Preface

Classical mechanics was developed by many of the greatest minds of the past, including Archimedes, Galileo, Newton, Huygens, the Bernoullis,[1] Lagrange, Gauss, Jacobi, Hamilton, and Poincaré, among others.

Their efforts made classical mechanics a multifaceted gem which is beautiful whichever way one turns it.

From the practical side, a remarkable feature of mechanics is its ability to predict things by pure thought, starting with almost nothing. For example, knowing only the gravitational acceleration $g \approx 10m/s^2$ near the Earth's surface, and Newton's laws, we can predict the frequency of oscillations of a pendulum, or the speed of precession of a spinning top, or the period of an orbiting satellite knowing its distance to the Earth's center, etc.

On the theoretical side, classical mechanics interlaces with almost every branch of mathematics: Euclidean geometry, differential equations, dynamical systems, differential geometry, topology, algebra, number theory.

[1] There was considerable rivalry between Johann and Jacob Bernoulli; ironically, nowadays some people confuse the two brothers, or even think they were the same person.

From yet another direction, mechanics actually explains and even suggests some theorems of geometry and calculus (see [**14**] for more details and references).

In such an old and fundamental subject it is difficult to say something that is simultaneously original (O), interesting (I) and correct (C). When pressed, I tried to satisfy (I) and (C) at a minimum. On occasion, (O) may have been satisfied, at least in a pedagogical sense.

Intended audience. This text is suitable for courses from junior to beginning/intermediate graduate, on the topics in the title. Each section corresponds roughly to one lecture, and it is not unreasonable to expect to cover 2 to 4 chapters in a semester. This means that the book contains enough material for more than one course. Many of the 180 problems (most with solutions or hints) can be used in lectures; in fact, much of the pleasure in the subject is derived from problems. I also hope that some ideas or problems may seem new not only to students but to specialists as well — at least in a pedagogical sense.

Physical and mathematical background. I assume that the reader has been exposed to beginning courses in physics, in vector calculus, and in basic linear algebra. Nevertheless, I recall some of the key concepts and facts as the need arises.

Some highlights of this book. 1. *Ideas come before formulas.* As an example, some texts *define* kinetic energy as $mv^2/2$, never explaining what it *really* is (it is the work required to bring the mass to speed v, see page 10 for details), and why, for instance, not $mv^2/3$, or mv^3? The reason is hidden from the reader, who deserves to be told *why*, and not only *how*. I tried to avoid such abuse throughout.

Chapter 8 is especially devoted to explaining the motivation behind Hamiltonian mechanics so that nothing is pulled out of thin air.

2. Having kept in mind Einstein's quote: "If you can't explain it to a six year old, you don't understand it yourself", I followed, as best I could, his advice: "Everything must be made as simple as possible. But not simpler."

3. I describe an *equivalence* between the dynamics of particles and the statics of springs (page 45). This equivalence is fruitful in a few ways: (i) it yields the Euler–Lagrange equation and Liouville's theorem with almost no work, at least in special cases; (ii) it gives a surprisingly obvious physical interpretation to these abstract mathematical statements, and (iii) it shows *why* Liouville's theorem holds, giving the bare essence of the reason for the result, not hidden by many steps.

4. Chapter 8 explains some fundamental ideas of Hamiltonian mechanics from one basic principle. Just as with the kinetic energy example above, many of the concepts in that field are often "pulled out of thin air".[2] Instead, I tried to show how one basic starting point leads to all of the above concepts automatically.

5. The table on page 280 shows a revealing analogy between Hamiltonian dynamics and statics of springs. In my mind, this analogy could be taken as an unspoken guiding principle of Hamiltonian formalism. Hamiltonian mechanics turns out to be a kind of "spring theory".

6. Some miscellaneous items: (i) a new heuristic way to minimize integrals $\int F(y)ds$ (page 188), (ii) some fun problems, such as one on finding the center of curvature using a bike, or on the hydrostatic nature of the tension in hanging cables, or on an interesting property of a mobile, etc., (iii) a remarkably short proof, due to Lagrange, of the ellipticity of planetary motions.

