated by the German mathematician Felix Klein in 1872 in a famous lecture at Erlangen. In that lecture (for an English translation, see [10]), he enunciated his views on geometries in the framework of what became known as the "Erlangen programme".

There is no doubt that all the geometries known in the times of Klein satisfy the property that he gave in his lecture, and so do all the geometries that were developed since then. However, this property can hardly be said to *characterize* geometries: it is much too broad. Thus, in the sense of the formal definition from the previous subsection, the permutation group is a geometry, and so is any topological space, any abstract group, even any set.

Nevertheless, we will stick to the notion of geometry given in 1.4.1 for want of a more precise formal definition. Such a definition, if it existed, would require supplying $(X : G)$ with additional structures (besides the action of G on X), but it is unclear at this time what these structures ought to be. If one looks at such branches of mathematics as global differential geometry, geometric topology, and differential topology, there appears to be no consensus among the experts about where geometry ends and topology begins in those fields.

The definition in 1.4.1 may be too broad, but it has the advantage of being extremely succinct and leading to the definition of a very natural category.

1.4.3. Morphisms. According to the general philosophy underlying the category language, a morphism from one geometry to another should be defined as a mapping of the set of points of one geometry to the set of points of the other that respects the actions of the corresponding transformation groups. More precisely, given two geometries $(G : X)$ and $(H : Y)$, a *morphism* (or an *equivariant map*) is any pair (α, f) consisting of a homomorphism of transformation groups $\alpha : G \to H$ and a mapping of sets $f : X \to Y$ such that

$$\boxed{f(xg) = (f(x))(\alpha(g))} \tag{1.4}$$

for all $x \in X$ and all $g \in G$. This definition is typical of the category approach in mathematics: at first glance, the boxed formula makes no sense at all (no wonder category theory is called abstract nonsense), but actually the definition is perfectly natural.

To see this, let us take any point $x \in X$ and let an arbitrary transformation $g \in G$ act on x, taking it to the point $xg \in X$. Under the map $f : X \to Y$, the point x is taken to the point $f(x) \in Y$ and the point xg is taken to the point $f(xg) \in Y$. How are these two points related? What transformation (if any) takes $f(x)$ to $f(xg)$? Clearly, if the pair of maps (f, α) respects the action of the transformation groups in X and Y, it must be none other than $\alpha(g)$, and this is precisely what the boxed formula says.

To check that the reader has really understood this definition, we suggest that she/he prove that $\alpha(\mathbf{1})=\mathbf{1}$ for any morphism (f, α).

1.4.4. Isomorphic geometries. In any mathematical theory, isomorphic objects are those which are equivalent, i.e., are not distinguished in the theory. Thus isomorphic linear spaces are not distinguished in linear algebra, sets of the same cardinality (i.e., sets for which there exists a bijective map) are equivalent in set theory, isomorphic fields are not distinguished in abstract algebra, congruent triangles are the same in Euclidean plane geometry, and so on. What geometries should be considered equivalent? We hope that the following definition will seem natural to the reader.

Two geometries $(X : G)$ and $(Y : H)$ are called *isomorphic*, if there exist a bijection $f : X \to Y$ and an isomorphism $\alpha : G \to H$ such that

$$f(xg) = (f(x))(\alpha(g)) \quad \text{for all} \quad x \in X \quad \text{and all} \quad g \in G.$$

In the definition, the displayed formula is a repetition of relation (1.4), so it expresses the requirement that an isomorphism be a morphism (must satisfy the equivariance condition, i.e., respect the action of the transformation groups), the conditions on α and f say that they are equivalences, so what this definition is saying is that $(X : G)$ and $(Y : H)$ are the same.

At this stage we have no meaningful examples of isomorphic geometries. They will abound in what follows. For instance, we will see (in Chapter 10) that the Poincaré half-plane model of hyperbolic geometry is isomorphic to the Cayley–Klein disk model.

1.4.5. Subgeometries. What are subobjects in the category of geometries? The reader who is acquiring a feel for the category language should have no difficulty in coming up with the following definition. A geometry $(G : X)$ is said to be a *subgeometry* of the geometry $(H : Y)$ if X is a subset of Y, G is a subgroup of H, and the pair $(\mathrm{id}_X, \mathrm{id}_G)$, where $\mathrm{id}_G : g \mapsto g \in H$ and $\mathrm{id}_X : x \mapsto x \in Y$ are the identities, is a morphism of geometries.

A closely related definition is the following. An *embedding* (or *injective morphism*) of the geometry $(X : G)$ to the geometry $(Y : H)$ is a morphism (f, α) such that $\alpha : G \to H$ is a monomorphism and $f : X \to Y$ is injective.

Examples of subgeometries and embeddings of geometries can easily be deduced from the examples of subgroups of transformation groups in Subsection 1.3.4.

1.5. Some philosophical remarks

The examples in Section 1.2 (square, cube, circle) were taken from elementary school geometry. This was done *to motivate* the choice of the action of the corresponding transformation group. But now, in the example of the cube, let us forget school geometry: instead of the cube I^3 with its vertices, edges, faces, angles, interior points and other structure, consider the abstract set of points $\{A, B, C, D, A', B', C', D'\}$ and define the "isometries" of this "cube" as a set of 48 bijections; for example, the "rotation by 270 degrees" about the vertical axis is the bijection

$$A \mapsto B,\ B \mapsto C,\ C \mapsto D,\ D \mapsto A,\ A' \mapsto B',\ B' \mapsto C',\ D' \mapsto A',$$

and the 47 other "isometries" are defined similarly. Then (still forgetting school geometry), we can *define* vertices, edges (AB is an edge, but AC' is not), faces, prove that all edges are congruent, all faces are congruent, the "cube" can "rotate" about each vertex, etc.). The result is the *intrinsic geometry of the set of vertices* of the cube.

