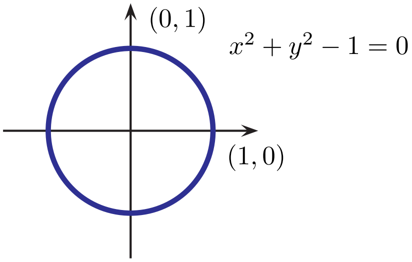
Chapter 1

Conics

Linear algebra studies the simplest type of geometric objects, such as straight lines and planes. Straight lines in the plane are the zero sets of linear, or first degree, polynomials, such as $\{(x, y) \in \mathbb{R}^2 : 3x + 4y - 1 = 0\}$. However, there are far more plane curves than just straight lines. Higher degree polynomials define other plane curves, and these are where we begin our exploration of algebraic geometry.

We start by looking at conics, which are the zero sets of second degree polynomials. The quintessential conic is the circle:

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 1 = 0\}.$$



Despite their apparent simplicity, an understanding of second degree equations and their solution sets is the beginning of much of algebraic geometry. By the end of the chapter, we will have developed some beautiful mathematics.

1.1. Conics over the Reals

The goal of this section is to understand the basic properties of conics in the real plane $\mathbb{R}^2$. In particular, we will see how to graph these conics.

For second degree polynomials, you can usually get a fairly good graph of the corresponding curve by just drawing it "by hand." The first series of exercises will lead you through this process. Our goal is to develop basic techniques for thinking about curves without worrying about too many technical details.

We start with the polynomial $P(x, y) = y - x^2$ and want to look at its zero set

$$C = \{(x, y) \in \mathbb{R}^2 : P(x, y) = 0\}.$$

We also denote this set by $\mathrm{V}(P)$.

Exercise 1.1.1. Show that for any $(x, y) \in C$, we also have

$$(-x, y) \in C.$$

Thus the curve C is symmetric about the y-axis.

Exercise 1.1.2. Show that if $(x, y) \in C$, then we have $y \geq 0$.

Exercise 1.1.3. Show that for every $y \geq 0$, there is a point $(x, y) \in C$ with this y-coordinate. Now, for points $(x, y) \in C$, show that if y goes to infinity, then one of the corresponding x-coordinates also approaches infinity while the other corresponding x-coordinate must approach negative infinity.

The last two exercises show that the curve C is unbounded in the positive and negative x-directions, unbounded in the positive y-direction, but bounded in the negative y-direction. This means that we can always find $(x, y) \in C$ so that x is arbitrarily large, in either the positive or negative directions, y is arbitrarily large in the positive direction, but that there is a number M (in this case 0) such that $y \geq M$ (in this case $y \geq 0$).

Exercise 1.1.4. Sketch the curve $C = \{(x, y) \in \mathbb{R}^2 : P(x, y) = 0\}$.

Conics that have these symmetry and boundedness properties and look like this curve C are called *parabolas*. Of course, we could have analyzed the curve $\{(x, y) : x - y^2 = 0\}$ and made similar observations, but with the roles of x and y reversed. In fact, we could have shifted, stretched, and rotated our parabola many ways and still retained these basic features.

We now perform a similar analysis for the plane curve

$$C = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \right\}.$$

Exercise 1.1.5. Show that if $(x, y) \in C$, then the three points $(-x, y)$, $(x, -y)$, and $(-x, -y)$ are also on C. Thus the curve C is symmetric about both the x- and y-axes.

Exercise 1.1.6. Show that for every $(x, y) \in C$, we have $|x| \leq 2$ and $|y| \leq 3$.

This shows that the curve C is bounded in both the positive and negative x- and y-directions.

Exercise 1.1.7. Sketch $C = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \right\}$.

Conics that have these symmetry and boundedness properties and look like this curve C are called *ellipses*.

There is a third type of conic. Consider the curve

$$C = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 - 4 = 0\}.$$

Exercise 1.1.8. Show that if $(x, y) \in C$, then the three points $(-x, y)$, $(x, -y)$, and $(-x, -y)$ are also on C. Thus the curve C is also symmetric about both the x- and y-axes.

Exercise 1.1.9. Show that if $(x, y) \in C$, then we have $|x| \geq 2$.

These exercises show that the curve C has two connected components. Intuitively, this means that C is composed of two distinct pieces that do not touch.

Exercise 1.1.10. Show that the curve C is unbounded in the positive and negative x-directions and also unbounded in the positive and negative y-directions.

Exercise 1.1.11. Sketch $C = \left\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 - 4 = 0\right\}$.

Conics that have these symmetry, connectedness, and boundedness properties are called *hyperbolas*.

In the following exercise, the goal is to sketch many concrete conics.

Exercise 1.1.12. Sketch the graph of each of the following conics in $\mathbb{R}^2$. Identify which are parabolas, ellipses, or hyperbolas.

(1) $\mathrm{V}(x^2 - 8y)$
(2) $\mathrm{V}(x^2 + 2x - y^2 - 3y - 1)$
(3) $\mathrm{V}(4x^2 + y^2)$
(4) $\mathrm{V}(3x^2 + 3y^2 - 75)$
(5) $\mathrm{V}(x^2 - 9y^2)$
(6) $\mathrm{V}(4x^2 + y^2 - 8)$
(7) $\mathrm{V}(x^2 + 9y^2 - 36)$
(8) $\mathrm{V}(x^2 - 4y^2 - 16)$
(9) $\mathrm{V}(y^2 - x^2 - 9)$

A natural question arises in the study of conics. If we have a second degree polynomial, how can we determine whether its zero set is an ellipse, hyperbola, parabola, or something else in $\mathbb{R}^2$? Suppose we have the following polynomial:

$$P(x, y) = ax^2 + bxy + cy^2 + dx + ey + h.$$

Are there conditions on a, b, c, d, e, h that determine what type of conic $\mathrm{V}(P)$ is?

Whenever we have a polynomial in more than one variable, a useful technique is to treat P as a polynomial in a single variable whose coefficients are themselves polynomials in the remaining variables.

Exercise 1.1.13. Express the polynomial $P(x, y) = ax^2 + bxy + cy^2 + dx + ey + h$ in the form

$$P(x, y) = Ax^2 + Bx + C$$

where A, B, and C are polynomials in y. What are A, B, and C?

Since we are interested in the zero set $\mathrm{V}(P)$, we want to find the roots of $Ax^2 + Bx + C = 0$ in terms of y. As we know from high school algebra the roots of the quadratic equation $Ax^2 + Bx + C = 0$ are

$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

To determine the number of real roots, we need to look at the sign of the discriminant

$$\Delta_x = B^2 - 4AC.$$

Exercise 1.1.14. Treating $P(x, y) = ax^2 + bxy + cy^2 + dx + ey + h$ as a polynomial in the variable x, show that the discriminant is

$$\Delta_x(y) = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4ah).$$

Exercise 1.1.15.

(1) Suppose $\Delta_x(y_0) < 0$. Explain why there is no point on $\mathrm{V}(P)$ whose y-coordinate is y_0.

(2) Suppose $\Delta_x(y_0) = 0$. Explain why there is exactly one point on $\mathrm{V}(P)$ whose y-coordinate is y_0.

(3) Suppose $\Delta_x(y_0) > 0$. Explain why there are exactly two points on $\mathrm{V}(P)$ whose y-coordinate is y_0.

This exercise demonstrates that in order to understand the set $\mathrm{V}(P)$ we need to understand the set $\{y : \Delta_x(y) \geq 0\}$. We will first see how for parabolas we expect the scalar $b^2 - 4ac$ to be zero.

Exercise 1.1.16. Suppose $b^2 - 4ac = 0$. Suppose further that $2bd - 4ae > 0$.

(1) Show that $\Delta_x(y) \geq 0$ if and only if $y \geq \dfrac{4ah - d^2}{2bd - 4ae}$.

(2) Conclude that if $b^2 - 4ac = 0$ and $2bd - 4ae > 0$, then $\mathrm{V}(P)$ is a parabola.

Notice that if $b^2 - 4ac \neq 0$, then $\Delta_x(y)$ is itself a quadratic function in y, and the features of the set over which $\Delta_x(y)$ is nonnegative is determined by its quadratic coefficient.

Exercise 1.1.17. Suppose $b^2 - 4ac < 0$.

(1) Show that one of the following occurs:
   (a) $\{y \mid \Delta_x(y) \geq 0\} = \emptyset$,
   (b) $\{y \mid \Delta_x(y) \geq 0\} = \{y_0\}$,
   (c) there exist real numbers α and β, $\alpha < \beta$, such that

$$\{y \mid \Delta_x(y) \geq 0\} = \{y \mid \alpha \leq y \leq \beta\}.$$

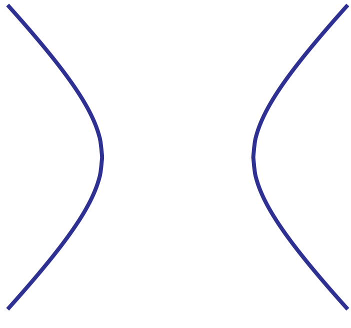
(2) Conclude that $\mathrm{V}(P)$ is either empty, a point, or an ellipse.

Exercise 1.1.18. Suppose $b^2 - 4ac > 0$.

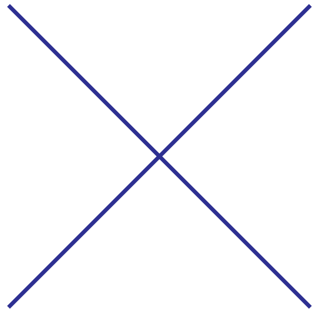
(1) Show that one of the following occurs:
   (a) $\{y \mid \Delta_x(y) \geq 0\} = \mathbb{R}$ and $\Delta_x(y) \neq 0$,
   (b) $\{y \mid \Delta_x(y) = 0\} = \{y_0\}$ and $\{y \mid \Delta_x(y) > 0\} = \{y \mid y \neq y_0\}$,
   (c) there exist real numbers α and β, $\alpha < \beta$, such that

$$\{y \mid \Delta_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\}.$$

(2) If $\{y \mid \Delta_x(y) \geq 0\} = \mathbb{R}$, show that $\mathrm{V}(P)$ is a hyperbola opening left and right:



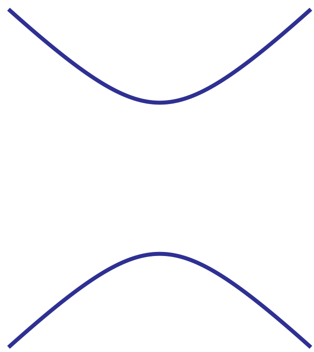
(3) If $\{y \mid \Delta_x(y) = 0\} = \{y_0\}$ and there is a point on $\mathrm{V}(P)$ with y-coordinate equal to y_0, show that $\mathrm{V}(P)$ is two lines intersecting in a point:



(4) If there are two real numbers α and β, $\alpha < \beta$, such that

$$\{y \mid \Delta_x(y) \geq 0\} = \{y \mid y \leq \alpha\} \cup \{y \mid y \geq \beta\},$$

show that $\mathrm{V}(P)$ is a hyperbola opening up and down:



Above we decided to treat P as a function of x, but we could have treated P as a function of y, $P(x, y) = A'y^2 + B'y + C'$, each of whose coefficients is now a polynomial in x.

Exercise 1.1.19. Show that the discriminant of $A'y^2 + B'y + C' = 0$ is

$$\Delta_y(x) = (b^2 - 4ac)x^2 + (2be - 4cd)x + (e^2 - 4ch).$$

Note that the quadratic coefficient is again $b^2 - 4ac$, so our observations from above are the same in this case as well. In the preceding exercises we were intentionally vague about some cases. For example, we do not say anything about what happens when $b^2 - 4ac = 0$ and $2bd - 4ae = 0$. This is an example of a "degenerate" conic. We treat degenerate conics later in this chapter, but for now it suffices to note that if $b^2 - 4ac = 0$, then $\mathrm{V}(P)$ is neither an ellipse nor a hyperbola. If $b^2 - 4ac < 0$, then $\mathrm{V}(P)$ is neither a parabola nor a hyperbola. And if $b^2 - 4ac > 0$, then $\mathrm{V}(P)$ is neither a parabola nor an ellipse. This leads us to the following theorem.

Theorem 1.1.20. Suppose $P(x, y) = ax^2 + bxy + cy^2 + dx + ey + h$. If $\mathrm{V}(P)$ is a parabola in $\mathbb{R}^2$, then $b^2 - 4ac = 0$; if $\mathrm{V}(P)$ is an ellipse in $\mathbb{R}^2$, then $b^2 - 4ac < 0$; and if $\mathrm{V}(P)$ is a hyperbola in $\mathbb{R}^2$, then $b^2 - 4ac > 0$.

In general, it is not immediately clear whether a given conic $C = \mathrm{V}(ax^2 + bxy + cy^2 + dx + e + h)$ is an ellipse, hyperbola, or parabola. When the coefficient $b = 0$, then it is much easier to determine what type of curve C is.

Corollary 1.1.1. Suppose $P(x, y) = ax^2 + cy^2 + dx + ey + h$. If $\mathrm{V}(P)$ is a parabola in $\mathbb{R}^2$, then $ac = 0$; if $\mathrm{V}(P)$ is a hyperbola in $\mathbb{R}^2$, then $ac < 0$, i.e. a and c have opposite signs; and if $\mathrm{V}(P)$ is a ellipse in $\mathbb{R}^2$, then $ac > 0$, i.e. a and c have the same sign.

