Preface

In the process of writing a mathematics book, an author has to make a variety of decisions. The central theme of the book must be followed by deciding on a potential audience to whom the book is directed, and then a choice of style for presenting the material must be made.

This book began as a collection of additional notes given to students participating in the courses taught by the author at Tulane University. These courses included:

• **The calculus sequence**. This is the typical two-semester course on differential and integral calculus. The standard book used at Tulane is J. Stewart [**281**]. The author has used M. Spivak [**277**] for the Honor section, which is slightly more advanced than the regular one.

• **Discrete mathematics**. This course introduces students to mathematical induction and provides a glimpse of number theory. This is the first time where the students are exposed to proofs. The books used in the past include M. Aigner [**4**] and K. Rosen [**256**].

• **Combinatorics**. This is a one-semester course that includes basic counting techniques, recurrences, combinatorial identities, and the ideas behind bijective proofs. The author has used a selection of texts, including T. Andreescu and Z. Feng [**17**], A. Benjamin and J. Quinn [**46**], M. Bona [**58**], and R. Brualdi [**82**].

• **Number theory**. This is also a one-semester course covering the basics of the subject: primality and factorization, congruences, diophantine equations, continued fractions, primitive roots, and quadratic reciprocity. The texts used by the author include G. H. Hardy and E. M. Wright [**160**], K. Ireland and M. Rosen [**178**], K. Rosen [**257**], and J. H. Silverman [**274**].

• **Real analysis**. This is one of the few required courses for a mathematics major. It introduces the student to the real line and all of its properties. Sequences and completeness, the study of the real line, continuity, and compactness form the bulk of the course. The author has employed O. Hijab [**168**] and E. Landau [**192**] to reflect his opinion that this class should be calculus "well done".

• **Experimental mathematics**. This is a course created by the author. Notwithstanding that it has been taught only three times to date, it has been received very well by our students. The material in the course includes an introduction by the author to the symbolic language `Mathematica` and to `Maple` by a second instructor. The topics have included an introduction to computer proofs in the Wilf-Zeilberger style, recurrences, symbolic integration, and graph theory. Beyond using the computer as a number cruncher, the course has employed symbolic packages to discover new mathematical patterns and relationships, to create impressive graphics to expose mathematical structure, and to suggest approaches to formal proofs. The author has used the texts by J. M. Borwein and D. H. Bailey [**69**] and the second volume written jointly with R. Girgensohn [**70**] as well as the lecture notes [**36**] from a course in experimental mathematics given by the authors at the Joint Mathematics Meetings in San Antonio. The volume by M. Petkovsek, H. Wilf, and D. Zeilberger [**247**] has been used to lecture on automatic proofs. The audience for this class has consisted of students majoring in mathematics and some others who wandered into the class because they heard that the topics were interesting.

The author has always placed a special effort in his lectures to point out that material covered in a specific class is part of a bigger picture. This book is a product of notes written for these courses.

Naturally, most of the topics covered appear in the literature. It is the point of view that is new.

The present book contains a variety of topics that at first reading might appear to be disconnected. It is the author's hope that in the end it will all fit together. The author finds some results in elementary mathematics particularly appealing and some material has been written to supply background towards a specific goal. No effort has been made to be systematic and there is no claim about the topics that do not appear here.

The reader will find here examples that include the following:

Evaluation of finite sums. These are used as examples to practice induction, to provide combinatorial interpretations, and to introduce the reader to ideas behind **automatic proofs**. The questions dealing with the evaluation of

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

and the analogous problem for

$$\sum_{k=0}^{n} \binom{n}{k}^3$$

are in the background.

Prime factorization. The fundamental theorem of arithmetic states that every natural number has a unique decomposition as a product of primes. There are some beautiful results that describe this prime factorization for interesting sequences. These are expressed in terms of the so-called valuation $\nu_p(m)$, the highest power of the prime p that divides m. Among them is Legendre's formula for **factorials**, usually given as a series

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor,$$

but it can also be reinterpreted as

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1}.$$

Here $s_p(n)$ is the sum of digits of n written in base p.

The book explores properties of these prime factorizations for many classical sequences that come from combinatorics. These include **binomial coefficients**, which are encountered in the most elementary counting problems; **Catalan numbers**, which count the number of ways to place parentheses to group symbols in a sequence of numbers; the **Fibonacci numbers**, which count the ways to cover a board with squares and dominoes; the **Stirling numbers**, which count the number of ways to split a set of n elements into k nonempty parts; and many others. The patterns of the valuations of these sequences are sometimes remarkably beautiful and, in many cases, still need to be explained. Some of these families of numbers are unexpectedly related. For instance, computing the residues of the binomial coefficient $\binom{2p-1}{p-1}$ modulo powers of the prime p, the reader will encounter the **harmonic numbers** $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ and the **Bernoulli numbers** defined by the expansion of $x/(e^x - 1)$.