This geometry is not the same as the geometry (I^3: Sym(I^3)) of the cube described in Subsection (1.2.3). Of course the group G acting in these two geometries is the *same group* of order 48, but it acts on two *different sets*: the (infinite) set of points of the cube I^3 and the (finite) set of its 8 vertices $A, B, C, D, A', B', C', D'$. Thus the algebra of the two situations is the same, but the geometry is different. The geometry of the solid cube I^3 is of course much richer than the geometry of the vertex set of the cube. For example, we can define line segments inside the cube, establish their congruence, etc.

Note also that the geometric properties of the cube I^3 *regarded as a subset of Euclidean space* $\mathbb{R}^3$ are richer than its properties coming from its own geometry (I^3 : Sym(I^3)), e.g., segments of the same length inside the cube, which are always congruent in the geometry of $\mathbb{R}^3$, don't have to be congruent in the geometry of the cube!

Another example: the set of three points $\{A, B, C\}$ with two transformations, namely the identity and the "reflection"

$$A \mapsto A,\ B \mapsto C,\ C \mapsto B$$

is of course a geometry in the sense of Klein. What should it be called? An appropriate title, as the reader will no doubt agree, is "the intrinsic geometry of the vertex set of the isosceles triangle".

1.6. Problems

1.1. List all the elements (indicating their orders) of the symmetry group (i.e., isometry group) of the equilateral triangle. List all its subgroups. How many elements are there in the group of motions (i.e., orientation-preserving isometries) of the equilateral triangle.

1.2. Answer the same questions as in Problem 1.1 for

(a) the regular pyramid with four lateral faces.

(b) the regular tetrahedron;

(c) the cube;

(d)* the dodecahedron;[2]

(e)* the icosahedron;

(f) the regular n-gon (i.e., the regular polygon of n sides); consider the cases of odd and even n separately;

1.3. Embed the geometry of the motion group of the square into the geometry of the motion group of the cube, and the geometry of the circle into the geometry of the sphere.

1.4. For what values of n and m can the geometry of the regular n-gon be embedded in the geometry of the regular m-gon?

1.5. Let G be the symmetry group of the regular tetrahedron. Find all its subgroups of order 2 and describe their action geometrically.

1.6. Let G^+ be the group of motions of the cube. Indicate four subsets of the cube on which G^+ acts by all possible permutations.

1.7. Find a minimal system of generators (i.e., a set of elements such that any element can be represented as the product of some elements from this set) for the symmetry group of

(a) the regular tetrahedron;

(b) the cube.

[2] Here and in what follows the asterisk after a problem number means that the problem is not easy and should be regarded as a challenge.

1.8. Describe the fundamental domains of the symmetry group of

(a) the cube;

(b) the icosahedron;

(c) the regular tetrahedron.

1.9. Describe the Möbius band as a subset of $\mathbb{R}P^2$.

1.10. Show that the composition of two reflections of the sphere in planes passing through its center is a rotation. Determine the axis of rotation and, if the angle between the planes is given, the angle of rotation.

1.11. Given two rotations of the sphere, describe their composition.

Chapter 7

The Poincaré Disk Model of Hyperbolic Geometry

In this chapter, we begin our study of the most popular of the non-Euclidean geometries – hyperbolic geometry, concentrating on the case of dimension two. We avoid the intricacies of the axiomatic approach and define hyperbolic plane geometry via the beautiful Poincaré disk model, which is the geometry of the disk determined by the action of a certain transformation group acting on the disk (namely, the group generated by reflections in circles orthogonal to the boundary of the disk).

In order to describe the model, we need some facts from Euclidean plane geometry, which should be studied in high school, but, unfortunately, in most cases, are not. So we begin by recalling some properties of inversion (which will be the main ingredient of the transformation group of our geometry) and some constructions related to orthogonal circles in the Euclidean plane. We then establish the basic facts of hyperbolic plane geometry and finally digress, following Poincaré's argumentation from his book *Science et Hypothèse* (for the English version, see [12]) about epistomological questions relating this geometry (and other geometries) to the physical world.

7.1. Inversion and orthogonal circles

7.1.1. Inversion and its properties. The main tool that we will need in this chapter is inversion, a classical transformation from elementary plane geometry. Denote by $\mathcal{R}$ the plane $\mathbb{R}^2$ with an added extra point (called the *point at infinity* and denoted by ∞). The set $\mathcal{R} := \mathbb{R}^2 \cup \infty$ can also be interpreted as the complex numbers $\mathbb{C}$ with the "point at infinity" added; it is then called the *Riemann sphere* and denoted by $\overline{\mathbb{C}}$.

An *inversion* of center $O \in \mathbb{R}^2$ and radius $r > 0$ is the transformation of $\mathcal{R}$ that maps each point M to the point N on the ray OM so that

$$\boxed{|OM| \cdot |ON| = r^2} \tag{7.1}$$

and interchanges the points O and ∞. Sometimes inversions are called *reflections* with respect to the *circle of inversion*, i.e., the circle of radius r centered at O.

