Chapter 1

One Degree of Freedom

1. The setup

In this chapter we consider the simplest class of mechanical systems: a point mass confined to a straight line or a curve; Figure 1 gives three examples. Such systems are referred to as the systems with *one degree of freedom*[1].

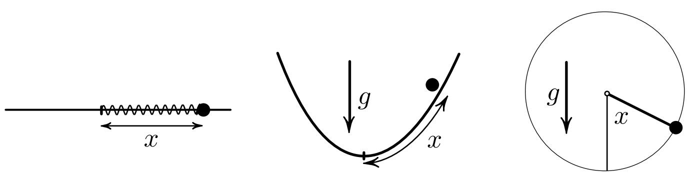


Figure 1. One-degree-of-freedom systems.

One-degree-of-freedom systems are "building blocks" for a larger class of more complex mechanical systems — the so-called completely integrable systems. Such systems (Kepler's problem is an example) reduce to a collection of decoupled one-degree-of-freedom systems. Many fundamental ideas and concepts in mechanics can be illustrated already on one-degree-of-freedom systems, as we do in this chapter.

[1] More generally, the degree of freedom of a mechanical system is the number of quantities which define the position of all the particles in the system. Other examples of one-degree-of-freedom systems include a rigid body spinning on a given axis (with the angle playing the role of coordinate), or a wheel rolling on a straight line without sliding. Higher degree of freedom systems are discussed in the next chapter. Examples of those include a particle in the plane (two coordinates define its position), a double pendulum (two angles define the positions of the masses), or a particle constrained to move on a surface in space.

2. Equations of motion

In this section I state Newton's second law, and then derive the equations of motion for each of the examples in Figure 1. An alternative way to derive the equations of motion, discovered by Lagrange and used more commonly, is described in Section 7.

Newton's second law for a point mass. Consider a particle of mass m subject to *net force* $\mathbf{F}$. By net force, also called the *resultant* force, one means the vector sum of *all* forces acting on the particle.[2] Newton's law states that the vector acceleration $\mathbf{a}$ of the particle is caused by the net force $\mathbf{F}$ and is proportional to that force:

$$m\mathbf{a} = \mathbf{F}, \quad \text{or} \quad \mathbf{a} = \frac{1}{m}\mathbf{F}, \tag{1.1}$$

where the coefficient of proportionality m is referred to as the (inertial) *mass*. In other words, the particle accelerates in the direction of the net force and with intensity proportional to the force and inverse proportional to the mass.

Exercise 1.1. A particle moves in space under the influence of a force $\mathbf{F}$. Must the particle's velocity $\mathbf{v}$ point in the same direction as $\mathbf{F}$?

Answer. No. For a flying projectile, for instance, $\mathbf{F}$ and $\mathbf{v}$ are not aligned since one points straight down while the other is tangent to the trajectory.

Motion on the line under a frictionless force. We consider a particle constrained to a straight line, which we take to be the x-axis. We assume that the particle located at x is subject to a force $F(x)$ acting in the direction of the x-axis. Note that this force is assumed to be independent of the velocity $\dot{x}$; the friction is thus excluded. Since the particle's acceleration is $a = \frac{d^2}{dt^2}x = \ddot{x}$, we can rewrite Newton's law as

$$\boxed{m\ddot{x} = F(x)} \tag{1.2}$$

Here $x = x(t)$ is a function of time, so that Newton's law becomes an ordinary differential equation for the unknown function $x(t)$. We now consider several important examples.

[2] One of the most common mistakes in mechanics is forgetting to include all of the forces in Newton's law.

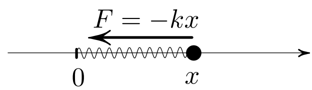


Figure 2. The harmonic oscillator. The picture shows a zero length spring, i.e., 0 is the equilibrium.

1. The free fall. The simplest example $F =$ const. was solved by Galileo in his study of free fall. Actually, Galileo's main discovery in this area was not the solution of the differential equation $\ddot{x} = -g$, but rather his realization that m does not enter this equation.