Elementary functions. The student finds a variety of functions in the beginning sequence of calculus. This book contains a study of polynomials, rational functions, exponentials/logarithms, and trigonometric functions. The author has made an effort to illustrate properties of these functions that hopefully the reader will find appealing. A variety of examples of polynomials appear in the book. These include **Bernoulli polynomials**, which provide the evaluation of well-known examples such as

$$\sum_{k=1}^{n} k^2 = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}$$

and the less well known

$$\sum_{k=1}^{n} k^8 = \frac{n^9}{9} + \frac{n^8}{2} + \frac{2n^7}{3} - \frac{7n^5}{15} + \frac{2n^3}{9} - \frac{n}{30}.$$

It is easy to see that the sum of ath powers from 1 to n is a polynomial in n of degree $a + 1$. The properties of the coefficients are a different story. These coefficients, the so-called **Bernoulli numbers**, are always rational numbers and their denominators can be described in relatively simple terms. It is a remarkable fact, one that the author always enjoyed, that the numerators are related to Fermat's last theorem, now established by A. Wiles, namely that the equation $x^n + y^n = z^n$ has no solutions in nonzero integers when $n \geq 3$.

Roots of polynomials are encountered by the student in the courses preparing him or her for the calculus sequence. This book treats the cases of degree 3 and 4 by expressing the roots by radicals and then in terms of one trigonometric function. In future courses the student will most likely learn that, in general, **there is no expression by radicals for the solution of a quintic equation**. This celebrated result of Abel and Galois is one of the jewels of the nineteenth century and it must be studied. On the other hand, it is possible to express the roots of any polynomial equation in terms of more advanced functions. Elliptic functions suffice for degree 5 or 6 and the so-called theta functions produce solutions for higher degrees. This is a beautiful subject that also deserves to be studied. From this point of view, the best way to solve a cubic equation is via trigonometric functions. This is an overstatement: the student should be aware of both methods.

Rational functions are introduced as coming from **generating functions** of sequences that obey linear recurrences with constant coefficients. The main example is that of the Fibonacci numbers, defined by $F_n = F_{n-1} + F_{n-2}$, with initial conditions $F_0 = 0$ and $F_1 = 1$. Their generating function is

$$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}.$$

The classical question of how to integrate a rational function is probably the first time that a student seriously deals with this class. The method of partial fractions, which provides the solution to this question, is discussed in detail. One of the earliest evaluations of an integral in closed form, the **Wallis formula**

$$\int_0^{\infty} \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}} \binom{2m}{m}$$

is established by a variety of methods.

This book contains two examples of transformations on the class of rational functions. The first example is

$$R(x) \mapsto \frac{R(\sqrt{x}) - R(-\sqrt{x})}{2\sqrt{x}}$$

and it originated from an attempt by the author to develop a new method of integration. The map above comes from separating the integrand R into its even and odd parts. Iterating this transformation to a function of the type $x^j/(x^m-1)$ has unexpected number theoretical properties. The second type of transformation comes from the rational function $R(x) = \dfrac{x^2-1}{2x}$, coming from the relation between $\cot\theta$ and $\cot 2\theta$. Using this rational function as a change of variable, one finds remarkable identities among integrals. These are the so-called ***Landen transformations***, which produce identities of the type

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = \int_{-\infty}^{\infty} \frac{dx}{a_1x^2+b_1x+c_1}$$

where

$$a_1 = \frac{2ac}{c+a}, \quad b_1 = \frac{b(c-a)}{c+a}, \quad c_1 = \frac{(c+a)^2-b^2}{2(c+a)}.$$

It turns out that iterating this procedure gives a sequence (a_n, b_n, c_n) and a number L such that $a_n \to L$, $b_n \to 0$, and $c_n \to L$. The invariance of the integral leads to the identity

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = \frac{\pi}{L}.$$

This shows that the integral may be computed numerically by computing the sequence $\{a_n\}$. These types of ideas extend to the computation of **the integral of any rational function on** $\mathbb{R}$. Details appear in Chapter 15.

Transcendental functions appearing in the book include some of elementary type such as **exponential**, **trigonometric**, and **hyperbolic** functions. These complete the class of elementary functions treated in the calculus sequence. The goal is to provide interesting properties of these functions and to describe connections with combinatorics, number theory, and interesting numbers. For instance, the reader will see that the expansion of some trigonometric functions around the origin involves the Bernoulli numbers mentioned in the context of evaluation of power sums.

The basic constants of analysis, e and π, are discussed in detail. Their irrationality is established by a systematic method. It is intriguing that the irrationality of $e + \pi$ is still an open problem. Their **continued fractions**, which provide optimal approximations by rational numbers, are established. There are some beautiful integral evaluations related to this topic. An example is given by

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx = \frac{22}{7} - \pi,$$

which proves that $\pi \neq \frac{22}{7}$. There is a marked difference in the behavior of these continued fractions. The patterns for e are quite regular, while those for π remain a mystery. The appearance of e in a combinatorial setting is given by the counting of permutations of n objects that do not fix a single one of them. This is the classical **derangement number**. Its behavior for large n is related to e. This is totally unexpected.