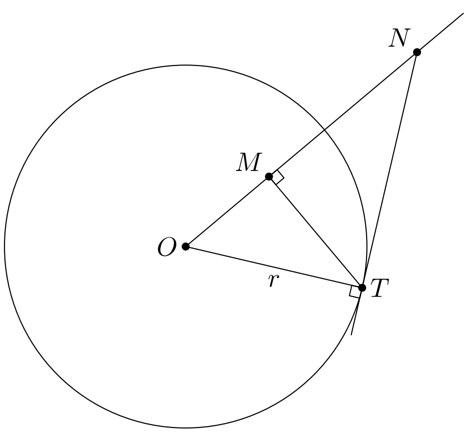

Figure 7.1. Inversion $|OM| \cdot |ON| = r^2$.

There is a simple geometric way of constructing the image of a point M under an inversion of center O and radius r: draw the circle of inversion, lower the perpendicular to OM from M to its intersection point T with the circle and construct the tangent to the circle at T to its intersection point N with the ray OM; then N will be the image of M under the given inversion. Indeed, the two right triangles OMT and OTN are similar (they have a common acute angle at O), and therefore

$$\frac{|OM|}{|OT|} = \frac{|OT|}{|ON|},$$

and since $|OT| = r$, we obtain (7.1).

If the extended plane $\mathcal{R}$ is interpreted as the Riemann sphere $\overline{\mathbb{C}}$, then an example of an inversion (of center O and radius 1) is the map $z \mapsto 1/\overline{z}$, where the bar over z denotes complex conjugation.

It follows immediately from the definition that inversions are bijections of $\mathcal{R} = \overline{\mathbb{C}}$ that leave the points of the circle of inversion in place, "turn the circle inside out" in the sense that points inside the circle are taken to points outside it (and vice versa), and are *involutions* (i.e., the composition of an inversion with itself is the identity). Further, inversions possess the following important properties.

(i) *Inversions map any circle or straight line orthogonal to the circle of inversion into itself.* Look at Figure 7.2, which shows two orthogonal circles $\mathcal{C}_O$ and $\mathcal{C}_I$ of centers O and I, respectively.

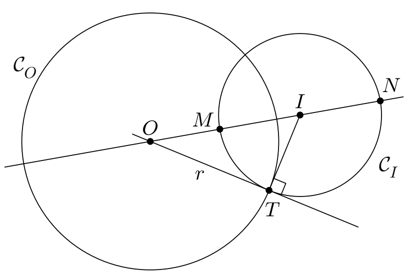

Figure 7.2. Orthogonal circles.

It follows from the definition of orthogonality that the tangent from the center O of $\mathcal{C}_O$ to the other circle $\mathcal{C}_I$ passes through the intersection point T of the two circles. Now let us consider the inversion of center O and radius $r = |OT|$. According to property (i) above, it takes the circle $\mathcal{C}_I$ to itself; in particular, the point M is mapped to N, the point T (as well as the other intersection point of the two circles) stays in place, and the two arcs of $\mathcal{C}_I$ cut out by $\mathcal{C}_O$ are interchanged. Note further that, vice versa, the inversion in the circle $\mathcal{C}_I$ transforms $\mathcal{C}_O$ in an analogous way.

(ii) *Inversions map any circle or straight line into a circle or straight line.* In particular, lines passing through the center of inversion are mapped to themselves (but are "turned inside out" in the sense that O goes to ∞ and vice versa, while the part of the line inside the circle of inversion goes to the outside part and vice versa); circles passing through the center of inversion are taken to straight lines, while straight lines not passing through the center of inversion are taken to circles passing through that center (see Figure 7.3).

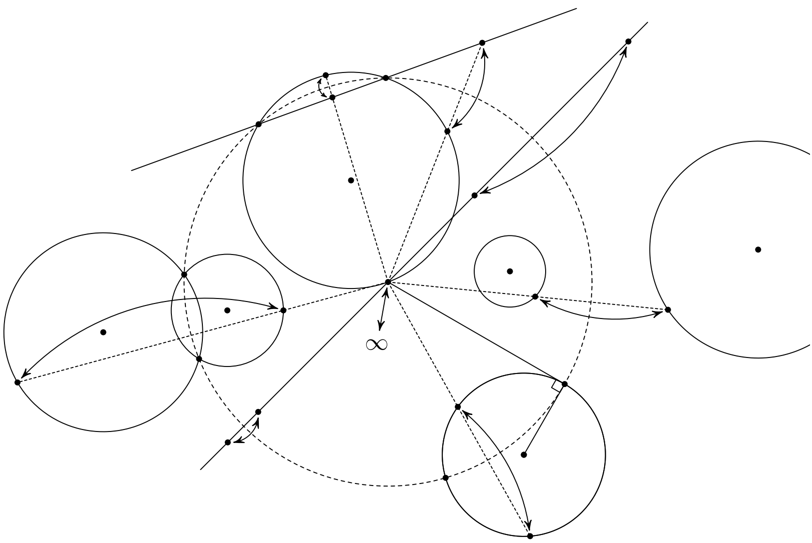

Figure 7.3. Images of circles and lines under inversion.

(iii) *Inversions preserve* (*the measure of*) *angles;* here by the measure of an angle formed by two intersecting curves we mean the ordinary (Euclidean) measure of the angle formed by their tangents at the intersection point.