2. The harmonic oscillator. The next simplest example, where $F = -kx$ ($k =$ const. > 0), is referred to as the *harmonic oscillator*, Figure 2. The minus sign indicates that the force is *restoring.* We can think of $F = -kx$ as the tension of a *linear zero length spring*, i.e., of a spring whose relaxed length is zero or, if one prefers, as the deflection from the relaxed length of a Hookean spring with nonzero relaxed length.[3] Equation (1.2) with $F = -kx$ becomes

$$\ddot{x} + \omega^2 x = 0, \quad \text{where} \quad \omega^2 = \frac{k}{m}. \tag{1.3}$$

The general solution of this ordinary differential equation is of the form $x(t) = A\cos(\omega t - \varphi)$, where A (the amplitude) and φ (the phase) are arbitrary constants which can be determined from the initial data.

3. A bead on a wire is an entire class of examples leading to (1.2). Figure 3(A) shows a bead sliding without friction on a rigid wire in the vertical plane under the influence of gravity. To write the equation of motion, let us use the arc length parameter s, the distance along the curve from some chosen point to the bead. Projecting Newton's law (1.1) onto the tangent to the wire we get the scalar equation

$$ma = F, \tag{1.4}$$

where $a = \ddot{s}$ is the tangential acceleration of the bead, and where F is the sum of projections of all the forces acting on the bead upon the tangent to the wire. As the figure illustrates, of the two forces acting on the bead the reaction $\mathbf{N}$ contributes zero (since there is no friction). The only contributing force is the projection of gravity:

[3]Some springs come pre-stressed in such a way that $F = -kx$ holds reasonably well as long as the spring is actually stretched.

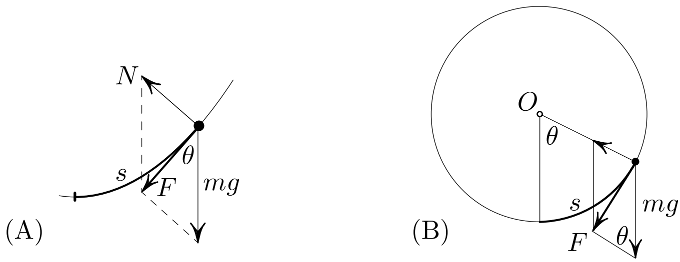


Figure 3. The bead on a wire; the circular wire corresponds to the pendulum.

$-mg \sin\theta$, where θ is the angle between the tangent and the vertical. Note that $\theta = \theta(s)$ is a function of s determined by the shape of the wire. Summarizing, (1.4) becomes

$$\boxed{\ddot{s} = -g \sin\theta(s)} \tag{1.5}$$

This equation is of the same form as (1.2), except that s is a coordinate along a curve. We now consider two shapes of the track: the circle and the cycloid.

4. The pendulum. The circular wire in Figure 3(B) need not be a physical wire: the bead can be constrained to the circle by a weightless rod hinged at O, giving us a pendulum. We have $s = L\theta$ if both s and the angle θ are measured from the same point on the circle, and where L is the length of the rod. Substituting this into (1.5) we get the differential equation for the angle θ (more convenient to use than s) of the pendulum:

$$\ddot{\theta} = -\frac{g}{R} \sin\theta. \tag{1.6}$$

5. Huygens's pendulum. The usual pendulum, used as a clock, has one shortcoming: its period depends on the amplitude. Huygens discovered how to fix this problem — one of the nicest discoveries in the history of calculus. First, Huygens showed that the cycloid (see Figure 4) is that special curve for which *the period of the bead's oscillations is independent of the amplitude* (see Problem 1.18 for a hint to a short proof). Second, Huygens showed how to make a mass travel on a cycloid in a practical way, Figure 5: Consider the piece ACB of the cycloid from Figure 4, where A and B are the lowest

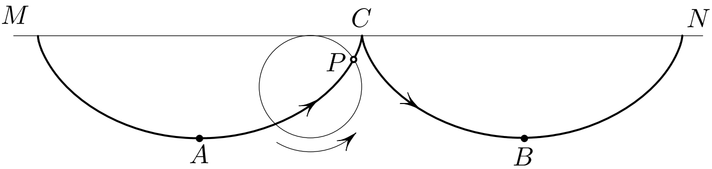


Figure 4. A circular wheel rolls on the line MN without sliding. The cycloid is a curve described by a point P on the rolling wheel. In our discussion MN is horizontal and the wheel is below MN.