1.2. Changes of Coordinates

The goal of this section is to show that, in $\mathbb{R}^2$, any ellipse can be transformed into any other ellipse, any hyperbola into any other hyperbola, and any parabola into any other parabola.

Here we start to investigate what it could mean for two conics to be *the same*; thus we start to solve an equivalence problem for conics. Intuitively, two curves are the same if we can shift, stretch, or rotate one to obtain the other. Cutting or gluing, however, is not allowed.

Our conics live in the real plane $\mathbb{R}^2$. In order to describe conics as the zero sets of second degree polynomials, we first must choose a coordinate system for the plane. Different choices for these coordinates will give different polynomials, even for the same curve. (To make this concrete, imagine 10 people separately go to a blank blackboard, put a dot on it to correspond to an origin, and then draw two axes. There will be 10 quite different coordinate systems chosen.)

Consider the two coordinate systems:

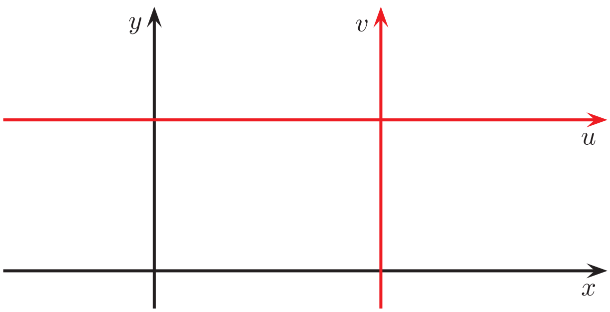


Figure 1. xy- and uv-coordinate systems.

Suppose there is a dictionary between these coordinate systems, given by

$$u = x - 3,$$
$$v = y - 2.$$

Then the circle of radius 4 has either the equation

$$u^2 + v^2 - 16 = 0$$

or the equation

$$(x-3)^2 + (y-2)^2 - 16 = 0,$$

which is the same as $x^2 - 6x + y^2 - 4y - 3 = 0$.

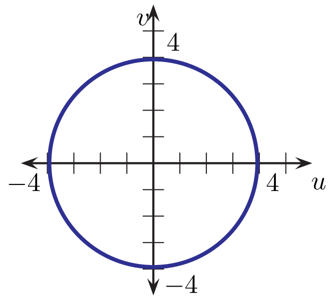


Figure 2. Circle of radius 4 centered at the origin in the uv-coordinate system.

These two coordinate systems differ only by where you place the origin.

Coordinate systems can also differ in their orientation. Consider two coordinate systems where the dictionary between the coordinate systems is:

$$u = x - y$$
$$v = x + y.$$

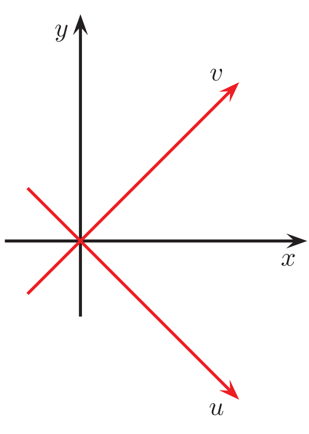


Figure 3. xy- and uv-coordinate systems with different orientations.

Coordinate systems can also vary by the chosen units of length. Consider two coordinate systems where the dictionary between the coordinate systems is:

$$u = 2x$$
$$v = 3y.$$

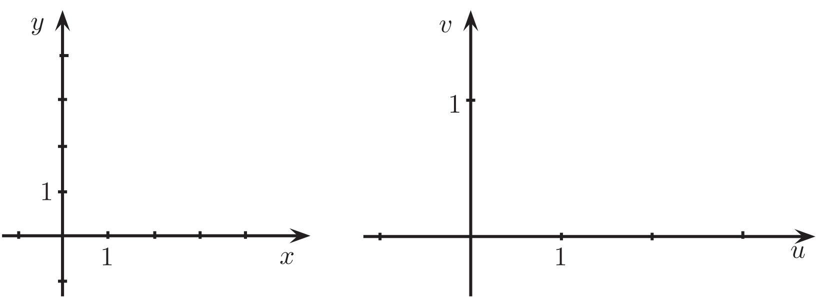


Figure 4. xy- and uv-coordinate systems with different units.

All of these possibilities are captured in the following.

Definition 1.2.1. A *real affine change of coordinates* in the real plane, $\mathbb{R}^2$, is given by

$$u = ax + by + e$$
$$v = cx + dy + f,$$

where $a, b, c, d, e, f \in \mathbb{R}$ and

$$ad - bc \neq 0.$$

In matrix language, we have

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix},$$

where $a, b, c, d, e, f \in \mathbb{R}$, and

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

Exercise 1.2.1. Show that the origin in the xy-coordinate system agrees with the origin in the uv-coordinate system if and only if $e = f = 0$. Thus the constants e and f describe translations of the origin.

Exercise 1.2.2. Show that if $u = ax + by + e$ and $v = cx + dy + f$ is a change of coordinates, then the inverse change of coordinates is

$$x = \left(\frac{1}{ad - bc}\right)(du - bv) - \left(\frac{1}{ad - bc}\right)(de - bf)$$
$$y = \left(\frac{1}{ad - bc}\right)(-cu + av) - \left(\frac{1}{ad - bc}\right)(-ce + af).$$

(This is why we require that $ad - bc \neq 0$.) There are two ways of working this problem. One method is to just start fiddling with the equations. The second is to translate the change of coordinates into the matrix language and then use a little linear algebra.

It is also common for us to change coordinates multiple times, but we need to ensure that a composition of real affine changes of coordinates is a real affine change of coordinates.

Exercise 1.2.3. Show that if

$$u = ax + by + e$$
$$v = cx + dy + f$$

and

$$s = Au + Bv + E$$
$$t = Cu + Dv + F$$

are two real affine changes of coordinates from the xy-plane to the uv-plane and from the uv-plane to the st-plane, respectively, then the composition from the xy-plane to the st-plane is a real affine change of coordinates.

We frequently go back and forth between using a change of coordinates and its inverse. For example, suppose we have the ellipse

$V(x^2 + y^2 - 1)$ in the xy-plane. Under the real affine change of coordinates

$$u = x + y$$
$$v = 2x - y,$$

this ellipse becomes $V(5u^2 - 2uv + 2v^2 - 9)$ in the uv-plane (verify this). To change coordinates from the xy-plane to the uv-plane, we use the inverse change of coordinates

$$x = \frac{1}{3}u + \frac{1}{3}v$$
$$y = \frac{2}{3}u - \frac{1}{3}v.$$

Since any affine transformation has an inverse transformation, we will not worry too much about whether we are using a transformation or its inverse in our calculations. When the context requires care, we will make the distinction.

Exercise 1.2.4. For each pair of ellipses, find a real affine change of coordinates that maps the ellipse in the xy-plane to the ellipse in the uv-plane.

(1) $V(x^2 + y^2 - 1)$, $V(16u^2 + 9v^2 - 1)$
(2) $V((x - 1)^2 + y^2 - 1)$, $V(16u^2 + 9(v + 2)^2 - 1)$
(3) $V(4x^2 + y^2 - 6y + 8)$, $V(u^2 - 4u + v^2 - 2v + 4)$
(4) $V(13x^2 - 10xy + 13y^2 - 1)$, $V(4u^2 + 9v^2 - 1)$

We can apply a similar argument for hyperbolas.

Exercise 1.2.5. For each pair of hyperbolas, find a real affine change of coordinates that maps the hyperbola in the xy-plane to the hyperbola in the uv-plane.

(1) $V(xy - 1)$, $V(u^2 - v^2 - 1)$
(2) $V(x^2 - y^2 - 1)$, $V(16u^2 - 9v^2 - 1)$
(3) $V((x - 1)^2 - y^2 - 1)$, $V(16u^2 - 9(v + 2)^2 - 1)$
(4) $V(x^2 - y^2 - 1)$, $V(v^2 - u^2 - 1)$
(5) $V(8xy - 1)$, $V(2u^2 - 2v^2 - 1)$

Now we move on to parabolas.

Exercise 1.2.6. For each pair of parabolas, find a real affine change of coordinates that maps the parabola in the xy-plane to the parabola in the uv-plane.

(1) $\mathrm{V}(x^2 - y)$, $\mathrm{V}(9v^2 - 4u)$

(2) $\mathrm{V}((x-1)^2 - y)$, $\mathrm{V}(u^2 - 9(v+2))$

(3) $\mathrm{V}(x^2 - y)$, $\mathrm{V}(u^2 + 2uv + v^2 - u + v - 2)$

(4) $\mathrm{V}(x^2 - 4x + y + 4)$, $\mathrm{V}(4u^2 - (v+1))$

(5) $\mathrm{V}(4x^2 + 4xy + y^2 - y + 1)$, $\mathrm{V}(4u^2 + v)$

The preceding three problems suggest that we can transform ellipses to ellipses, hyperbolas to hyperbolas, and parabolas to parabolas by way of real affine changes of coordinates. This turns out to be the case. Suppose $C = \mathrm{V}(ax^2 + bxy + cy^2 + dx + ey + h)$ is a conic in $\mathbb{R}^2$. Our goal in the next several exercises is to show that if C is an ellipse, we can transform it to $\mathrm{V}(x^2 + y^2 - 1)$; if C is a hyperbola, we can transform it to $\mathrm{V}(x^2 - y^2 - 1)$; and if C is a parabola, we can transform it to $\mathrm{V}(x^2 - y)$. We will pass through a series of real affine transformations and appeal to Exercise 1.2.3. This result ensures that the final composition of our individual transformations is the real affine transformation we seek. This composition is, however, a mess, so we won't write it down explicitly. We will see in Section 1.11 that we can organize this information much more efficiently by using tools from linear algebra.

We begin with ellipses. Suppose $\mathrm{V}(ax^2 + bxy + cy^2 + dx + ey + h)$ is an ellipse in $\mathbb{R}^2$. Our first transformation will be to remove the xy term, i.e. to find a real affine transformation that will align our given curve with the coordinate axes. By Theorem 1.1.20 we know that $b^2 - 4ac < 0$.

Exercise 1.2.7. Explain why if $b^2 - 4ac < 0$, then $ac > 0$.

Exercise 1.2.8. Show that under the real affine transformation

$$x = \sqrt{\frac{c}{a}}u + v$$

$$y = u - \sqrt{\frac{a}{c}}v,$$

the ellipse $\mathrm{V}(ax^2 + bxy + cy^2 + dx + ey + h)$ in the xy-plane becomes an ellipse in the uv-plane whose defining equation is $Au^2 + Cv^2 + Du + Ev + H = 0$. Find A and C in terms of a, b, c. Show that if $b^2 - 4ac < 0$, then $A \neq 0$ and $C \neq 0$.

Now we have a new ellipse $\mathrm{V}(Au^2 + Cv^2 + Du + Ev + H)$ in the uv-plane. If our original ellipse already had $b = 0$, then we would have skipped the previous step and gone directly to this one.

Exercise 1.2.9. Show that there exist constants R, S, and T such that the equation

$$Au^2 + Cv^2 + Du + Ev + H = 0$$

can be rewritten in the form

$$A(u - R)^2 + C(v - S)^2 - T = 0.$$

Express R, S, and T in terms of A, C, D, E, and H.

To simplify notation, we revert to using x and y as our variables instead of u and v, but we keep in mind that we are not really still working in our original xy-plane. This is a convenience to avoid subscripts. Without loss of generality we can assume $A, C > 0$, since if $A, C < 0$ we could simply multiply the above equation by -1 without affecting the conic. Note that we assume that our original conic is an ellipse, i.e., it is nondegenerate. A consequence of this is that $T > 0$.

Exercise 1.2.10. Suppose $A, C > 0$. Find a real affine change of coordinates that maps the ellipse

$$\mathrm{V}(A(x - R)^2 + C(y - S)^2 - T),$$

to the circle

$$\mathrm{V}(u^2 + v^2 - 1).$$

Hence, we have found a composition of real affine changes of coordinates that transforms any ellipse $\mathrm{V}(ax^2+bxy+cy^2+dx+ey+h)$ to the circle $\mathrm{V}(u^2+v^2-1)$.

We want a similar process for parabolas. Suppose $\mathrm{V}(ax^2+bxy+cy^2+dx+ey+h)$ is a parabola in $\mathbb{R}^2$. We want to show, by direct algebra, that there is a change of coordinates that takes this parabola to

$$\mathrm{V}(u^2-v).$$

By Theorem 1.1.20 we know that $b^2-4ac=0$. As before our first task is to eliminate the xy term. Suppose first that $b \neq 0$. Since $b^2>0$ and $4ac=b^2$ we know $ac>0$, so we can repeat Exercise 1.2.8.

Exercise 1.2.11. Consider the values A and C found in Exercise 1.2.8. Show that if $b^2-4ac=0$, then either $A=0$ or $C=0$, depending on the signs of a, b, c. [Hint: Recall, $\sqrt{\alpha^2}=-\alpha$ if $\alpha<0$.]

Since either $A=0$ or $C=0$ we can assume $C=0$ without loss of generality. Then $A \neq 0$, for our curve is a parabola and not a straight line, so our transformed parabola is $\mathrm{V}(Au^2+Du+Ev+H)$ in the uv-plane. If our original parabola already had $b=0$, then we also know, since $b^2-4ac=0$, that either $a=0$ or $c=0$, so we could have skipped ahead to this step.

Exercise 1.2.12. Show that there exist constants R and T such that the equation

$$Au^2+Du+Ev+H=0$$

can be rewritten as

$$A(u-R)^2+E(v-T)=0.$$

Express R and T in terms of A, D, E, and H.