The (elementary) proofs of properties (i)–(iii) are left to the reader (see Problems 7.1–7.3).

7.1.2. Construction of orthogonal circles. We have already noted the important role that orthogonal circles play in inversion (see 7.1.1 (i)). Here we will describe several constructions of orthogonal circles that will be used in subsequent sections.

Lemma 7.1.3. *Let A be a point inside a circle $\mathcal{C}$ centered at some point O; then there exists a circle orthogonal to $\mathcal{C}$ such that the reflection in this circle takes A to O.*

Proof. From A draw the perpendicular to line OA to its intersection T with the circle $\mathcal{C}$ (see Figure 7.4).

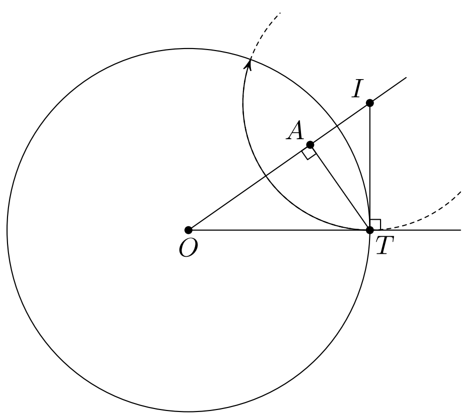

Figure 7.4. Inversion taking an arbitrary point A to O.

Draw the tangent to $\mathcal{C}$ at T to its intersection at I with OA. Then the circle of radius IT centered at I is the one we need. Indeed, the similar right triangles IAT and ITO yield $|IA|/|IT| = |IT|/|IO|$, whence we obtain $|IA| \cdot |IO| = |IT|^2$, which means that O is the reflection of A in the circle of radius $|IT|$ centered at I, as required. □

Corollary 7.1.4. (i) *Let A and B be points inside a circle $\mathcal{C}_0$ not lying on the same diameter; then there exists a unique circle orthogonal to $\mathcal{C}_0$ and passing through A and B.*

(ii) *Let A be a point inside a circle $\mathcal{C}_0$ and P a point on $\mathcal{C}_0$, with A and P not lying on the same diameter; then there exists a unique circle orthogonal to $\mathcal{C}_0$ passing through A and P.*

(iii) *Let P and Q be points on a circle $\mathcal{C}_0$ of center O such that PQ is not a diameter; then there exists a unique circle $\mathcal{C}$ orthogonal to $\mathcal{C}_0$ and passing through P and Q.*

(iv) *Let A be a point inside a circle $\mathcal{C}_0$ of center O and $\mathcal{D}$ a circle orthogonal to $\mathcal{C}_0$; then there exists a unique circle $\mathcal{C}$ orthogonal to both $\mathcal{C}_0$ and $\mathcal{D}$ and passing through A.*

Proof. To prove (i), we describe an effective step-by-step construction, which can be carried out by ruler and compass, yielding the required circle. The construction is shown on Figure 7.5, with the numbers in parentheses near each point indicating at which step the point was obtained.

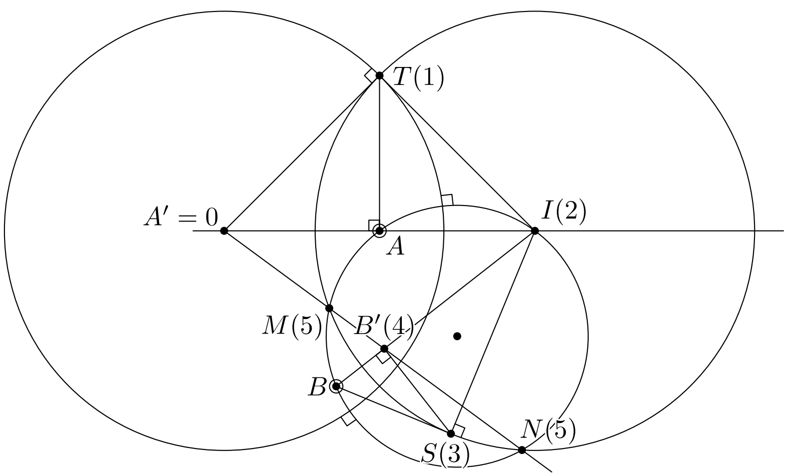

Figure 7.5. Circle orthogonal to $\mathcal{C}_0$ containing A, B.

First, we apply Lemma 7.1.3, to define an inversion φ taking A to the center O of the given circle; to do this, we lower a perpendicular from A to OA to its intersection T (1) with $\mathcal{C}$, then draw the perpendicular to OT from T to its intersection I (2) with OA; the required inversion is centered at I and is of radius $|IT|$. Joining B and I, we construct the tangent BS (3) to the circle of the inversion φ and find the image B' (4) of B under φ by dropping a perpendicular from S to IB.

Next, we draw the line $B'O$ and obtain the intersection points M, N of this line with the circle of the inversion φ. Finally, we draw the circle $\mathcal{C}$ passing through the points M, N, I. Then $\mathcal{C}$ "miraculously" passes through A and B and is orthogonal to $\mathcal{C}_0$! Of course, there is no miracle in this: $\mathcal{C}$ passes through A and B because it is the inverse image under φ of the line OB' (see 7.1.1(ii)), it is orthogonal to $\mathcal{C}_0$ since so is OB' (see 7.1.1(iii)).