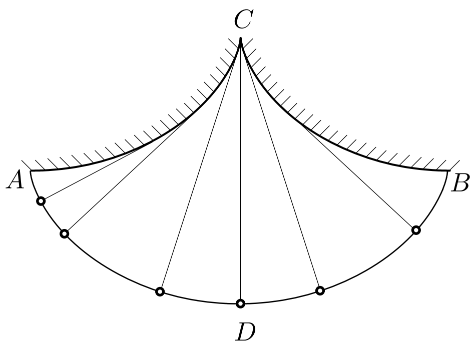


Figure 5. Hugyens's pendulum: CA and CB are arcs of a cycloid; the string attached at C has the same length as these arcs. Then the free end of the string will trace a congruent cycloid, and, moreover, the period of the resulting pendulum will be in dependent of the amplitude.

points, and treat this piece as an obstacle impenetrable by a string attached at C. The length of the string is chosen to be the same as that of the semi-arcs CA and CB. If we hang a weight at the free end of the string and let it swing, part of the string will hug the obstacle arcs and part will be straight; we obtained a pendulum of variable length. Huygens's second discovery was that *the path of the weight is a congruent cycloid(!)*

High quality grandfather clocks have special suspension mechanisms based on Huygens's discovery.

Remark 1.1. Note that the string in Figure 5 is normal to ADB and tangent to CA or CB. In other words, *the arc ACB is the envelope*

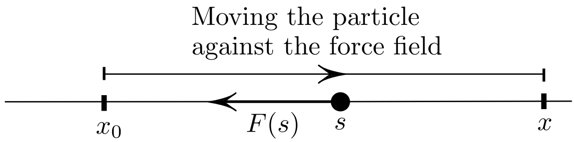


Figure 6. Potential energy for one-degree-of-freedom systems.

of the family of lines normal to ADB. Such an envelope of the family of normal lines to a planar curve is called the *evolute* of this curve. Huygens therefore showed that *the evolute of a cycloid* (ADB) *is a congruent cycloid* (ACB).[4]

3. Potential energy

The setting. Figure 6 shows a point mass on the line in a force field $F(x)$: the particle whose position is x is subject to the force $F(x)$. The mass-spring system is a prime example; in that case, $F(x) = -kx$, where x is the position of the particle relative to the equilibrium.

Intuitively, potential energy of the point mass at x is the work that I must do *against* the force $F(x)$ to bring the particle to the location x from some given reference location x_0. That is, I must apply force $-F(s)$ at s, in order to balance $F(s)$, thus dragging the particle from x_0 to x. This suggests the formal definition.

Definition. Potential energy $U(x)$ at x of a point mass in the force field $F(x)$ is defined as

$$U(x) = -\int_{x_0}^{x} F(s)\, ds \tag{1.7}$$

Remark 1.2. One might wonder whether the above motivation of the definition is imprecise: when I apply force $-F(s)$ to the mass, the net force on the mass becomes $-F(s) + F(s) = 0$, so why would it move at all? To answer this question, the mass will move, by inertia, *if* given an arbitrarily small initial speed, since the net force $= 0$ everywhere. So technically, to move the mass from x_0 to x I must spend work $U(x)$ plus an arbitrarily small quantity. The formal

[4] See Problems 1.19 and 1.20 on page 55.

definition (1.7) avoids this point, which is a plus, but does not seem motivated on its own, which is a minus.