As above we revert our notation to x and y with the same caveat as before. Multiplying our equation by -1 if necessary, we may assume $A>0$.

Exercise 1.2.13. Suppose $A>0$ and $E \neq 0$. Find a real affine change of coordinates that maps the parabola

$$\mathrm{V}(A(x-R)^2-E(y-T))$$

to the parabola

$$V(u^2 - v).$$

Hence, we have found a real affine change of coordinates that transforms any parabola $V(ax^2 + bxy + cy^2 + dx + ey + h)$ to the parabola $V(u^2 - v)$.

Finally, suppose $V(ax^2 + bxy + cy^2 + dx + ey + h)$ is a hyperbola in $\mathbb{R}^2$. We want to show that there is a change of coordinates that takes this hyperbola to

$$V(u^2 - v^2 - 1).$$

By Theorem 1.1.20 we know that $b^2 - 4ac > 0$. Suppose first that $b \neq 0$. Unlike before, we can now have $ac > 0$, $ac < 0$, or $ac = 0$.

Exercise 1.2.14. Suppose $ac > 0$. Use the real affine transformation in Exercise 1.2.8 to transform C to a conic in the uv-plane. Find the coefficients of u^2 and v^2 in the resulting equation and show that they have opposite signs.

Now for the $ac < 0$ case.

Exercise 1.2.15. Suppose $ac < 0$ and $b \neq 0$. Use the real affine transformation

$$x = \sqrt{-\frac{c}{a}}u + v$$
$$y = u - \sqrt{-\frac{a}{c}}v$$

to transform C to a conic in the uv-plane of the form

$$Au^2 + Cv^2 + Du + Ev + H = 0.$$

Find the coefficients of u^2 and v^2 in the resulting equation and show that they have opposite signs.

Note that in the case when $ac < 0$ and $b = 0$, then a and c have opposite signs and the hyperbola is already of the form

$$ax^2 + cy^2 + dx + ey + f = 0.$$

Exercise 1.2.16. Suppose $ac = 0$ (so $b \neq 0$). Since either $a = 0$ or $c = 0$, we can assume $c = 0$. Use the real affine transformation

$$\begin{aligned} x &= u + v \\ y &= \left(\frac{1-a}{b}\right) u - \left(\frac{1+a}{b}\right) v \end{aligned}$$

to transform $\mathrm{V}(ax^2 + bxy + dx + ey + h)$ to a conic in the uv-plane of the form

$$\mathrm{V}(u^2 - v^2 + Du + Ev + H).$$

In all three cases we find that the hyperbola can be transformed to $\mathrm{V}(Au^2 - Cv^2 + Du + Ev + H)$ in the uv-plane, with both A and C positive. We can now complete the transformation of the hyperbola as we did above with parabolas and ellipses.

Exercise 1.2.17. Show that there exist constants R, S and T so that

$$Au^2 - Cv^2 + Du + Ev + H = A(u - R)^2 - C(v - S)^2 - T.$$

Express R, S, and T in terms of A, C, D, E, and H.

We are assuming that we have a hyperbola. Hence $T \neq 0$, since otherwise we would have just two lines through the origin. If $T < 0$, then we can multiply the equation $A(u - R)^2 - C(v - S)^2 - T = 0$ through by -1 and then interchange u with v. Thus we can assume that our original hyperbola has become

$$\mathrm{V}(A(u - R)^2 - C(v - S)^2 - T)$$

with A, C and T all positive.

Exercise 1.2.18. Suppose $A, C, T > 0$. Find a real affine change of coordinates that maps the hyperbola

$$\mathrm{V}(A(x - R)^2 - C(y - S)^2 - T),$$

to the hyperbola

$$\mathrm{V}(u^2 - v^2 - 1).$$

We have now shown that in $\mathbb{R}^2$ we can find a real affine change of coordinates that will transform any ellipse to $\mathrm{V}(x^2 + y^2 - 1)$, any hyperbola to $\mathrm{V}(x^2 - y^2 - 1)$, and any parabola to $\mathrm{V}(x^2 - y)$. Thus we have three classes of smooth conics in $\mathbb{R}^2$. Our next task is to show that these are distinct, that is, that we cannot transform an ellipse to a parabola and so on.

Exercise 1.2.19. Give an intuitive argument, based on the number of connected components, for the fact that no ellipse can be transformed into a hyperbola by a real affine change of coordinates.

Exercise 1.2.20. Show that there is no real affine change of coordinates

$$u = ax + by + e$$
$$v = cx + dy + f$$

that transforms the ellipse $\mathrm{V}(x^2 + y^2 - 1)$ to the hyperbola $\mathrm{V}(u^2 - v^2 - 1)$.

Exercise 1.2.21. Give an intuitive argument, based on boundedness, for the fact that no parabola can be transformed into an ellipse by a real affine change of coordinates.

Exercise 1.2.22. Show that there is no real affine change of coordinates that transforms the parabola $\mathrm{V}(x^2 - y)$ to the circle $\mathrm{V}(u^2 + v^2 - 1)$.

Exercise 1.2.23. Give an intuitive argument, based on the number of connected components, for the fact that no parabola can be transformed into a hyperbola by a real affine change of coordinates.

Exercise 1.2.24. Show that there is no real affine change of coordinates that transforms the parabola $\mathrm{V}(x^2 - y)$ to the hyperbola $\mathrm{V}(u^2 - v^2 - 1)$.

Definition 1.2.2. Two conics are *equivalent under a real affine change of coordinates* if the defining polynomial for one of the conics can be transformed via a real affine change of coordinates into the defining polynomial of the other conic.

Combining all of the work in this section, we have just proven the following theorem.

Theorem 1.2.25. Under a real affine change of coordinates, all ellipses in $\mathbb{R}^2$ are equivalent, all hyperbolas in $\mathbb{R}^2$ are equivalent, and all parabolas in $\mathbb{R}^2$ are equivalent. Further, these three classes of conics are distinct; no conic of one class can be transformed via a real affine change of coordinates to a conic of a different class.

In Section 1.11 we will revisit this theorem using tools from linear algebra. This approach will yield a cleaner and more straightforward proof than the one we have in the current setting. The linear algebraic setting will also make all of our transformations simpler, and it will become apparent how we arrived at the particular transformations.

1.3. Conics over the Complex Numbers

The goal of this section is to see how, under a complex affine changes of coordinates, all ellipses and hyperbolas are equivalent, while parabolas are still distinct.

While it is certainly natural to begin with the zero set of a polynomial $P(x, y)$ as a curve in the real plane $\mathbb{R}^2$, polynomials also have roots over the complex numbers. In fact, throughout mathematics it is almost always easier to work over the complex numbers than over the real numbers. This can be seen in the solutions given by the quadratic equation $x^2+1=0$, which has no solutions if we require $x \in \mathbb{R}$ but does have the two solutions, $x = \pm i$, in the complex numbers $\mathbb{C}$.

Exercise 1.3.1. Show that the set

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 + 1 = 0\}$$

is empty but that the set

$$C = \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 + 1 = 0\}$$

is not empty. In fact, show that given any complex number x there must exist a $y \in \mathbb{C}$ such that

$$(x, y) \in C.$$

Then show that if $x \neq \pm i$, then there are two distinct values $y \in \mathbb{C}$ such that $(x, y) \in C$, while if $x = \pm i$ there is only one such y.

Thus if we use only real numbers, some zero sets of second degree polynomials will be empty. This does not happen over the complex numbers.

Exercise 1.3.2. Let

$$P(x, y) = ax^2 + bxy + cy^2 + dx + ey + f,$$

with $a \neq 0$. Show that for any value $y \in \mathbb{C}$, there must be at least one $x \in \mathbb{C}$, but no more than two such x's, such that

$$P(x, y) = 0.$$

[Hint: Write $P(x, y) = Ax^2 + Bx + C$ as a function of x whose coefficients A, B, and C are themselves functions of y, and use the quadratic formula. As mentioned before, this technique will be used frequently.]

Thus for any second order polynomial, its zero set is non-empty provided we work over the complex numbers.

But even more happens. We start with:

Exercise 1.3.3. Let $C = \mathrm{V}\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) \subset \mathbb{C}^2$. Show that C is unbounded in both x and y. (Over the complex numbers $\mathbb{C}$, being unbounded in x means, given any number M, there will be a point $(x, y) \in C$ such that $|x| > M$. Compare this result to Exercise 1.1.6.)

Hyperbolas in $\mathbb{R}^2$ come in two pieces. In $\mathbb{C}^2$, it can be shown that hyperbolas are connected, meaning there is a continuous path from any point to any other point. The following shows this for a specific hyperbola.

Exercise 1.3.4. Let $C = \mathrm{V}(x^2 - y^2 - 1) \subset \mathbb{C}^2$. Show that there is a continuous path on the curve C from the point $(-1, 0)$ to the point $(1, 0)$, despite the fact that no such continuous path exists in $\mathbb{R}^2$. (Compare this exercise with Exercise 1.1.9.)

These two exercises demonstrate that in $\mathbb{C}^2$ ellipses are unbounded (just like hyperbolas and parabolas) and hyperbolas are connected (just like ellipses and parabolas). Thus the intuitive arguments in Exercises 1.2.19, 1.2.21, and 1.2.23 no longer work in $\mathbb{C}^2$. We have even more.

Exercise 1.3.5. Show that if $x = u$ and $y = iv$, then the circle $\{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 1\}$ transforms into the hyperbola $\{(u, v) \in \mathbb{C}^2 : u^2 - v^2 = 1\}$.

Definition 1.3.1. A *complex affine change of coordinates* in the complex plane $\mathbb{C}^2$ is given by

$$\begin{aligned} u &= ax + by + e \\ v &= cx + dy + f, \end{aligned}$$

where $a, b, c, d, e, f \in \mathbb{C}$ and

$$ad - bc \neq 0.$$

Exercise 1.3.6. Show that if $u = ax + by + e$ and $v = cx + dy + f$ is a change of coordinates, then the inverse change of coordinates is

$$\begin{aligned} x &= \left(\frac{1}{ad - bc}\right)(du - bv) - \left(\frac{1}{ad - bc}\right)(de - bf) \\ y &= \left(\frac{1}{ad - bc}\right)(-cu + av) - \left(\frac{1}{ad - bc}\right)(-ce + af). \end{aligned}$$

This proof should look almost identical to the solution of Exercise 1.2.2.

Definition 1.3.2. Two conics are *equivalent under a complex affine change of coordinates* if the defining polynomial for one of the conics can be transformed via a complex affine change of coordinates into the defining polynomial for the other conic.

Exercise 1.3.7. Use Theorem 1.2.25 together with the new result of Exercise 1.3.5 to conclude that all ellipses and hyperbolas are equivalent under complex affine changes of coordinates.

Parabolas, though, are still different.

Exercise 1.3.8. Show that the circle $\{(x, y) \in \mathbb{C}^2 : x^2 + y^2 - 1 = 0\}$ is not equivalent under a complex affine change of coordinates to the parabola $\{(u, v) \in \mathbb{C}^2 : u^2 - v = 0\}$.

We now want to look more directly at $\mathbb{C}^2$ in order to understand more clearly why the class of ellipses and the class of hyperbolas are different as real objects but the same as complex objects. We start by looking at $\mathbb{C}$. Algebraic geometers regularly use the variable x for a complex number. Complex analysts more often use the variable z, which allows a complex number to be expressed in terms of its real and imaginary parts

$$z = x + iy,$$

where x is the real part of z and y is the imaginary part.

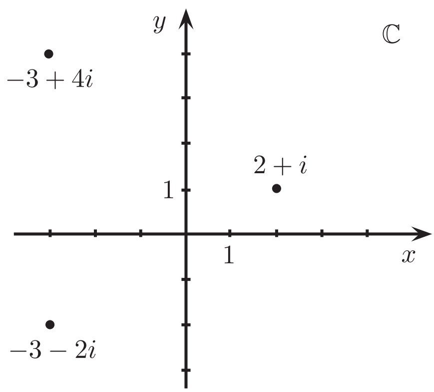


Figure 5. Points in the complex plane.

Similarly, an algebraic geometer will usually use (x, y) to denote points in the complex plane $\mathbb{C}^2$ while a complex analyst will instead use (z, w) to denote points in the complex plane $\mathbb{C}^2$. Here the complex analyst will write

$$w = u + iv.$$

There is a natural bijection from $\mathbb{C}^2$ to $\mathbb{R}^4$ given by

$$(z, w) = (x + iy, u + iv) \to (x, y, u, v).$$

In the same way, there is a natural bijection from $\mathbb{C}^2 \cap \{(x, y, u, v) \in \mathbb{R}^4 : y = 0, v = 0\}$ to the real plane $\mathbb{R}^2$, given by

$$(x + 0i, u + 0i) \to (x, 0, u, 0) \to (x, u).$$

Likewise, there is a similar natural bijection from $\mathbb{C}^2 = \{(z, w) \in \mathbb{C}^2\} \cap \{(x, y, u, v) \in \mathbb{R}^4 : y = 0, u = 0\}$ to $\mathbb{R}^2$, given this time by

$$(x + 0i, 0 + vi) \to (x, 0, 0, v) \to (x, v).$$

One way to think about conics in $\mathbb{C}^2$ is to consider two-dimensional slices of $\mathbb{C}^2$. Let

$$C = \{(z, w) \in \mathbb{C}^2 : z^2 + w^2 = 1\}.$$

Exercise 1.3.9. Give a bijection from

$$C \cap \{(x + iy, u + iv) : x, u \in \mathbb{R}, y = 0, v = 0\}$$

to the real circle of unit radius in $\mathbb{R}^2$. (Thus a real circle in the plane $\mathbb{R}^2$ can be thought of as a real slice of the complex curve C.)