7. I included a geometrical discussion of Pontryagin's Maximum Principle of optimal control. This topic really belongs to Hamiltonian mechanics/ray optics and so is a natural fit for this book. It is remarkable that the Maximum Principle is, loosely speaking, a version of Huygens's principle. Optimal control is a standard topic in engineering courses but is rarely taught to mathematicians and physicists; I tried to bring this neglected child of Hamiltonian mechanics, often abused in engineering literature, back home. Many engineering

[2]These concepts, which the reader is not expected to know at this stage, include the Legendre transform, Hamilton–Jacobi equations, Liouville's theorem, Poincaré integral invariants, Noether's theorem.

texts effectively hide the geometry of Pontryagin's principle; I tried to bring this simple geometry to the surface.[3]

Analogies in general.[4] The use of analogies goes back to Archimedes who interpreted geometrical objects mechanically and then used mechanical intuition to discover new geometrical theorems. Later examples include the so-called Kirchhoff's kinetic analogy — actually, the *equivalence* — between the dynamics of a free rigid body on the one hand and statics of elastic rods on the other, Poincaré's analogy of phase flow of a Hamiltonian system to a fluid flow (see Arnold [**1**] for more details), Riemann's interpretation of analytic functions in terms of fluid flow, and much, much more. Aubry's discovery, simultaneously with Mather, of the Aubry–Mather theory (1982) — one of the key advances of dynamical systems in the twentieth century — was driven by a mechanical analogy. Analogy unifies our understanding by showing that seemingly unrelated things are, in certain aspects, the same.

Analogies in this book. In that vein, the dynamics of a single particle is analogous (in fact, equivalent) to the statics of a Hookean spring (see the table on page 45). For a general mechanical system (not just a single particle), there is also an analogy different from the one just mentioned (see the table on page 280). Each of these analogies opens a new view — they certainly did for me when I realized them. Some things that are not obvious and abstract for one side of the analogy are obvious and palpable for the other, as the table just mentioned illustrates.

Outline of the book. The book consists of four parts: (I) Dynamics (Chapters 1–3), (II) Variational Principles (Chapters 4–6), (III) Optimal control (Chapter 7) and (IV) Foundations of Hamiltonian mechanics (Chapter 8).

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[3] The texts [**15**] and [**2**] give a very nice and a much more extensive presentation of these topics than this book.

[4] The subtitle of a remarkable recent book "Surfaces and Essences: Analogy as the Fuel and Fire of Thinking" by Hofstadter and Sander represents accurately, I think, the pivotal role of analogy in all thinking, and in particular, in classical mechanics.

Originally, I was planning Chapter 8 as a small book, but the manuscript grew, thanks in part to Sergei Gelfand's encouragement and despite my attempts to keep it short. This main chapter ended up being the last one in the book, which may seem an injustice. Here is a slightly more detailed outline of the book.

Chapters 1–3: Dynamics. Chapter 1 deals with one-dimensional motion, introducing many key ideas of mechanics with a minimum of technicalities. Chapter 2 discusses several degrees of freedom, including Kepler's problem and vibrations. Discussion of free rigid body motion (such as a tumbling asteroid, or a football, Chapter 3) completes the first part of the book.

Chapters 4–7: Calculus of Variations, Optimal Control. This topic is presented in four parts: First, variational principles of mechanics (Chapter 4); second, some classical problems from calculus of variations (Chapter 5); third, the Jacobi criterion for the minimum — a kind of a second derivative test for a minimum (Chapter 6); and fourth, Pontryagin's Maximum Principle of optimal control (Chapter 7). This principle turns out to be essentially a restatement of Huygens's principle.

Chapter 8: Hamiltonian approach motivated. This chapter contains a heuristic view of Hamiltonian and Lagrangian mechanics. Ever since having learned classical mechanics in school, I had a nagging suspicion that many texts on mechanics leave something very fundamental unspoken. Why, for example, does one introduce the Hamiltonian, or the Legendre transform, or the momentum, or the symplectic 1-form $p\,dx$?[5] The best explanation I saw was that the end — the beauty of the resulting theory — justifies the unmotivated means. It turns out, however, that *if we start the presentation of Hamiltonian mechanics with the right question, then all the concepts mentioned above fall into our laps automatically and unavoidably;* most of the theory begins to look simple and natural. This presentation is given in the ten short sections of Chapter 8. Section 11 of this chapter gives a static analog of the preceding theory; with

[5] All of these concepts are introduced in the first chapter in a conventional way, and then again in Chapter 8 in a way that shows the inevitability of these concepts.

this analogy we realize that many of the seemingly abstract concepts have an elementary physically palpable explanation. It is remarkable that, in hindsight, much of the Hamiltonian theory would have been obvious to Archimedes, if only it were restated in terms of a certain static mechanical model. Finally, the last section of Chapter 8 gives a quick description of the analogy of mechanics to optics. This traditional analogy is explained beautifully in Gelfand and Fomin's Calculus of Variations [**8**] (Appendix 1); see also Arnold's Mechanics [**1**].