Uniqueness is obvious in the case $A = O$ and follows in the general case by 7.1.1(ii)-(iii).

The proof of (ii) is analogous: we send A to O by an inversion φ, join O and $\varphi(P)$ and continue the argument as above.

To prove (iii), construct lines OP and OQ, draw perpendiculars to these lines from P and Q, respectively, and denote by I their intersection point. Then the circle of radius $|IP|$ centered at I is the required one. Its uniqueness is easily proved by contradiction.

To prove (iv), we again use Lemma 7.1.3 to construct an inversion φ that takes $\mathcal{C}_0$ to itself and sends A to O. From the point O, we draw the (unique) ray $\mathcal{R}$ orthogonal to $\varphi(\mathcal{L})$. Then the circle $\varphi^{-1}(\mathcal{R})$ is the required one. □

7.2. Definition of the disk model

7.2.1. The disk model of the *hyperbolic plane* is the geometry $(\mathbb{H}^2 : \mathcal{M})$ whose points are the points of the open disk

$$\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\},$$

and whose transformation group $\mathcal{M}$ is the group generated by reflections in all the circles orthogonal to the boundary circle

$$\mathbb{A} := \{(x, y) : x^2 + y^2 = 1\}$$

of $\mathbb{H}^2$, and by reflections in all the diameters of the circle $\mathbb{A}$. Now $\mathcal{M}$ is indeed a transformation group of $\mathbb{H}^2$: the discussion in 7.1.1

implies that a reflection of the type considered takes points of $\mathbb{H}^2$ to points of $\mathbb{H}^2$ and, being its own inverse, we have the implication $\varphi \in \mathcal{M} \Longrightarrow \varphi^{-1} \in \mathcal{M}$.

We will often call $\mathbb{H}^2$ the *hyperbolic plane*. The boundary circle $\mathbb{A}$ (which is not part of the hyperbolic plane) is called the *absolute*.

7.2.2. We will see later that $\mathcal{M}$ is actually the isometry group of hyperbolic geometry with respect to the *hyperbolic distance*, which will be defined in the next chapter. We will see that although the Euclidean distance between points of $\mathbb{H}^2$ is always less than 2, the hyperbolic plane is unbounded with respect to the hyperbolic distance. Endpoints of a short segment (in the Euclidean sense!) near the absolute are very far away from each other in the sense of hyperbolic distance.

Figure 7.6 gives an idea of what an isometric transformation (the simplest one – a reflection in a line) does to a picture. Note that from our Euclidean point of view, the reflection changes the size and the shape of the picture, whereas from the hyperbolic point of view, the size and shape of the image is exactly the same as that of the original. It should also be clear that *hyperbolic reflections reverse orientation*, e.g., the image of a right hand under reflection will look like a left hand, but of somewhat different size and shape.

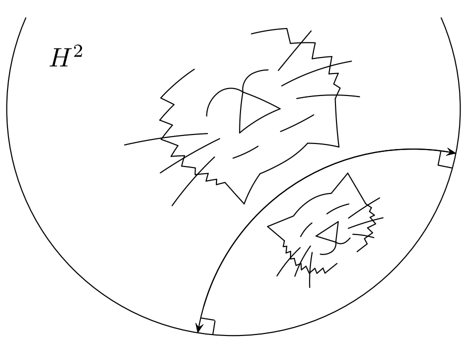

Figure 7.6. An isometry in the hyperbolic plane.

7.3. Points and lines in the hyperbolic plane

7.3.1. First we define *points of the hyperbolic plane* simply as points of the open disk $\mathbb{H}^2$. We then define the *lines* on the hyperbolic plane as the intersections with $\mathbb{H}^2$ of the (Euclidean) circles orthogonal to the absolute as well as the diameters (without endpoints) of the absolute (see Figure 7.7).

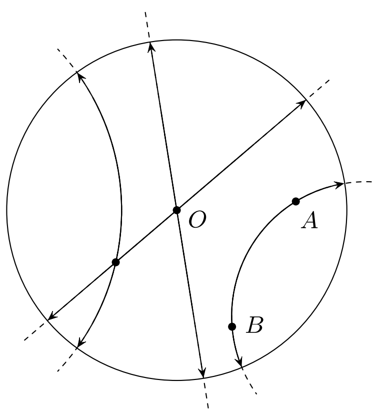

Figure 7.7. Lines on the hyperbolic plane.

Note that the endpoints of the arcs and the diameters do not belong to the hyperbolic plane: they lie in the absolute, whose points are not points of our geometry.

Thus the hyperbolic plane, as well as the lines in it, is not compact. Its compactification (or closure) is the compact disk $\overline{\mathbb{H}}^2$.

Figure 7.7 shows that some lines intersect in one point, others have no common points, and none have two common points (unlike lines in spherical geometry). This is not surprising, because we have the following statement.

Theorem 7.3.2. *One and only one line passes through any pair of distinct points of the hyperbolic plane.*

Proof. The theorem immediately follows from Corollary 7.1.4(i). □

7.4. Perpendiculars

7.4.1. Two lines in $\mathbb{H}^2$ are called *perpendicular* if they are orthogonal in the sense of elementary Euclidean geometry. When both are diameters, they are perpendicular in the usual sense, when both are arcs of circles, they have perpendicular tangents at the intersection point, when one is an arc and the other, a diameter, then the diameter is perpendicular to the tangent to the arc at the intersection point.