Taking a different real slice of C will yield not a circle but a hyperbola.

Exercise 1.3.10. Give a bijection from

$$C \cap \{(x + iy, u + iv) \in \mathbb{R}^4 : x, v \in \mathbb{R}, y = 0, u = 0\}$$

to the hyperbola $\mathrm{V}(x^2 - v^2 - 1)$ in $\mathbb{R}^2$.

Thus the single complex curve C contains both real circles and real hyperbolas.

1.4. The Complex Projective Plane $\mathbb{P}^2$

The goal of this section is to introduce the complex projective plane $\mathbb{P}^2$, which is the natural ambient space (with its higher dimensional analog $\mathbb{P}^n$) for much of algebraic geometry. In $\mathbb{P}^2$, we will see that all ellipses, hyperbolas, and parabolas are equivalent.

In $\mathbb{R}^2$ all ellipses are equivalent, all hyperbolas are equivalent, and all parabolas are equivalent under real affine changes of coordinates. Further, these classes of conics are distinct in $\mathbb{R}^2$. When we move to $\mathbb{C}^2$, ellipses and hyperbolas are equivalent under complex affine changes of coordinates, but parabolas remain distinct. The next step is to describe a larger plane in which all three classes are equivalent.

First, we will define the complex projective plane $\mathbb{P}^2$ and discuss some of its basic properties. While it may not be immediately clear from this definition, we will see how $\mathbb{C}^2$ naturally lives in $\mathbb{P}^2$. Further, the extra points in $\mathbb{P}^2$ that are not in $\mathbb{C}^2$ can be viewed as "points at infinity." Then we will look at the projective analogue of change of coordinates and see how we can view all ellipses, hyperbolas, and parabolas as equivalent.

Definition 1.4.1. Define a relation $\sim$ on points in $\mathbb{C}^3 - \{(0,0,0)\}$ as follows: $(x,y,z) \sim (u,v,w)$ if and only if there exists $\lambda \in \mathbb{C} - \{0\}$ such that $(x,y,z) = (\lambda u, \lambda v, \lambda w)$.

Exercise 1.4.1.

(1) Show that $(2, 1+i, 3i) \sim (2-2i, 2, 3+3i)$.

(2) Show that $(1,2,3) \sim (2,4,6) \sim (-2,-4,-6) \sim (-i,-2i,-3i)$.

(3) Show that $(2, 1+i, 3i) \not\sim (4, 4i, 6i)$.

(4) Show that $(1,2,3) \not\sim (3,6,8)$.

Exercise 1.4.2. Show that $\sim$ is an equivalence relation. (Recall that an *equivalence relation* $\sim$ on a set X satisfies the conditions (i) $a \sim a$ for all $a \in X$, (ii) $a \sim b$ implies $b \sim a$, and (iii) $a \sim b$ and $b \sim c$ implies $a \sim c$.)

Exercise 1.4.3. Suppose that $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ and that $x_1 = x_2 \neq 0$. Show that $y_1 = y_2$ and $z_1 = z_2$.

Exercise 1.4.4. Suppose that $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ with $z_1 \neq 0$ and $z_2 \neq 0$. Show that

$$(x_1, y_1, z_1) \sim \left(\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1\right) = \left(\frac{x_2}{z_2}, \frac{y_2}{z_2}, 1\right) \sim (x_2, y_2, z_2).$$

Let $(x : y : z)$ denote the equivalence class of (x,y,z), i.e. $(x : y : z)$ is the following set

$$(x : y : z) = \left\{(u,v,w) \in \mathbb{C}^3 - \{(0,0,0)\} : (x,y,z) \sim (u,v,w)\right\}.$$

Exercise 1.4.5.

(1) Find the equivalence class of $(0,0,1)$.

(2) Find the equivalence class of $(1,2,3)$.

Exercise 1.4.6. Show that the equivalence classes $(1 : 2 : 3)$ and $(2 : 4 : 6)$ are equal as sets.

Definition 1.4.2. The *complex projective plane* $\mathbb{P}^2(\mathbb{C})$ is the set of equivalence classes of the points in $\mathbb{C}^3 - \{(0,0,0)\}$. This is often written as

$$\mathbb{P}^2(\mathbb{C}) = \left(\mathbb{C}^3 - \{(0,0,0)\}\right) \Big/ \sim .$$

The set of points $\{(x:y:z) \in \mathbb{P}^2(\mathbb{C}) : z = 0\}$ is called the *line at infinity*. We will write $\mathbb{P}^2$ to mean $\mathbb{P}^2(\mathbb{C})$ when the context is clear.

Let $(a,b,c) \in \mathbb{C}^3 - \{(0,0,0)\}$. Then the complex line through this point and the origin $(0,0,0)$ can be defined as all points, (x,y,z), satisfying

$$x = \lambda a,\ y = \lambda b,\ \text{and } z = \lambda c$$

for any complex number λ. Here λ can be thought of as an independent parameter.

Exercise 1.4.7. Explain why the elements of $\mathbb{P}^2$ can intuitively be thought of as complex lines through the origin in $\mathbb{C}^3$.

Exercise 1.4.8. If $c \neq 0$, show, in $\mathbb{C}^3$, that the line $x = \lambda a,\ y = \lambda b,\ z = \lambda c$ intersects the plane $\{(x,y,z) : z = 1\}$ in exactly one point. Show that this point of intersection is $\left(\frac{a}{c}, \frac{b}{c}, 1\right)$.

In the next several exercises we will use

$$\mathbb{P}^2 = \{(x:y:z) \in \mathbb{P}^2 : z \neq 0\} \cup \{(x:y:z) \in \mathbb{P}^2 : z = 0\}$$

to show that $\mathbb{P}^2$ can be viewed as the union of $\mathbb{C}^2$ with the line at infinity.

Exercise 1.4.9. Show that the map $\phi : \mathbb{C}^2 \to \{(x:y:z) \in \mathbb{P}^2 : z \neq 0\}$ defined by $\phi(x,y) = (x:y:1)$ is a bijection.

Exercise 1.4.10. Find a map from $\{(x:y:z) \in \mathbb{P}^2 : z \neq 0\}$ to $\mathbb{C}^2$ that is the inverse of the map ϕ in Exercise 1.4.9.

The maps ϕ and ϕ^{-1} in Exercises 1.4.9 and 1.4.10 show us how to view $\mathbb{C}^2$ inside $\mathbb{P}^2$. Now we show how the set $\{(x:y:z) \in \mathbb{P}^2 : z = 0\}$ corresponds to directions towards infinity in $\mathbb{C}^2$.

Exercise 1.4.11. Consider the line $\ell = \{(x,y) \in \mathbb{C}^2 : ax + by + c = 0\}$ in $\mathbb{C}^2$. Assume $a, b \neq 0$. Explain why, as $|x| \to \infty$, $|y| \to \infty$. (Here, $|x|$ is the modulus of x.)

Exercise 1.4.12. Consider again the line ℓ. We know that a and b cannot both be 0, so we will assume without loss of generality that $b \neq 0$.

(1) Show that the image of ℓ in $\mathbb{P}^2$ under ϕ is the set
$$\{(bx : -ax - c : b) : x \in \mathbb{C}\}.$$

(2) Show that this set equals the following union.
$$\{(bx : -ax - c : b) : x \in \mathbb{C}\} = \{(0 : -c : b)\} \cup \left\{\left(1 : -\frac{a}{b} - \frac{c}{bx} : \frac{1}{x}\right)\right\}.$$

(3) Show that as $|x| \to \infty$, the second set in the above union becomes
$$\{(1 : -\frac{a}{b} : 0)\}.$$

Thus, the points $(1 : -\frac{a}{b} : 0)$ are directions toward infinity and the set $\{(x : y : z) \in \mathbb{P}^2 : z = 0\}$ is the *line at infinity.*

If a point $(a : b : c)$ in $\mathbb{P}^2$ is the image of a point $(x, y) \in \mathbb{C}^2$ under the map from $\phi : \mathbb{C}^2 \to \mathbb{P}^2$, we say that (a, b, c) are *homogeneous coordinates* for (x, y). Notice that homogeneous coordinates for a point $(x, y) \in \mathbb{C}^2$ are not unique. For example, the points $(2 : -3 : 1)$, $(10 : -15 : 5)$, and $(2 - 2i : -3 + 3i : 1 - i)$ all provide homogeneous coordinates for $(2, -3)$.

In order to consider zero sets of polynomials in $\mathbb{P}^2$, a little care is needed. We start with:

Definition 1.4.3. A polynomial is *homogeneous* if every monomial term has the same total degree, that is, if the sum of the exponents in every monomial is the same. The *degree* of the homogeneous polynomial is the total degree of any of its monomials. An equation is homogeneous if every non-zero monomial has the same total degree.

Exercise 1.4.13. Explain why the following polynomials are homogeneous, and find each degree.

(1) $x^2 + y^2 - z^2$

(2) $xz - y^2$

(3) $x^3 + 3xy^2 + 4y^3$

(4) $x^4 + x^2y^2$

Exercise 1.4.14. Explain why the following polynomials are not homogeneous.

(1) $x^2 + y^2 - z$

(2) $xz - y$

(3) $x^2 + 3xy^2 + 4y^3 + 3$

(4) $x^3 + x^2y^2 + x^2$

Exercise 1.4.15. Show that if the homogeneous equation $Ax+By+Cz = 0$ holds for the point (x, y, z) in $\mathbb{C}^3 - \{(0, 0, 0)\}$, then it holds for every point of $\mathbb{C}^3$ that belongs to the equivalence class $(x : y : z)$ in $\mathbb{P}^2$.

Exercise 1.4.16. Show that if the homogeneous equation $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz = 0$ holds for the point (x, y, z) in $\mathbb{C}^3 - \{(0, 0, 0)\}$, then it holds for every point of $\mathbb{C}^3$ that belongs to the equivalence class $(x : y : z)$ in $\mathbb{P}^2$.

Exercise 1.4.17. State and prove the generalization of the previous two exercises for any degree n homogeneous equation $P(x, y, z) = 0$.

Exercise 1.4.18. Consider the non-homogeneous equation $P(x, y, z) = x^2+2y+2z = 0$. Show that $(2, -1, -1)$ satisfies this equation. Find a point of the equivalence class $(2 : -1 : -1)$ that does not satisfy the equation.

Thus the zero set of a non-homogeneous polynomial is not well-defined in $\mathbb{P}^2$. These exercises demonstrate that the only zero sets of polynomials that are well-defined on $\mathbb{P}^2$ are homogeneous polynomials.

To study the behavior at infinity of a curve in $\mathbb{C}^2$, we would like to extend the curve to $\mathbb{P}^2$. Thus, we want to be able to pass from zero sets of polynomials in $\mathbb{C}^2$ to zero sets of homogeneous polynomials in $\mathbb{P}^2$. This motivates our next step, a method to *homogenize* polynomials.

We start with an example. For any point $(x : y : z) \in \mathbb{P}^2$ with $z \neq 0$ we have $(x : y : z) = \left(\frac{x}{z} : \frac{y}{z} : 1\right)$, which we identify, via ϕ^{-1} from Exercise 1.4.10, with the point $\left(\frac{x}{z}, \frac{y}{z}\right) \in \mathbb{C}^2$.

Under this identification, the polynomial $P(x, y) = y - x - 2$ maps to $P(x, y, z) = \frac{y}{z} - \frac{x}{z} - 2$. Since $P(x, y, z) = 0$ and $zP(x, y, z) = 0$ have the same zero set if $z \neq 0$ we clear the denominator and, with an abuse of notation, consider the homogeneous polynomial $P(x, y, z) = y - x - 2z$. The zero set of $P(x, y, z) = y - x - 2z$ in $\mathbb{P}^2$ corresponds to the zero set of $P(x, y) = y - x - 2 = 0$ in $\mathbb{C}^2$ precisely when $z = 1$.

Similarly, the polynomial $x^2 + y^2 - 1$ maps to $\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1$. Again, clear the denominators to obtain the homogeneous polynomial $x^2 + y^2 - z^2$, whose zero set $\mathrm{V}(x^2 + y^2 - z^2) \subset \mathbb{P}^2$ corresponds to the zero set $\mathrm{V}(x^2 + y^2 - 1) \subset \mathbb{C}^2$ when $z = 1$.

Definition 1.4.4. Let $P(x, y)$ be a degree n polynomial defined over $\mathbb{C}^2$. The corresponding homogeneous polynomial defined over $\mathbb{P}^2$ is

$$P(x, y, z) = z^n P\left(\frac{x}{z}, \frac{y}{z}\right).$$

This method is called the *homogenization* of $P(x, y)$.

In a similar manner, we can homogenize an equation.

Exercise 1.4.19. Homogenize the following equations. Then find the point(s) where the curves intersect the line at infinity.

(1) $ax + by + c = 0$

(2) $x^2 + y^2 = 1$

(3) $y = x^2$

(4) $x^2 + 9y^2 = 1$

(5) $y^2 - x^2 = 1$

Exercise 1.4.20. Show that in $\mathbb{P}^2$, any two distinct lines will intersect in a point. Notice this implies that parallel lines in $\mathbb{C}^2$, when embedded in $\mathbb{P}^2$, intersect at the line at infinity.

Exercise 1.4.21. Once we have homogenized an equation, the original variables x and y are no more important than the variable z. Suppose we regard x and z as the original variables in our homogenized equation. Then the image of the xz-plane in $\mathbb{P}^2$ would be $\{(x : y : z) \in \mathbb{P}^2 : y = 1\}$.