The author has chosen two examples of nonelementary transcendental functions to illustrate some of their properties. The first one is the **gamma function** $\Gamma(x)$ (and its logarithmic derivative: the **digamma function** $\psi(x) = \Gamma'(x)/\Gamma(x)$) introduced by Euler, and the second one is the **Riemann zeta function** $\zeta(s)$ coming from questions dealing with the distribution of prime numbers.

Irrationality questions are considered throughout the book. The irrationality of some special numbers is presented in detail. These include $\sqrt{2}$, e, π, and also $\zeta(2) = \pi^2/6$ and $\zeta(3)$. This last constant does not admit a simpler representation. Its irrationality, recently established by R. Apéry, is discussed in the last chapter. It is unknown whether it is a rational multiple of π^3. The arithmetic properties of the **Euler constant**, defined by $\gamma = -\Gamma'(1)$, are still unknown. Some details on this question are described.

Symbolic computations are seen as an essential ingredient of this book. `Mathematica` examples are included to illustrate the capabilities of this language. In an earlier draft of the book there was a separate chapter describing the methods developed by Sister Celine and by W. Gosper Jr. and the WZ-theory created by H. Wilf and D. Zeilberger. The final draft incorporates these techniques into the flow of the book.

This book started as a collection of notes written for a variety of courses. The author has tried to give credit to the authors of the various notes. It is very likely that some of them have been missed. **My apologies to those ignored or misquoted**. There are many books that have been used to obtain the information presented here. These are the author's favorite ones, starting with the classic **Modern Analysis** by E. T. Whittaker and G. N. Watson [**311**]; the basic treatment on automatic proofs can be found in the text by M. Petkovsek, H. Wilf, and D. Zeilberger [**247**]; and the book by R. Graham, D. Knuth, and O. Patashnik [**145**] is a great source for a class in discrete mathematics. The best introduction to the issues of symbolic computation is given in the text by M. Kauers and P. Paule [**181**]. From there the reader should consult the information provided on the website

`http://www.risc.jku.at/`

from RISC (the Research Institute for Symbolic Computation) at the Johannes Kepler University in Linz, Austria, and

`http://carma.newcastle.edu.au`

from the Priority Research Centre for Computer-Assisted Research Mathematics and Its Applications (CARMA) at the University of Newcastle, Australia. The order in which these two centers were listed does not imply a preference by the author.

Experimental mathematics is a relatively new name for an old approach to doing mathematics. It seems optimal to quote the gurus of the field about their opinions on what experimental mathematics is.

In [**143**], H. Wilf describes the path from experiment to theory in mathematics as follows: *"... it begins with wondering what a par-*

ticular situation looks like in detail; it continues with some computer experiments to show the structure of that situation for a selection of small values of the parameters of the problem; and then comes the human part: the mathematician gazes at the computer output, attempting to see and to codify some patterns. If this seems fruitful, then the final step requires the mathematician to prove that the apparent pattern is really there, and it is not a shimmering mirage above the desert sands."

D. Zeilberger in [**322**] and in many of his other articles and opinions proposes to *eliminate the human factor* in the mathematical experience. Perhaps *eliminate* is too strong of a word and *minimize* is more pleasant. But Doron does not mince words, so neither does the author. The reality is that more and more mathematics is becoming part of the computer experience. The author remembers spending hours of valuable high school time extrapolating tables of logarithms. Then came the hand calculator....

J. M. Borwein and D. H. Bailey [**69**] enumerate the role of computing in mathematics: (i) gaining insight and *intuition*; (ii) *discovering* new relationships; (iii) *visualizing* mathematical principles; (iv) *testing* and especially *falsifying* conjectures; (v) *exploring* a possible result to see if it *merits* formal proof; (vi) *suggesting* approaches for formal proof; (vii) *computing* replacing lengthy hand derivations; (viii) *confirming* analytically derived results. A nice collection of examples illustrating this point of view of mathematics is provided in [**34, 35**].

The author has tried to follow this list in the context of undergraduate material. Aside from that, he has aimed to present elementary results from a novel point of view, hoping to motivate the reader to learn more about standard subjects. It is clear that some statements have shorter proofs than the one presented in the text. The students have always been in the background of the writing, so sometimes it is instructive to present a complicated proof: the point illustrated is that often that is all you can do. This is fine. If you find a nicer argument later, even better.

Most of the topics are from **elementary mathematics** with occasional hints on how it connects to more sophisticated subjects. Part of the motivation for bringing together a large collection of notes on diverse topics was to provide the reader with some fun while learning interesting pieces of mathematics. *It was a lot of fun to write the book. Hopefully the reader will enjoy part of it.*

The final version of this book has been improved by many comments received by colleagues, students, and friends. A partial list is

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