Theorem 7.4.2. *There is one and only one line passing through a given point and perpendicular to a given line.*

Proof. The theorem immediately follows from Corollary 7.1.4(iv). □

7.5. Parallels and nonintersecting lines

7.5.1. Let l be a line and P be a point of the hyperbolic plane $\mathbb{H}^2$ not contained in the line l. Denote by A and B the points at which l intersects the absolute. Consider the lines $k = PA$ and $m = PB$ and denote their second intersection points with the absolute by A' and B'. Clearly, the lines k and m do not intersect l. Moreover, any line passing through P between k and m (i.e., any line containing P and joining the arcs AA' and BB') does not intersect l. The lines APA' and BPB' are called *parallels* to l through P, and the lines between them are called *nonintersecting lines* with l.

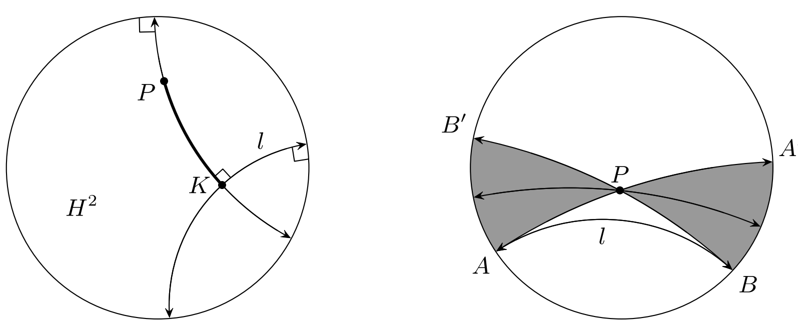

Figure 7.8. Perpendiculars and parallels.

We have proved the following statement.

Theorem 7.5.2. *There are infinitely many lines passing through a given point P not intersecting a given line l if $P \notin l$. These lines are all located between the two parallels to l.* □

This theorem contradicts Euclid's famous *Fifth Postulate*, which, in its modern formulation, says that one and only one parallel to a given line passes through a given point. For more than two thousand years, many attempts to prove that the Fifth Postulate follows from Euclid's other postulates (which, unlike the Fifth Postulate, were simple and intuitively obvious) were made by mathematicians and philosophers. Had such a proof been found, Euclidean geometry could have been declared to be an absolute truth both from the physical and the philosophical point of view, it would have been an example of a set of facts that the German philosopher Kant included in the category of *synthetic apriori.* For two thousand years, the naive belief among scientists in the absolute truth of Euclidean geometry made it difficult for the would be discoverers of other geometries to realize that they had found something worthwhile. Thus the appearance of a consistent geometry in which the Fifth Postulate does not hold was not only a crucial development in the history of mathematics, but one of the turning points in the philosophy of science. In this connection, see the discussion in Chapter 11.

7.6. Sum of the angles of a triangle

7.6.1. Consider three points A, B, C not on one line. The three segments AB, BC, CA (called *sides*) form a *triangle* with *vertices* A, B, C. The *angles* of the triangle, measured in radians, are defined as equal to the (Euclidean measure of the) angles between the tangents to the sides at the vertices.

Theorem 7.6.2. *The sum of the angles α, β, γ of a triangle ABC is less than two right angles:*

$$\boxed{\alpha + \beta + \gamma < \pi}.$$

Proof. In view of Lemma 7.1.3, we can assume, without loss of generality, that A is O (the center of $\mathbb{H}^2$). But then if we compare the hyperbolic triangle OBC with the Euclidean triangle OBC, we see that they have the same angle at O, but the Euclidean angles at B and C are larger than their hyperbolic counterparts (look at Fig. 7.9), which implies the claim of the theorem. $\square$

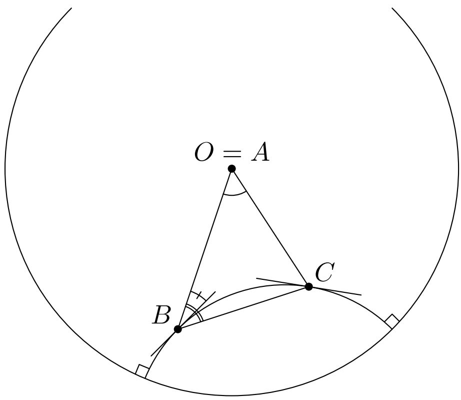

Figure 7.9. Sum of the angles of a hyperbolic triangle.

It is easy to see that very small triangles have angles sums very close to π; in fact, *the least upper bound of the angle sum of hyperbolic triangles is exactly* π. Further, *the greatest lower bound of these sums is 0.* To see this, divide the absolute into three equal arcs by three points P, Q, R and construct three circles orthogonal to the absolute passing through the pairs of points P and Q, Q and R, R and P. These circles exist by Corollary 7.1.4(iii). Then all the angles of the "triangle" PQR are zero, so its angle sum is zero. Of course, PQR is not a real triangle in our geometry (its vertices, being on the absolute, are not points of $\mathbb{H}^2$), but if we take three points P', Q', R' close enough to P, Q, R, then the angle sum of triangle $P'Q'R'$ will be less than any prescribed $\varepsilon > 0$.