(1) Homogenize the equations for the parallel lines $y = x$ and $y = x + 2$.

(2) Now regard x and z as the original variables, and set $y = 1$ to sketch the image of the lines in the xz-plane.

(3) Explain why the lines in part (2) meet at the x-axis.

1.5. Projective Changes of Coordinates

The goal of this section is to define a projective change of coordinates and then show that all ellipses, hyperbolas, and parabolas are equivalent under projective changes of coordinates.

Earlier we described a complex affine change of coordinates from points $(x, y) \in \mathbb{C}^2$ to points $(u, v) \in \mathbb{C}^2$ by setting $u = ax + bx + e$ and $v = cx + dy + f$. We will define the analogue for changing homogeneous coordinates $(x : y : z) \in \mathbb{P}^2$ to homogeneous coordinates $(u : v : w) \in \mathbb{P}^2$. We need the change of coordinates equations to be both homogeneous and linear.

Definition 1.5.1. A *projective change of coordinates* is given by

$$\begin{aligned} u &= a_{11}x + a_{12}y + a_{13}z \\ v &= a_{21}x + a_{22}y + a_{23}z \\ w &= a_{31}x + a_{32}y + a_{33}z, \end{aligned}$$

where the $a_{ij} \in \mathbb{C}$ and

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \neq 0.$$

In matrix language

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $A = (a_{ij})$, $a_{ij} \in \mathbb{C}$, and $\det A \neq 0$.

For any affine change of coordinates, there is a corresponding projective change of coordinates as seen in the following:

Exercise 1.5.1. For the complex affine change of coordinates

$$\begin{aligned} u &= ax + by + e \\ v &= cx + dy + f, \end{aligned}$$

where $a, b, c, d, e, f \in \mathbb{C}$ and $ad - bc \neq 0$, show that

$$\begin{aligned} u &= ax + by + ez \\ v &= cx + dy + fz \\ w &= z \end{aligned}$$

is the corresponding projective change of coordinates.

Definition 1.5.2. Two conics in $\mathbb{P}^2$ are *equivalent under a projective change of coordinates*, or *projectively equivalent*, if the defining homogeneous polynomial for one of the conics can be transformed into the defining polynomial for the other conic via a projective change of coordinates.

By Exercise 1.5.1, if two conics in $\mathbb{C}^2$ are equivalent under a complex affine change of coordinates, then the corresponding conics in $\mathbb{P}^2$ will still be equivalent, but now under a projective change of coordinates.

Exercise 1.5.2. Let $C_1 = \mathrm{V}(x^2 + y^2 - 1)$ be an ellipse in $\mathbb{C}^2$ and let $C_2 = \mathrm{V}(u^2 - v)$ be a parabola in $\mathbb{C}^2$. Homogenize the defining polynomials for C_1 and C_2 and show that the projective change of coordinates

$$\begin{aligned} u &= ix \\ v &= y + z \\ w &= y - z \end{aligned}$$

transforms the ellipse in $\mathbb{P}^2$ into the parabola in $\mathbb{P}^2$.

Exercise 1.5.3. Use the results of Section 1.3, together with the above problem, to show that, under a projective change of coordinates, all ellipses, hyperbolas, and parabolas are equivalent in $\mathbb{P}^2$.

1.6. The Complex Projective Line $\mathbb{P}^1$

The goal of this section is to define the complex projective line $\mathbb{P}^1$ and show that it can be viewed topologically as a sphere. In the next section we will use this to show that ellipses, hyperbolas, and parabolas are also topologically spheres.

We start with the definition of $\mathbb{P}^1$.

Definition 1.6.1. Define an equivalence relation $\sim$ on points in $\mathbb{C}^2 - \{(0,0)\}$ as follows: $(x,y) \sim (u,v)$ if and only if there exists $\lambda \in \mathbb{C} - \{0\}$ such that $(x,y) = (\lambda u, \lambda v)$. Let $(x : y)$ denote the equivalence class of (x,y). The *complex projective line* $\mathbb{P}^1$ is the set of equivalence classes of points in $\mathbb{C}^2 - \{(0,0)\}$. That is,

$$\mathbb{P}^1 = \left(\mathbb{C}^2 - \{(0,0)\}\right) \Big/ \sim .$$

The point $(1 : 0)$ is called the *point at infinity.*

The next series of problems are direct analogues of problems for $\mathbb{P}^2$.

Exercise 1.6.1. Suppose that $(x_1, y_1) \sim (x_2, y_2)$ and that $x_1 = x_2 \neq 0$. Show that $y_1 = y_2$.

Exercise 1.6.2. Suppose that $(x_1, y_1) \sim (x_2, y_2)$ with $y_1 \neq 0$ and $y_2 \neq 0$. Show that

$$(x_1, y_1) \sim \left(\frac{x_1}{y_1}, 1\right) = \left(\frac{x_2}{y_2}, 1\right) \sim (x_2, y_2).$$

Exercise 1.6.3. Explain why the elements of $\mathbb{P}^1$ can intuitively be thought of as complex lines through the origin in $\mathbb{C}^2$.

Exercise 1.6.4. If $b \neq 0$, show that the line $x = \lambda a$, $y = \lambda b$ will intersect the line $\{(x,y) : y = 1\}$ in exactly one point. Show that this point of intersection is $\left(\frac{a}{b}, 1\right)$.

We have that

$$\mathbb{P}^1 = \left\{(x : y) \in \mathbb{P}^1 : y \neq 0\right\} \cup \{(1 : 0)\}.$$

Exercise 1.6.5. Show that the map $\phi : \mathbb{C} \to \{(x : y) \in \mathbb{P}^1 : y \neq 0\}$ defined by $\phi(x) = (x : 1)$ is a bijection.

Exercise 1.6.6. Find a map from $\{(x : y) \in \mathbb{P}^1 : y \neq 0\}$ to $\mathbb{C}$ that is the inverse of the map ϕ in Exercise 1.6.5.

The maps ϕ and ϕ^{-1} in Exercises 1.6.5 and 1.6.6 show us how to view $\mathbb{C}$ inside $\mathbb{P}^1$. Now we want to see how the extra point $(1 : 0)$ will correspond to the point at infinity of $\mathbb{C}$.

Exercise 1.6.7. Consider the map $\phi : \mathbb{C} \to \mathbb{P}^1$ given by $\phi(x) = (x : 1)$. Show that as $|x| \to \infty$, we have $\phi(x) \to (1 : 0)$.

Hence we can think of $\mathbb{P}^1$ as the union of $\mathbb{C}$ and a single point at infinity. Now we want to see how we can regard $\mathbb{P}^1$ as a sphere, which means we want to find a homeomorphism between $\mathbb{P}^1$ and a sphere. A *homeomorphism* is a continuous map with a continuous inverse. Two spaces are topologically equivalent, or homeomorphic, if we can find a homeomorphism from one to the other. We know that the points of $\mathbb{C}$ are in one-to-one correspondence with the points of the real plane $\mathbb{R}^2$, so we will first work in $\mathbb{R}^2 \subset \mathbb{R}^3$. Specifically, identify $\mathbb{R}^2$ with the xy-plane in $\mathbb{R}^3$ via $(x, y) \mapsto (x, y, 0)$. Let S^2 denote the unit sphere in $\mathbb{R}^3$ centered at the origin. This sphere is given by the equation

$$x^2 + y^2 + z^2 = 1.$$

Exercise 1.6.8. Let p denote the point $(0, 0, 1) \in S^2$, and let ℓ denote the line through p and the point $(x, y, 0)$ in the xy-plane, whose parametrization is given by

$$\gamma(t) = (1 - t)(0, 0, 1) + t(x, y, 0),$$

i.e.,

$$\ell = \{(tx, ty, 1 - t) \mid t \in \mathbb{R}\}.$$

(1) ℓ clearly intersects S^2 at the point p. Show that there is exactly one other point of intersection q.
(2) Find the coordinates of q.
(3) Define the map $\psi : \mathbb{R}^2 \to S^2 - \{p\}$ to be the map that takes the point (x, y) to the point q. Show that ψ is a continuous bijection.

(4) Show that as $\sqrt{x^2+y^2} \to \infty$, we have $\psi(x,y) \to p$. Thus as we move away from the origin in $\mathbb{R}^2$, $\psi(x,y)$ moves toward the North Pole.

The above argument does establish a homeomorphism, but it relies on coordinates and an embedding of the sphere in $\mathbb{R}^3$. We now give an alternative method for showing that $\mathbb{P}^1$ is a sphere that does not rely as heavily on coordinates.

If we take a point $(x : y) \in \mathbb{P}^1$, then we can choose a representative for this point of the form $\left(\frac{x}{y} : 1\right)$, provided $y \neq 0$, and a representative of the form $\left(1 : \frac{y}{x}\right)$, provided $x \neq 0$.

Exercise 1.6.9. Determine which point(s) in $\mathbb{P}^1$ do **not** have two representatives of the form $(x : 1) = \left(1 : \frac{1}{x}\right)$.

Our construction needs two copies of $\mathbb{C}$. Let U denote the first copy of $\mathbb{C}$, whose elements are denoted by x. Let V be the second copy of $\mathbb{C}$, whose elements we'll denote y. Further let $U^* = U - \{0\}$ and $V^* = V - \{0\}$.

Exercise 1.6.10. Map $U \to \mathbb{P}^1$ via $x \mapsto (x : 1)$ and map $V \to \mathbb{P}^1$ via $y \mapsto (1 : y)$. Show that $(x : 1) \mapsto \left(1 : \frac{1}{x}\right)$ is a natural one-to-one map from U^* onto V^*.

The next two exercises have quite a different flavor than most of the problems in the book. The emphasis is not on calculations but on the underlying intuitions.

Exercise 1.6.11. A sphere can be split into a neighborhood of its northern hemisphere and a neighborhood of its southern hemisphere. Show that a sphere can be obtained by correctly gluing together two copies of $\mathbb{C}$.

Exercise 1.6.12. Put together the last two exercises to show that $\mathbb{P}^1$ is topologically equivalent to a sphere.

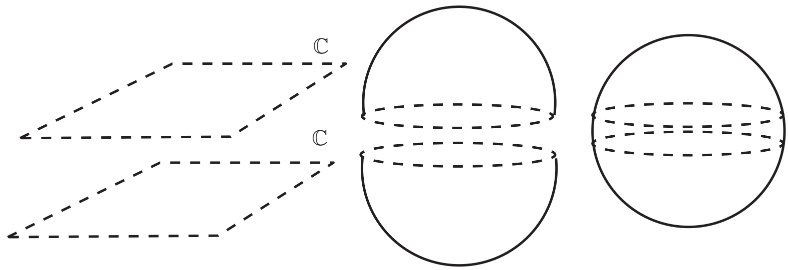


Figure 6. Gluing copies of $\mathbb{C}$ together.

1.7. Ellipses, Hyperbolas, and Parabolas as Spheres

The goal of this section is to show that there is always a bijective polynomial map from $\mathbb{P}^1$ to any ellipse, hyperbola, or parabola. Since we showed in the last section that $\mathbb{P}^1$ is topologically equivalent to a sphere, this means that all ellipses, hyperbolas, and parabolas are spheres.

We start with rational parameterizations of conics. While we will consider conics in the complex plane $\mathbb{C}^2$, we often draw these conics in $\mathbb{R}^2$. Part of learning algebraic geometry is developing a sense for when the real pictures capture what is going on in the complex plane.

Consider a conic $C = \{(x,y) \in \mathbb{C}^2 : P(x,y) = 0\} \subset \mathbb{C}^2$, where $P(x,y)$ is a second degree polynomial. Our goal is to parametrize C with polynomial or rational maps. This means we want to find a map $\phi : \mathbb{C} \to C \subset \mathbb{C}^2$, given by $\phi(\lambda) = (x(\lambda), y(\lambda))$ such that $x(\lambda)$ and $y(\lambda)$ are polynomials or rational functions. In the case of a parabola, for example when $P(x,y) = x^2 - y$, it is easy to find a bijection from $\mathbb{C}$ to the conic C.

Exercise 1.7.1. Find a bijective polynomial map from $\mathbb{C}$ to the conic $C = \{(x,y) \in \mathbb{C}^2 : x^2 - y = 0\}$.

Sometimes it may be easy to find a parametrization but not one that is rational.

Exercise 1.7.2. Let $C = \mathrm{V}(x^2 + y^2 - 1)$ be an ellipse in $\mathbb{C}^2$. Find a trigonometric parametrization of C. [Hint: Think high school trigonometry.]

This exercise gives a parameterization for the circle, but in algebraic geometry we restrict our parameterizations to polynomial or rational maps. We develop a standard method, similar to the method developed in Exercise 1.6.8, to find such a parameterization below.

Exercise 1.7.3. Consider the ellipse $C = \mathrm{V}(x^2 + y^2 - 1) \subset \mathbb{C}^2$ and let p denote the point $(0, 1) \in C$.

(1) Parametrize the line segment from p to the point $(\lambda, 0)$ on the complex line $y = 0$ as in Exercise 1.6.8.

(2) This line segment clearly intersects C at the point p. Show that if $\lambda \neq \pm i$, then there is exactly one other point of intersection. Call this point q.

(3) Find the coordinates of $q \in C$.

(4) Show that if $\lambda = \pm i$, then the line segment intersects C only at p.