Classical mechanics as a branch of mathematics. Classical mechanics deals with idealized objects, such as "point masses", "rigid bodies", "rods". These objects are imaginary approximations of actual physical things. For example, in studying planetary motion, we may treat the Earth as a point. In this respect classical mechanics is a sister of geometry, which also deals with idealized objects; to the geometer a "point" is not an ink dot on paper but an imaginary dot of zero size. When applying classical mechanics in practice one must of course remember that, for instance, the "rigid body" of classical mechanics is not exactly rigid in reality.[6] A "point mass" is a good approximation unless we look at it too closely. Classical mechanics serves very well except at microscopic scales (quantum mechanics) or at macroscopic ones (relativity). We assume, for example, that the position of a point mass can be precisely defined, that inertial frames exist, and that the time is well defined throughout the reference frame. These statements are strictly true only in the idealized world of classical mechanics.

The mystery of variational principles. (The reader who has not seen variational principles should skip this paragraph, or else see page 167 first.) Newton's laws (N) are equivalent to variational principles (V), as proven in many texts on classical mechanics. Logically, there is nothing to complain about: (N) is taken as axioms (suggested by

[6] An interesting mistake along these lines was made with the first Explorer satellite. The orbiting satellite, shaped as an elongated cylinder, was given an original spin around its long axis; unexpectedly, it developed wobbles and after several hours ended up tumbling end-over-end, contrary to what the theory of rigid body dynamics predicts (Chapter 4). It was soon realized that the satellite was not exactly rigid (it had flexible antennas), so that the theory did not quite apply.

experiments and by physical intuition, itself the result of experiments we do as part of physical activity), (V) is proven; the picture seems complete. But there is a mystery about (V): how can a mindless projectile "know" to take the path of least action?[7] After all, in order to choose the path of least action, doesn't one have to compare the action of this path to the actions of other paths? That a brainless projectile "knows" about nearby paths and chooses the "best" seems so surprising that the logical proof (N) $\Rightarrow$ (V) looks a bit legalistic and insufficient. One senses that something else must be going on. Yet no book on classical mechanics that I know of mentions this question.[8] But it is precisely this question that stimulated Feynman in his remarkable discovery of path integrals ([**7**], [**6**]). Variational principles become much less mysterious through the idea of phase cancellation, as explained briefly in the next paragraph.

Classical mechanics leading to quantum mechanics. In relation to the previous item, it is remarkable that the Maupertuis' principle (discovered by Leibnitz around 1707), or Hamilton's principle ($\delta \int L\,dt = 0$), can in retrospect be seen as a loud hint at the wave nature of particles, available long before the first shoots of quantum mechanics appeared. In optics, Fermat's principle of least time ($\delta \int dt = 0$) is explained as follows: the waves travel between two points A and B along all possible paths; but the waves traveling along paths close to the path of least time arrive in sync, with nearly equal phases, and hence add up; by contrast, the waves traveling along noncritical paths arrive at B with disparate phases and thus cancel with their neighbors, contributing almost nothing at B. In other words, the light is not intelligent but rather omnipresent; some people mistook phase cancellation for intelligence. Fermat's principle ($\delta \int dt = 0$) is thus due to the wave nature of light, suggesting that Maupertuis' principle, or Hamilton's principle, is analogously due to wave nature of particles. The alternative would be to ascribe this principle to the supernatural [**6**]. In light of this (with apologies for the pun) it may seem striking that so much time passed from the formulation of Hamilton's principle ($\delta \int L\,dt = 0$) to the discovery of

[7] Loosely speaking, action is the integral of the difference between the kinetic and the potential energies. A precise definition is given by (1.22), on page 19.

[8] Perhaps due to the long tradition because the subject is so, uh, classical.

the wave nature of particles. Feynman's derivation of the Schrödinger equation using classical mechanics is given on page 286. The remarkable feature of this derivation is that it uses almost nothing besides classical mechanics and the analogy with electromagnetic optics to arrive at the Schrödinger equation.

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