7.7. Rotations and circles in the hyperbolic plane

We mentioned previously that distance between points of the hyperbolic plane will be defined later. Recall that the hyperbolic plane is

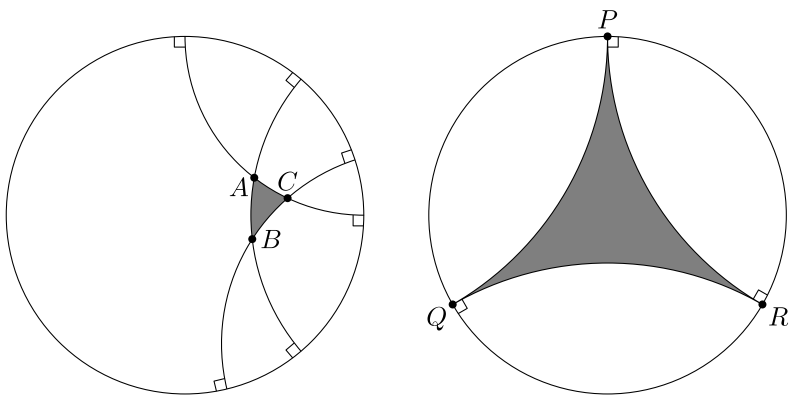

Figure 7.10. Ordinary triangle and "triangle" with angle sum 0.

the geometry $(\mathbb{H}^2 : \mathcal{M})$, in which, by definition, $\mathcal{M}$ is the transformation group generated by all reflections in all the lines of $\mathbb{H}^2$. If we take the composition of two reflections in two intersecting lines, then what we get should be a "rotation", but we can't assert that at this point, because we don't have any definition of rotation: the usual (Euclidean) definition of a rotation or even that of a circle cannot be given until distance is defined.

But the notions of rotation and of circle *can* be defined without appealing to distance in the following natural way: a *rotation* about a point $P \in \mathbb{H}^2$ is, by definition, the composition of any two reflections in lines passing through P. If I and A are distinct points of $\mathbb{H}^2$, then the (hyperbolic) *circle* of center I and radius IA is the set of images of A under all rotations about I.

Theorem 7.7.1. *A (hyperbolic) circle in the Poincaré disk model is a Euclidean circle, and vice versa, any Euclidean circle inside* $\mathbb{H}^2$ *is a hyperbolic circle in the geometry* $(\mathbb{H}^2 : \mathcal{M})$.

Proof. Let $\mathcal{C}$ be a circle of center I and radius IA in the geometry $(\mathbb{H}^2 : \mathcal{M})$. Using Lemma 7.1.3, we can send I to the center O of $\mathbb{H}^2$ by a reflection φ. Let ρ be a rotation about I determined by two lines l_1 and l_2. Then the lines $d_1 := \varphi(l_1)$ and $d_2 := \varphi(l_2)$ are diameters of the absolute and the composition of reflections in these diameters is a Euclidean rotation about O (and simultaneously a hyperbolic one). This rotation takes the point $\varphi(A)$ to a point on the circle $\mathcal{C}'$ of center O and radius $O\varphi(A)$, which is simultaneously a hyperbolic and Euclidean circle. Now by Corollary 7.1.4(i), the inverse image of $\varphi^{-1}(\mathcal{C}')$ will be a (Euclidean!) circle. But $\varphi^{-1}(\mathcal{C}')$ coincides with $\mathcal{C}$) by construction, so $\mathcal{C}$) is indeed a Euclidean circle in our model.

The proof of the converse assertion is similar and is left to the reader (see Problem 7.7). □

7.8. Hyperbolic geometry and the physical world

In his famous book *Science et Hypothèse*, Henri Poincaré describes the physics of a small "universe" and the physical theories that its inhabitants would create. The universe considered by Poincaré is Euclidean, plane (two-dimensional), and has the form of an open unit disk. Its temperature is 100° Farenheit at the center of the disk and decreases linearly to absolute zero at its boundary. The lengths of objects (including living creatures) are proportional to temperature.

How will a little flat creature endowed with reason and living in this disk describe the main physical laws of his universe? The first question he/she may ask could be: Is the world bounded or infinite? To answer this question, an expedition is organized; but as the expedition moves towards the boundary of the disk, the legs of the explorers become smaller, their steps shorter – they will never reach the boundary, and conclude that the world is infinite.

The next question may be: Does the temperature in the universe vary? Having constructed a thermometer (based on different expansion coefficients of various materials), scientists carry it around the universe and take measurements. However, since the lengths of all objects change similarly with temperature, the thermometer gives the same measurement all over the universe – the scientists conclude that the temperature is constant.

Then the scientists might study straight lines, i.e., investigate what is the shortest path between two points. They will discover that the shortest path is what we perceive to be the arc of the circle containing the two points and orthogonal to the boundary disk (this is because such a circular path brings the investigator nearer to the center of the disk, and thus increases the length of his steps). Further, they will find that the shortest path is unique and regard such paths as "straight lines".

Continuing to develop geometry, the inhabitants of Poincaré's little flat universe will decide that there is more than one parallel to a given line passing through a given point, the sum of angles of triangles is less than π, and obtain other statements of hyperbolic geometry.

Thus they will come to the conclusion that they live in an infinite flat universe with constant temperature governed by the laws of hyperbolic geometry. But this not true – their universe is a finite disk, its temperature is variable (tends to zero towards the boundary) and the underlying geometry is Euclidean, not hyperbolic!