Define the map $\widetilde{\psi} : \mathbb{C} \to C \subset \mathbb{C}^2$ by

$$\widetilde{\psi}(\lambda) = \left(\frac{2\lambda}{\lambda^2 + 1}, \frac{\lambda^2 - 1}{\lambda^2 + 1} \right).$$

But we want to work in projective space. This means that we have to homogenize our map.

Exercise 1.7.4. Show that the above map can be extended to the map

$$\psi : \mathbb{P}^1 \to \{(x : y : z) \in \mathbb{P}^2 : x^2 + y^2 - z^2 = 0\}$$

given by

$$\psi(\lambda : \mu) = (2\lambda\mu : \lambda^2 - \mu^2 : \lambda^2 + \mu^2).$$

Exercise 1.7.5.

(1) Show that the map ψ is one-to-one.

(2) Show that ψ is onto. [Hint: Consider two cases: $z \neq 0$ and $z = 0$. For $z \neq 0$ follow the construction given above. For

$z = 0$, find values of λ and μ to show that these points are given by ψ. How does this relate to Part 4 of Exercise 1.7.3?]

Since we already know that every ellipse, hyperbola, and parabola is projectively equivalent to the conic defined by $x^2 + y^2 - z^2 = 0$, we have, by composition, a one-to-one and onto map from $\mathbb{P}^1$ to any ellipse, hyperbola, or parabola.

However, we can construct such maps directly. Here is what we can do for any conic C. Fix a point p on C, and parametrize the line segment through p and the point $(\lambda, 0)$. We use this to determine another point on the curve C, and the coordinates of this point give us our map.

Exercise 1.7.6. For the following conics and the given point p, follow what we did for the conic $x^2 + y^2 - 1 = 0$ to find a rational map from $\mathbb{C}$ to the curve in $\mathbb{C}^2$ and then a one-to-one map from $\mathbb{P}^1$ onto the conic in $\mathbb{P}^2$.

(1) $x^2 + 2x - y^2 - 4y - 4 = 0$ with $p = (0, -2)$
(2) $3x^2 + 3y^2 - 75 = 0$ with $p = (5, 0)$
(3) $4x^2 + y^2 - 8 = 0$ with $p = (1, 2)$

1.8. Links to Number Theory

The goal of this section is to see how geometry can be used to find all primitive Pythagorean triples, a classic problem from number theory.

Overwhelmingly, in this book we are interested in working over the complex numbers. If instead we work over the integers or the rational numbers, some of the deepest questions in mathematics appear.

We want to see this approach in the case of conics. In particular we want to link the last section to the search for primitive Pythagorean triples. A *Pythagorean triple* is a triple, (x, y, z), of integers that satisfies the equation

$$x^2 + y^2 = z^2.$$

Exercise 1.8.1. Suppose (x_0, y_0, z_0) is a solution to $x^2 + y^2 = z^2$. Show that (mx_0, my_0, mz_0) is also a solution for any scalar m.

A *primitive Pythagorean triple* is a Pythagorean triple that cannot be obtained by multiplying another Pythagorean triple by an integer. The simplest example, after the trivial solution $(0, 0, 0)$, is $(3, 4, 5)$. These triples get their name from the attempt to find right triangles with integer length sides, x, y, and z. We will see that the previous section gives us a method to compute all possible primitive Pythagorean triples.

We first see how to translate the problem of finding integer solutions of $x^2+y^2 = z^2$ to finding rational number solutions to $x^2+y^2 = 1$.

Exercise 1.8.2. Let $(a, b, c) \in \mathbb{Z}^3$ be a solution to $x^2+y^2 = z^2$. Show that $c = 0$ if and only if $a = b = 0$.

This means that we can assume $c \neq 0$, since there is only one solution when $c = 0$.

Exercise 1.8.3. Show that if (a, b, c) is a Pythagorean triple with $c \neq 0$, then the pair of rational numbers $\left(\frac{a}{c}, \frac{b}{c}\right)$ is a solution to $x^2 + y^2 = 1$.

Exercise 1.8.4. Let $\left(\frac{a}{c_1}, \frac{b}{c_2}\right) \in \mathbb{Q}^2$ be a rational solution to $x^2 + y^2 = 1$. Find a corresponding Pythagorean triple.

Thus to find Pythagorean triples, we want to find the rational points on the curve $x^2 + y^2 = 1$. We denote these points as

$$C(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 : x^2 + y^2 = 1\}.$$

Recall from the last section, the parameterization

$$\widetilde{\psi} : \mathbb{Q} \to \{(x, y) \in \mathbb{Q}^2 : x^2 + y^2 = 1\}$$

given by

$$\lambda \mapsto \left(\frac{2\lambda}{\lambda^2 + 1}, \frac{\lambda^2 - 1}{\lambda^2 + 1}\right).$$

Exercise 1.8.5. Show that the above map $\widetilde{\psi}$ sends $\mathbb{Q} \to C(\mathbb{Q})$.

Extend this to a map $\psi : \mathbb{P}^1(\mathbb{Q}) \to C(\mathbb{Q}) \subset \mathbb{P}^2(\mathbb{Q})$ by

$$(\lambda : \mu) \mapsto (2\lambda\mu : \lambda^2 - \mu^2 : \lambda^2 + \mu^2),$$

where $\lambda, \mu \in \mathbb{Z}$. Since we know already that the map ψ is one-to-one by Exercise 1.7.5, this gives us a way to produce an infinite number of integer solutions to $x^2 + y^2 = z^2$.

We now want to show that the map ψ is onto, so that we actually obtain all Pythagorean triples.

Exercise 1.8.6.

(1) Show that $\psi : \mathbb{P}^1(\mathbb{Q}) \to C(\mathbb{Q}) \subset \mathbb{P}^2(\mathbb{Q})$ is onto.

(2) Show that every primitive Pythagorean triple is of the form $(2\lambda\mu, \lambda^2 - \mu^2, \lambda^2 + \mu^2)$.

Exercise 1.8.7. Find a rational point on the conic $x^2 + y^2 - 2 = 0$. Develop a parameterization and conclude that there are infinitely many rational points on this curve.

Exercise 1.8.8. By mimicking the above, find four rational points on each of the following conics.

(1) $x^2 + 2x - y^2 - 4y - 4 = 0$ with $p = (0, -2)$

(2) $3x^2 + 3y^2 - 75 = 0$ with $p = (5, 0)$

(3) $4x^2 + y^2 - 8 = 0$ with $p = (1, 2)$

Exercise 1.8.9. Show that the conic $x^2 + y^2 = 3$ has no rational points.

Diophantine problems are those where you try to find integer or rational solutions to a polynomial equation. The work in this section shows how we can approach such problems using algebraic geometry. For higher degree equations the situation is quite different and leads to the heart of a great deal of the current research in number theory.

1.9. Degenerate Conics

The goal of this section is to extend our study of conics from ellipses, hyperbolas, and parabolas to the "degenerate" conics: crossing lines and double lines.

Let $f(x, y, z)$ be any homogeneous second degree polynomial with complex coefficients. The overall goal of this chapter is to understand curves

$$C = \{(x : y : z) \in \mathbb{P}^2 : f(x, y, z) = 0\}.$$

Most of these curves will be various ellipses, hyperbolas, and parabolas. Now consider the second degree polynomial

$$f(x, y, z) = (-x+y+z)(2x+y+3z) = -2x^2+y^2+3z^2+xy-xz+4yz.$$

Exercise 1.9.1. Dehomogenize $f(x, y, z)$ by setting $z = 1$. Graph the curve

$$C(\mathbb{R}) = \{(x : y : z) \in \mathbb{P}^2 : f(x, y, 1) = 0\}$$

in the real plane $\mathbb{R}^2$.

The zero set of a second degree polynomial could be the union of crossing lines.

Exercise 1.9.2. Consider the two lines given by

$$(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0,$$

and suppose

$$\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \neq 0.$$

Show that the two lines intersect at a point where $z \neq 0$.

Exercise 1.9.3. Dehomogenize the equation in the previous exercise by setting $z = 1$. Give an argument that, as lines in the complex plane $\mathbb{C}^2$, they have distinct slopes.

Exercise 1.9.4. Again consider the two lines

$$(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0.$$

Suppose that

$$\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0$$

but that

$$\det \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix} \neq 0 \text{ or } \det \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} \neq 0.$$

Show that the two lines still have one common point of intersection, but that this point must have $z = 0$.

There is one other possibility. Consider the zero set

$$C = \{(x : y : z) \in \mathbb{P}^2 : (ax + by + cz)^2 = 0\}.$$

As a zero set, the curve C is geometrically the line

$$ax + by + cz = 0$$

but due to the exponent 2, we call C a *double line*.

Exercise 1.9.5. Let

$$f(x, y, z) = (a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z),$$

where at least one of $a_1, b_1,$ or c_1 is non-zero and at least one of the $a_2, b_2,$ or c_2 is non-zero. Show that the curve defined by $f(x, y, z) = 0$ is a double line if and only if

$$\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0, \quad \det \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix} = 0, \quad \text{and } \det \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} = 0.$$

We now want to show that any two crossing lines are equivalent under a projective change of coordinates to any other two crossing lines and any double line is equivalent under a projective change of coordinates to any other double line. This will yield that there are precisely three types of conics: the ellipses, hyperbolas, and parabolas; crossing lines; and double lines.

For the exercises that follow, assume that at least one of $a_1, b_1,$ or c_1 is non-zero and at least one of $a_2, b_2,$ or c_2 is non-zero.

Exercise 1.9.6. Consider the crossing lines

$$(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0,$$

with

$$\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \neq 0.$$

Find a projective change of coordinates from xyz-space to uvw-space so that the crossing lines become

$$uv = 0.$$

Exercise 1.9.7. Consider the crossing lines $(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z) = 0$, with

$$\det \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix} \neq 0.$$

Find a projective change of coordinates from xyz-space to uvw-space so that the crossing lines become

$$uv = 0.$$

Exercise 1.9.8. Show that there is a projective change of coordinates from xyz-space to uvw-space so that the double line $(ax+by+cz)^2 = 0$ becomes the double line

$$u^2 = 0.$$

Exercise 1.9.9. Argue that there are three distinct classes of conics in $\mathbb{P}^2$.

1.10. Tangents and Singular Points

The goal of this section is to develop the idea of singularity. We'll show that all ellipses, hyperbolas, and parabolas are smooth, while crossing lines and double lines are singular.

So far, we have not explicitly needed to use calculus; that changes in this section. We will use the familiar differentiation rules from real calculus.

Let $f(x, y)$ be a polynomial. Recall that if $f(a, b) = 0$, then a normal vector for the curve $f(x, y) = 0$ at the point (a, b) is given by the gradient vector

$$\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right).$$

A tangent vector to the curve at the point (a, b) is perpendicular to $\nabla f(a, b)$ and hence must have a dot product of zero with $\nabla f(a, b)$. This observation shows that the tangent line is given by

$$\left\{(x,y) \in \mathbb{C}^2 : \left(\frac{\partial f}{\partial x}(a,b)\right)(x-a) + \left(\frac{\partial f}{\partial y}(a,b)\right)(y-b) = 0\right\}.$$

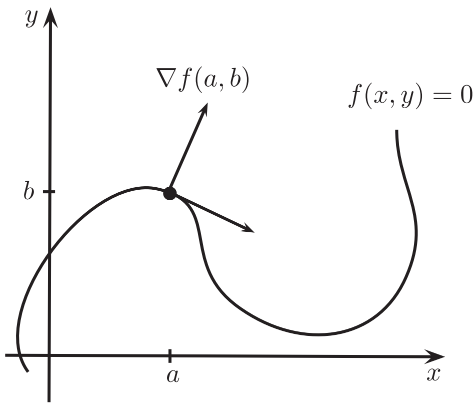


Figure 7. Gradient and tangent vectors.

Exercise 1.10.1. Explain why if both $\frac{\partial f}{\partial x}(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b) = 0$, then the tangent line is not well-defined at (a, b).

This exercise motivates the following definition.

Definition 1.10.1. A point $p = (a, b)$ on a curve $C = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}$ is said to be *singular* if

$$\frac{\partial f}{\partial x}(a,b) = 0 \text{ and } \frac{\partial f}{\partial y}(a,b) = 0.$$

A point that is not singular is called *smooth*. If there is at least one singular point on C, then the curve C is called *singular*. If there are no singular points on C, the curve C is called *smooth*.

Exercise 1.10.2. Show that the curve

$$C = \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 - 1 = 0\}$$

is smooth.

Exercise 1.10.3. Show that the pair of crossing lines

$$C = \{(x, y) \in \mathbb{C}^2 : (x + y - 1)(x - y - 1) = 0\}$$

has exactly one singular point. Give a geometric interpretation of this singular point.

Exercise 1.10.4. Show that every point on the double line

$$C = \{(x, y) \in \mathbb{C}^2 : (2x + 3y - 4)^2 = 0\}$$

is singular.

These definitions can also be applied to curves in $\mathbb{P}^2$.

Definition 1.10.2. A point $p = (a : b : c)$ on a curve $C = \{(x : y : z) \in \mathbb{P}^2 : f(x, y, z) = 0\}$, where $f(x, y, z)$ is a homogeneous polynomial, is said to be *singular* if

$$\frac{\partial f}{\partial x}(a, b, c) = 0, \quad \frac{\partial f}{\partial y}(a, b, c) = 0, \text{ and } \frac{\partial f}{\partial z}(a, b, c) = 0.$$

We have similar definitions, as before, for smooth point, smooth curve, and singular curve.

Exercise 1.10.5. Show that the curve

$$C = \{(x : y : z) \in \mathbb{P}^2 : x^2 + y^2 - z^2 = 0\}$$

is smooth.

Exercise 1.10.6. Show that the pair of crossing lines

$$C = \{(x : y : z) \in \mathbb{P}^2 : (x + y - z)(x - y - z) = 0\}$$

has exactly one singular point.

Exercise 1.10.7. Show that every point on the double line

$$C = \{(x : y : z) \in \mathbb{P}^2 : (2x + 3y - 4z)^2 = 0\}$$

is singular.

For homogeneous polynomials, there is a simple relationship among $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$, which is the goal of the next few exercises.

Exercise 1.10.8. For

$$f(x, y, z) = x^2 + 3xy + 5xz + y^2 - 7yz + 8z^2,$$

show that

$$2f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}.$$

Exercise 1.10.9. For

$$f(x, y, z) = ax^2 + bxy + cxz + dy^2 + eyz + hz^2,$$

show that

$$2f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}.$$

Exercise 1.10.10. Let $f(x, y, z)$ be a homogeneous polynomial of degree n. Show that

$$nf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}.$$

(This problem is quite similar to the previous two, but working out the details takes some work.)

Exercise 1.10.11. Use Exercise 1.10.10 to show that if $p = (a : b : c)$ satisfies

$$\frac{\partial f}{\partial x}(a, b, c) = \frac{\partial f}{\partial y}(a, b, c) = \frac{\partial f}{\partial z}(a, b, c) = 0,$$

then $p \in \mathrm{V}(f)$.

The notion of smooth curves and singular curves certainly extends beyond the study of conics. We will briefly discuss higher degree curves here. Throughout, we will see that *singular* corresponds to not having a well-defined tangent.

Exercise 1.10.12. Graph the curve

$$f(x, y) = x^3 + x^2 - y^2 = 0$$

in the real plane $\mathbb{R}^2$. What is happening at the origin $(0, 0)$? Find the singular points.

Exercise 1.10.13. Graph the curve

$$f(x, y) = x^3 - y^2 = 0$$

in the real plane $\mathbb{R}^2$. What is happening at the origin $(0, 0)$? Find the singular points.

For any two polynomials, $f_1(x, y)$ and $f_2(x, y)$, let $f(x, y) = f_1(x, y)f_2(x, y)$ be their product. We have

$$\mathrm{V}(f) = \mathrm{V}(f_1) \cup \mathrm{V}(f_2).$$

The picture of these curves is

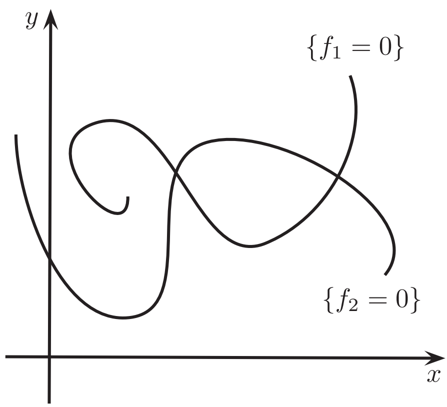


Figure 8. Curves $\mathrm{V}(f_1)$ and $\mathrm{V}(f_2)$.

From the picture, it seems that the curve $\mathrm{V}(f)$ should have singular points at the points of intersection of $\mathrm{V}(f_1)$ and $\mathrm{V}(f_2)$.

Exercise 1.10.14. Suppose that

$$f_1(a,b) = 0 \quad \text{and} \quad f_2(a,b) = 0$$

for a point $(a,b) \in \mathbb{C}^2$. Show that (a,b) is a singular point on $\mathrm{V}(f)$, where $f = f_1 f_2$.

While it is safe to say for higher degree curves and especially for higher dimensional algebraic geometric objects that "singularness" is far from understood, that is not the case for conics. A complete description is contained in the following theorem.

Theorem 1.10.15. All ellipses, hyperbolas, and parabolas are smooth curves. All conics that are crossing lines have exactly one singular point, namely the point of intersection of the two lines. Every point on a double line is singular.

We have seen specific examples for each of these. The proof of the theorem relies on the fact that under projective transformations there are three distinct classes of conics. We motivated the idea of projective changes of coordinates as just the relabeling of coordinate systems. Surely how we label points on the plane should not affect the lack of a well-defined tangent line. Hence a projective change of coordinates should not affect whether or not a point is smooth or singular. The next series of exercises proves this.

Consider a projective change of coordinates from xyz-space to uvw-space given by

$$\begin{aligned} u &= a_{11}x + a_{12}y + a_{13}z \\ v &= a_{21}x + a_{22}y + a_{23}z \\ w &= a_{31}x + a_{32}y + a_{33}z, \end{aligned}$$

where $a_{ij} \in \mathbb{C}$ and

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \neq 0.$$

In $\mathbb{P}^2$, with homogeneous coordinates $(u : v : w)$, consider a curve $C = \{(u : v : w) : f(u, v, w) = 0\}$, where f is a homogeneous polynomial. The (inverse) change of coordinates above gives a map from polynomials in $(u : v : w)$ to polynomials in $(x : y : z)$ described by

$$\begin{aligned} f(u, v, w) \mapsto & f(a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z, \\ & a_{31}x + a_{32}y + a_{33}z) = \widetilde{f}(x, y, z). \end{aligned}$$

The curve C corresponds to the curve $\widetilde{C} = \{(x : y : z) : \widetilde{f}(x, y, z) = 0\}$.

Exercise 1.10.16. Consider the curve

$$C = \{(u : v : w) \in \mathbb{P}^2 : u^2 - v^2 - w^2 = 0\}.$$

Suppose we have the projective change of coordinates given by

$$\begin{aligned} u &= x + y \\ v &= x - y \\ w &= z. \end{aligned}$$

Show that C corresponds to the curve

$$\widetilde{C} = \{(x : y : z) \in \mathbb{P}^2 : 4xy - z^2 = 0\}.$$

In other words, if $f(u,v,w) = u^2 - v^2 - w^2$, then $\widetilde{f}(x,y,z) = 4xy - z^2$.

Exercise 1.10.17. Suppose we have the projective change of coordinates given by

$$\begin{aligned} u &= x + y \\ v &= x - y \\ w &= x + y + z. \end{aligned}$$

If $f(u,v,w) = u^2 + uw + v^2 + vw$, find $\widetilde{f}(x,y,z)$.

Exercise 1.10.18. For a general projective change of coordinates given by

$$\begin{aligned} u &= a_{11}x + a_{12}y + a_{13}z \\ v &= a_{21}x + a_{22}y + a_{23}z \\ w &= a_{31}x + a_{32}y + a_{33}z \end{aligned}$$

and a polynomial $f(u,v,w)$, describe how to find the corresponding $\widetilde{f}(x,y,z)$.

We now want to show, under a projective change of coordinates, that singular points go to singular points and smooth points go to smooth points.

Exercise 1.10.19. Let

$$\begin{aligned} u &= a_{11}x + a_{12}y + a_{13}z \\ v &= a_{21}x + a_{22}y + a_{23}z \\ w &= a_{31}x + a_{32}y + a_{33}z \end{aligned}$$

be a projective change of coordinates. Show that $(u_0 : v_0 : w_0)$ is a singular point of the curve $C = \{(u : v : w) : f(u,v,w) = 0\}$ if and only if the corresponding point $(x_0 : y_0 : z_0)$ is a singular point of the corresponding curve $\widetilde{C} = \{(x : y : z) : \widetilde{f}(x,y,z) = 0\}$. (This is an exercise in the multi-variable chain rule; most people are not comfortable with the chain rule without a lot of practice. Hence the value of this exercise.)

Exercise 1.10.20. Use the previous exercise to prove Theorem 1.10.15.

1.11. Conics via Linear Algebra

The goal of this section is to show how to interpret conics via linear algebra. The linear algebra of symmetric 3×3 matrices will lead to straightforward proofs that, under projective changes of coordinates, all ellipses, hyperbolas, and parabolas are equivalent; all crossing line conics are equivalent; and all double lines are equivalent.

1.11.1. Conics via 3×3 Symmetric Matrices. We start by showing how to represent conics with symmetric 3×3 matrices. Consider the second degree homogeneous polynomial

$$\begin{aligned} f(x,y,z) &= x^2 + 6xy + 5y^2 + 4xz + 8yz + 9z^2 \\ &= x^2 + (3xy + 3yx) + 5y^2 + (2xz + 2zx) \\ &\qquad + (4yz + 4zy) + 9z^2 \\ &= \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 3 & 5 & 4 \\ 2 & 4 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

By using seemingly silly tricks such as $6xy = 3xy + 3yx$, we have written our initial second degree polynomial in terms of the symmetric 3×3 matrix

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 5 & 4 \\ 2 & 4 & 9 \end{pmatrix}.$$

There is nothing special about this particular second degree polynomial. We can write all homogeneous second degree polynomials $f(x,y,z)$ in terms of symmetric 3×3 matrices. (Recall that a matrix $A = (a_{ij})$ is symmetric if $a_{ij} = a_{ji}$ for all i and j. Since the transpose of A simply switches the row and column entries, $A^T = (a_{ji})$, A is symmetric if and only if $A = A^T$.)

Exercise 1.11.1. Write the following conics in the form

$$(x\ y\ z)\, A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

That is, find a symmetric matrix A for each quadratic equation.

(1) $x^2 + y^2 + z^2 = 0$
(2) $x^2 + y^2 - z^2 = 0$
(3) $x^2 - y^2 = 0$
(4) $x^2 + 2xy + y^2 + 3xz + z^2 = 0$

Symmetric matrices can be used to define second degree homogeneous polynomials with any number of variables.

Definition 1.11.1. A *quadratic form* is a homogeneous polynomial of degree two in any given number of variables. Given a symmetric $n \times n$ matrix A and $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, then $f(X) = X^T AX$ is a quadratic form.

Thus conics are defined by quadratic forms in three variables.

Exercise 1.11.2. Show that any conic

$$f(x, y, z) = ax^2 + bxy + cy^2 + dxz + eyz + hz^2$$

can be written as

$$\begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where A is a symmetric 3×3 matrix.

1.11.2. Change of Variables via Matrices. We want to see that a projective change of coordinates has a quite natural linear algebra interpretation.

Suppose we have a projective change of coordinates

$$\begin{aligned} u &= a_{11}x + a_{12}y + a_{13}z \\ v &= a_{21}x + a_{22}y + a_{23}z \\ w &= a_{31}x + a_{32}y + a_{33}z. \end{aligned}$$

The matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

that encodes the projective change of coordinates will be key.

Suppose $f(u,v,w)$ is a second degree homogeneous polynomial and let $\widetilde{f}(x,y,z)$ be the corresponding second degree homogeneous polynomial in the xyz-coordinate system. From the previous subsection, we know that there are two 3×3 symmetric matrices A and B such that

$$f(u,v,w) = \begin{pmatrix} u & v & w \end{pmatrix} A \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \widetilde{f}(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} B \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We want to find a relation between the three matrices M, A, and B.

Exercise 1.11.3. Let C be a 3×3 matrix and let X be a 3×1 matrix. Show that $(CX)^T = X^T C^T$.

Exercise 1.11.4. Let M be a projective change of coordinates

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and suppose

$$f(u,v,w) = \begin{pmatrix} u & v & w \end{pmatrix} A \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \widetilde{f}(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} B \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Show that

$$B = M^T A M.$$

As a pedagogical aside, if we were following the format of earlier problems, before stating the above theorem, we would have given some concrete exercises illustrating the general principle. We have chosen not to do that here. In part, it is to allow readers to come up with their own concrete examples, if necessary. The other part is that this entire section's goal is not only to link linear algebra with conics but also to (not so secretly) force readers to review some linear algebra.

Recall the following definitions from linear algebra.

Definition 1.11.2. We say that two $n \times n$ matrices A and B are *similar*, $A \sim B$, if there is an invertible $n \times n$ matrix C such that

$$A = C^{-1}BC.$$

Definition 1.11.3. An $n \times n$ matrix C is *orthogonal* if $C^{-1} = C^T$.

Definition 1.11.4. A matrix A has an eigenvalue λ if $Av = \lambda v$ for some non-zero vector v. The vector v is called an *eigenvector* with associated *eigenvalue* λ.

Exercise 1.11.5. Given a 3×3 matrix A, show that A has exactly three eigenvalues, counting multiplicity. (For this problem, it is fine to find the proof in a linear algebra text. After looking it up, close the book and try to reproduce the proof on your own. Repeat as necessary until you get it. This is, of course, another attempt by the authors to coax the reader into reviewing linear algebra.)

Exercise 1.11.6.

(1) Let A and B be two symmetric matrices, neither of which has a zero eigenvalue. Show there is an invertible 3×3 matrix C such that

$$A = C^T BC.$$

(2) Let A and B be two symmetric matrices, each of which has exactly one zero eigenvalue (with the other two eigenvalues being non-zero). Show that there is an invertible 3×3 matrix C such that

$$A = C^T BC.$$

(3) Now let A and B be two symmetric matrices, each of which has a zero eigenvalue with multiplicity two (and hence the remaining eigenvalue must be non-zero). Show that there is an invertible 3×3 matrix C such that

$$A = C^T BC.$$

(Again, it is fine to look up this deep result in a linear algebra text. Just make sure that you can eventually reproduce it on your own.)

Exercise 1.11.7.

(1) Show that the 3×3 matrix associated to the ellipse $\mathrm{V}(x^2 + y^2 - z^2)$ has three non-zero eigenvalues.

(2) Show that the 3×3 matrix associated to the two crossing lines $\mathrm{V}(xy)$ has one zero eigenvalue and two non-zero eigenvalues.