The philosophical conclusion of Poincaré's argument is not agnosticism – he goes further. The physical model described above, according to Poincaré, shows not only that the truth about the universe cannot be discovered, but that it makes no sense to speak of any "truth" or approximation of truth in science – pragmatically, the inhabitants of his physical model are perfectly right to use hyperbolic geometry as the foundation of their physics because it is convenient, and it is counterproductive to search for any abstract Truth which has no practical meaning anyway.

This conclusion has been challenged by other thinkers, but we will not get involved in this philosophical discussion.

7.9. Problems

7.1. Prove that inversion maps circles and straight lines to circles or straight lines.

7.2. Prove that inversion maps any circle orthogonal to the circle of inversion into itself.

7.3. Prove that inversion is conformal (i.e., it preserves the measure of angles).

7.4. Prove that if P is point lying outside a circle γ and A, B are the intersection points with the circle of a line l passing through P, then the product $|PA| \cdot |PB|$ (often called the *power of P with respect to γ*) does not depend on the choice of l.

7.5. Prove that if P is point lying inside a circle γ and A, B are the intersection points with the circle of a line l passing through P, then the product $|PA| \cdot |PB|$ (often called the *power of P with respect to γ*) does not depend on the choice of l.

7.6. Prove that inversion with respect to a circle orthogonal to a given circle $\mathcal{C}$ maps the disk bounded by $\mathcal{C}$ bijectively onto itself.

7.7. Prove that any Euclidean circle inside the disk model is also a hyperbolic circle. Does the ordinary (Euclidean) center coincide with its "hyperbolic center"?

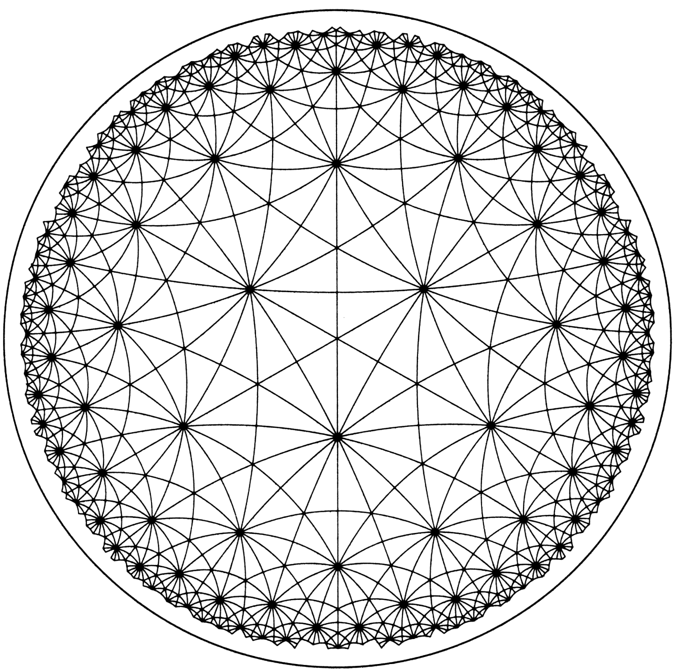

Figure 7.11. A pattern of lines in $\mathbb{H}^2$.

7.8. Study Figure 7.11. Does it demonstrate any tilings of $\mathbb{H}^2$ by regular polygons? Of how many sides? Do you discern a Coxeter geometry in this picture with "hyperbolic Coxeter triangles" as fundamental domains? What are their angles?

7.9. Prove that any inversion of $\overline{\mathbb{C}}$ preserves the cross ratio of four points:

$$\langle z_1, z_2, z_3, z_4 \rangle := \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2}.$$

7.10*. Using complex numbers, invent a formula for the distance between points on the Poincaré disk model and prove that "symmetry with respect to straight lines" (i.e., inversion) preserves this distance.

7.11. Prove that hyperbolic geometry is homogeneous in the sense that for any two flags (i.e., half-planes with a marked point on the boundary) there exists an isometry taking one flag to the other.

7.12. Prove that the hyperbolic plane (as defined via the Poincaré disk model) can be tiled by regular pentagons.

7.13. Define inversion (together with the center and the sphere of inversion) in Euclidean space $\mathbb{R}^3$, state and prove its main properties: inversion takes planes and spheres to planes or spheres, any sphere orthogonal to the sphere of inversion to itself, any plane passing through the center of inversion to itself.

7.14. Using the previous problem, prove that any inversion in $\mathbb{R}^3$ takes circles and straight lines to circles or straight lines.

7.15. Prove that any inversion in $\mathbb{R}^3$ is conformal (preserves the measure of angles).

7.16. Construct a model of hyperbolic space geometry on the open unit ball (use Problem 7.13).

7.17. Prove that there is a unique common perpendicular joining any two nonintersecting lines.

7.18. Let $A_\infty P$ and $A_\infty P'$ be two parallel lines (with A_∞ a point on the absolute). Given a point M on $A_\infty P$, we say that $M' \in A_\infty P'$ is the *corresponding point* to M if the angles $A_\infty MM'$ and $A_\infty M'M$ are equal. Prove that any point $M \in A_\infty P$ has a unique corresponding point on the line $A_\infty P'$.

7.19. The locus of all points corresponding to a point M on $A_\infty P$ and lying on all the parallels to $A_\infty P$ is known as a *horocycle*. What do horocycles look like in the Poincaré disk model?