(3) Finally, show that the 3×3 matrix associated to the double line $\mathrm{V}((x-y)^2)$ has a zero eigenvalue of multiplicity two and a non-zero eigenvalue.

Exercise 1.11.8. Based on the material of this section, give another proof that under projective changes of coordinates all ellipses, hyperbolas, and parabolas are the same, all crossing line conics are the same, and all double lines are the same.

1.11.3. Conics in $\mathbb{R}^2$. We have shown that all smooth conics can be viewed as the same in the complex projective plane $\mathbb{P}^2$. As we saw earlier, ellipses, hyperbolas, and parabolas are quite different in the real plane $\mathbb{R}^2$. There is a more linear-algebraic approach that captures these differences.

Let $f(x, y, z) = ax^2 + bxy + cy^2 + dxz + eyz + hz^2 = 0$, with $a, b, c, d, e, h \in \mathbb{R}$. Dehomogenize by setting $z = 1$, so that we are looking at the polynomial

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + h,$$

which can be written as

$$f(x,y) = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & h \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

In $\mathbb{P}^2$, the coordinates x, y, and z all play the same role. That is no longer the case after setting $z = 1$. The second order term of f,

$$ax^2 + bxy + cy^2,$$

determines whether we have an ellipse, hyperbola, or parabola.

Exercise 1.11.9. Explain why we need to consider only the second order terms. [Hint: We have already answered this question earlier in this chapter.]

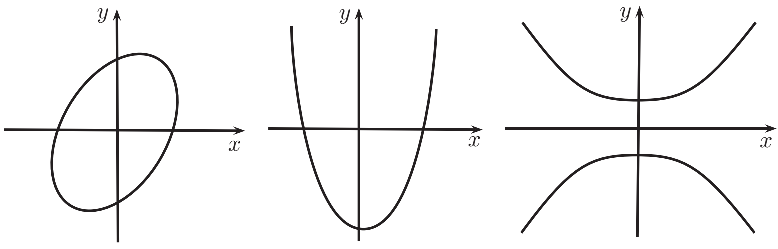


Figure 9. Three types of conics.

This suggests that the matrix

$$\begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$$

might be worth investigating.

Definition 1.11.5. The *discriminant* of a conic over $\mathbb{R}^2$ is

$$\Delta = -4 \det \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}.$$

Exercise 1.11.10. Find the discriminant of each of the following conics.

(1) $9x^2 + 4y^2 = 1$
(2) $9x^2 - 4y^2 = 1$
(3) $9x^2 - y = 0$

Exercise 1.11.11. Based on the previous exercise, describe the conic obtained if $\Delta = 0$, $\Delta < 0$, or $\Delta > 0$. State what the general result ought to be. (To rigorously prove it should take some time. In fact, if you have not seen this before, this type of problem will have to be spread out over a few days. We do not intend for you to spend all of your time on this problem; no, we intend for you to work on it for thirty minutes to an hour, put it aside, and then come back to it.)

Exercise 1.11.12. Consider the equation $ax^2+bxy+cy^2 = 0$, where all coefficients are real numbers. Dehomogenize the equation by setting $y = 1$. Solve the resulting quadratic equation for x. You should see a factor involving Δ in your solution. How does Δ relate to the discriminant used in the quadratic formula?

Exercise 1.11.13. The discriminant in the quadratic formula tells us how many (real) solutions a given quadratic equation in a single variable has. Classify a conic $\mathrm{V}(f(x, y))$ based on the number of solutions to its dehomogenized quadratic equation.

1.12. Duality

The first goal of this section is show that there is a duality between points and lines in the projective plane. The second goal of this section is to use duality to map any smooth curve in $\mathbb{P}^2$ to another curve called the dual curve in $\mathbb{P}^2$.

1.12.1. Duality in $\mathbb{P}^2$ between Points and Lines.

Given a triple of points $a, b, c \in \mathbb{C}$, not all zero, we have a line

$$L = \{(x : y : z) \in \mathbb{P}^2 : ax + by + cz = 0\}.$$

Exercise 1.12.1. Show that the line associated to $a_1 = 1, b_1 = 2, c_1 = 3$ is the same line as that associated to $a_2 = -2, b_2 = -4, c_2 = -6$.

Exercise 1.12.2. Show that the line associated to a_1, b_1, c_1 is the same as the line associated to a_2, b_2, c_2 if and only if there is a non-zero constant $\lambda \in \mathbb{C}$ such that $a_1 = \lambda a_2,\ b_1 = \lambda b_2,\ c_1 = \lambda c_2$.

Hence all representatives in the equivalence class for $(a : b : c) \in \mathbb{P}^2$ define the same line.

Exercise 1.12.3. Show that the set of all lines in $\mathbb{P}^2$ can be identified with $\mathbb{P}^2$ itself.

Even though the set of lines in $\mathbb{P}^2$ can be thought of as another $\mathbb{P}^2$, we want notation to be able to distinguish $\mathbb{P}^2$ as a set of points and $\mathbb{P}^2$ as the set of lines. Let $\mathbb{P}^2$ be our set of points and let $\widetilde{\mathbb{P}}^2$ denote the set of lines in $\mathbb{P}^2$. To help our notation, given $(a : b : c) \in \mathbb{P}^2$, let

$$L_{(a:b:c)} = \{(x : y : z) \in \mathbb{P}^2 : ax + by + cz = 0\}.$$

Then we define the map $\mathcal{D} : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ by

$$\mathcal{D}(L_{(a:b:c)}) = (a : b : c).$$

The $\mathcal{D}$ stands for *duality*.

Let us look for a minute at the equation of a line:

$$ax + by + cz = 0.$$

Though it is traditional to think of a, b, c as constants and x, y, z as variables, this is only a convention. Think briefly of x, y, z as fixed, and consider the set

$$M_{(x:y:z)} = \{(a : b : c) \in \widetilde{\mathbb{P}}^2 : ax + by + cz = 0\}.$$

Exercise 1.12.4. Explain in your own words why, given $(x_0 : y_0 : z_0) \in \mathbb{P}^2$, we can interpret $M_{(x_0:y_0:z_0)}$ as the set of all lines containing the point $(x_0 : y_0 : z_0)$.

We are beginning to see a duality between lines and points.

Let

$$\Sigma = \{((a : b : c), (x_0 : y_0 : z_0)) \in \widetilde{\mathbb{P}}^2 \times \mathbb{P}^2 : ax_0 + by_0 + cz_0 = 0\}.$$

There are two natural projection maps:

$$\pi_1 : \Sigma \to \widetilde{\mathbb{P}}^2$$

given by

$$\pi_1(((a : b : c), (x_0 : y_0 : z_0))) = (a : b : c)$$

and

$$\pi_2 : \Sigma \to \mathbb{P}^2$$

given by

$$\pi_2(((a : b : c), (x_0 : y_0 : z_0))) = (x_0 : y_0 : z_0).$$

Exercise 1.12.5. Show that both maps π_1 and π_2 are onto.

Exercise 1.12.6. Given a point $(a : b : c) \in \widetilde{\mathbb{P}}^2$, consider the set

$$\pi_1^{-1}(a : b : c) = \{((a : b : c), (x_0 : y_0 : z_0)) \in \Sigma\}.$$

Show that the set $\pi_2(\pi_1^{-1}(a : b : c))$ is identical to a set in $\mathbb{P}^2$ that we defined near the beginning of this section.

As evidence for a type of duality, show:

Exercise 1.12.7. Given a point $(x_0 : y_0 : z_0) \in \mathbb{P}^2$, consider the set

$$\pi_2^{-1}(x_0 : y_0 : z_0) = \{((a : b : c), (x_0 : y_0 : z_0)) \in \Sigma\}.$$

Show that the set $\pi_1(\pi_2^{-1}(x_0 : y_0 : z_0))$ is identical to a set in $\widetilde{\mathbb{P}}^2$ that we defined near the beginning of this section.

Exercise 1.12.8. Let $(1 : 2 : 3), (2 : 5 : 1) \in \widetilde{\mathbb{P}}^2$. Find

$$\pi_2(\pi_1^{-1}(1 : 2 : 3)) \cap \pi_2(\pi_1^{-1}(2 : 5 : 1)).$$

Explain why this is just a fancy way for finding the point of intersection of the two lines

$$x + 2y + 3z = 0$$

$$2x + 5y + z = 0.$$

As another piece of evidence for duality, consider:

Exercise 1.12.9. Let $(1 : 2 : 3), (2 : 5 : 1) \in \mathbb{P}^2$. Find

$$\pi_1(\pi_2^{-1}(1 : 2 : 3)) \cap \pi_1(\pi_2^{-1}(2 : 5 : 1)).$$

Explain that this is just a fancy way for finding the unique line containing the two points $(1 : 2 : 3), (2 : 5 : 1)$.

Principle 1.12.1. The *duality principle* for points and lines in the complex projective plane is that for any theorem for points and lines there is a corresponding theorem obtained by interchanging the words "points" and "lines".

Exercise 1.12.10. Use the duality principle to find the corresponding theorem to:

Theorem 1.12.11. Any two distinct points in $\mathbb{P}^2$ determine a unique line.

This duality extends to higher dimensional projective spaces. The following is a fairly open-ended exercise:

Exercise 1.12.12. Given $(x_0, y_0, z_0, w_0), (x_1, y_1, z_1, w_1) \in \mathbb{C}^4 - \{(0,0,0,0)\}$, define

$$(x_0, y_0, z_0, w_0) \sim (x_1, y_1, z_1, w_1)$$

if there exists a non-zero λ such that

$$x_0 = \lambda x_1, y_0 = \lambda y_1, z_0 = \lambda z_1, w_0 = \lambda w_1.$$

Define

$$\mathbb{P}^3 = \mathbb{C}^4 - \{(0,0,0,0)\}/\sim .$$

Show that the set of all planes in $\mathbb{P}^3$ can be identified with another copy of $\mathbb{P}^3$. Explain how the duality principle can be used to link the fact that three non-collinear points define a unique plane to the fact three planes with linearly independent normal vectors intersect in a unique point.

1.12.2. Dual Curves to Conics. Let $f(x, y, z)$ be a homogeneous polynomial and let

$$C = \{(x : y : z) \in \mathbb{P}^2 : f(x, y, z) = 0\}.$$

We know that the normal vector at a point $p = (x_0 : y_0 : z_0) \in C$ is

$$\nabla f(p) = \left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p) \right).$$

Further the tangent line at $p = (x_0 : y_0 : z_0) \in C$ is defined as

$$T_p(C) = \{(x : y : z) \in \mathbb{P}^2 : x\frac{\partial f}{\partial x}(p) + y\frac{\partial f}{\partial y}(p) + z\frac{\partial f}{\partial z}(p) = 0\}.$$

Recall from Section 1.10 that if f has degree n, then

$$nf(x, y, z) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}.$$

Exercise 1.12.13. Show for any $p = (x_0 : y_0 : z_0) \in C$, we have

$$\begin{aligned} T_p(C) \ = \{(x : y : z) \in \mathbb{P}^2 : (x - x_0)\frac{\partial f}{\partial x}(p) \\ + \ (y - y_0)\frac{\partial f}{\partial y}(p) + (z - z_0)\frac{\partial f}{\partial z}(p) = 0\}. \end{aligned}$$

Recall that $p \in C$ is smooth if the gradient

$$\nabla f(p) \neq (0,0,0).$$

Definition 1.12.1. For a smooth curve C, the *dual curve* $\widetilde{C}$ is the composition of the map, for $p \in C$,

$$p \mapsto T_p(C)$$

with the dual map

$$\mathcal{D} : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$$

from the last subsection. We also denote this map by $\mathcal{D}$. Then

$$\mathcal{D}(p) = \left(\frac{\partial f}{\partial x}(p) : \frac{\partial f}{\partial y}(p) : \frac{\partial f}{\partial z}(p) \right).$$

To make sense of this, we, of course, need some examples.

Exercise 1.12.14. For $f(x,y,z) = x^2+y^2-z^2$, let $C = \mathrm{V}(f(x,y,z))$. Show for any $(x_0 : y_0 : z_0) \in C$ that

$$\mathcal{D}(x_0 : y_0 : z_0) = (2x_0 : 2y_0 : -2z_0).$$

Show that in this case the dual curve $\widetilde{C}$ is the same as the original C.

Exercise 1.12.15. Consider $f(x,y,z) = x^2 - yz = 0$. Then for any $(x_0 : y_0 : z_0) \in C$, where $C = \mathrm{V}(f)$, show that

$$\mathcal{D}(x_0, y_0, z_0) = (2x_0 : -z_0 : -y_0).$$

Show that the image is in $\mathbb{P}^2$ by showing that $(2x_0, -z_0, -y_0) \neq (0,0,0)$. Letting $(u : v : w) = (2x : -z : -y)$, show that $u^2 - 4vw = 0$ defines the dual curve $\widetilde{C}$. Note that here $\widetilde{C} \neq C$.

Exercise 1.12.16. For $C = \mathrm{V}(x^2 + 4y^2 - 9z^2)$, show that the dual curve is

$$\widetilde{C} = \{(x : y : z) \in \mathbb{P}^2 : x^2 + \frac{1}{4}y^2 - \frac{1}{9}z^2 = 0\}.$$

Exercise 1.12.17. For $C = \mathrm{V}(5x^2 + 2y^2 - 8z^2)$, find the dual curve.

Exercise 1.12.18. For a line $L = \{(x : y : z) \in \mathbb{P}^2 : ax + by + cz\}$, find the dual curve. Explain why calling this set the "dual curve" might seem strange.