Remark 1.3. If F points from x_0 to x, then the work done against F is negative. For example, in moving a weight from the tabletop to the floor I do negative work; that is, the work is done for me.

Potential energy and force. Differentiating both sides of the definition (1.7) yields, by the Fundamental Theorem of Calculus[5]:

$$\boxed{F(x) = -U'(x)} \tag{1.8}$$

Thus the force can be recovered from the potential energy. Note that large force is characterized by steep changes of energy.

Many potential energies. The potential energy depends on the choice of the reference point x_0. Choosing a different x_0 amounts to changing U by an additive constant. This constant does not affect the force $F(x)$ since the derivative in (1.8) kills the constant.

A geometrical interpretation of (1.8). Figure 7 shows that the force acts "down the slope" of the graph of $U(x)$. Imagine gravity $g = 1$ pointing down in Figure 7. Then the tangential component of this gravity upon a bead sliding on the graph of U is $F_{\text{tang}} = -\sin\tan^{-1} U'(x) = -U'(x) + o(U'(x))$. For small $U'(x)$ the force $-U'(x)$ is close to the tangential component of gravity upon a bead on the wire. For large slopes $U'(x)$ the approximation fails, although it still does give the correct sign of the force.[6]

Potential energy is defined up to an arbitrary constant. Indeed, with a different choice of the reference location $\tilde{x}_0 \neq x_0$, the corresponding potential energy $\tilde{U}(x) = -\int_{\tilde{x}_0}^{x} F(s)\, ds$ differs from $U(x)$ by

$$U(x) - \tilde{U}(x) = \int_{\tilde{x}_0}^{x_0} F(s)\, ds = \text{const.},$$

a quantity independent of x. In fact, any of the anti-derivatives $U(x) = -\int F(x)$ (defined up to a constant) is a potential energy.

[5] A reminder: $\frac{d}{dx}\int_a^x F(s)\, ds = F(x)$. Differentiation undoes the integration.
[6] See Problem 1.3 on page 51

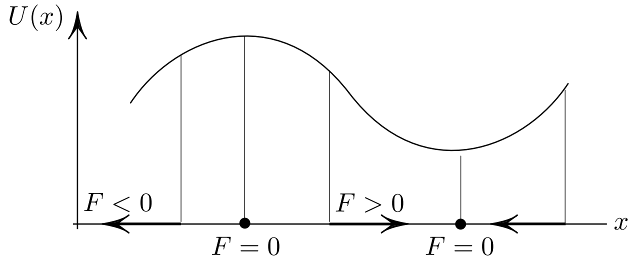


Figure 7. Force equals the negative slope of the potential energy graph.

Particle constrained to a curve. Our definition of the potential energy for a point mass on the straight line extends to a point mass constrained to a curve (Figure 1); definition (1.7) still applies, where x now stands for the arc length measured along the curve, and where $F(x)$ stands for the force in the tangential direction to the curve.

Examples of potential energy.

1. A Hookean spring. $U(x) = \int_0^x (-F(s))\, ds = \int_0^x (-(-ks))\, ds = \frac{1}{2}kx^2$, where x is the amount by which the spring was stretched from its relaxed length. Here is an elementary derivation of the result avoiding integrals: when pulling the spring from $x = 0$ to $x \neq 0$, I must apply the average force $\frac{0+kx}{2} = kx/2$; the work equals this force times the distance x, reproducing the above result.

2. Potential energy in a constant gravitational field. The particle is subject to the gravitational force $F(z) = -mg$ in the direction of the negative z-axis. The work required to move the particle from z_0 to z against F is $U(z) = (mg) \cdot (z - z_0) = mgh$, where h is the height of z above z_0.

3. Potential energy in the gravitational field of a star. Let us place the origin at the center of the star, Figure 8. The star's gravitational pull on a point mass is $F(x) = -\frac{k}{x^2}$ (k is a constant whose value is not important here). We treat the star as a point mass; otherwise we must take $x > R$, the radius of the star. Let us choose $x_0 = \infty$ for the reference "point". We have

$$U(x) = -\int_{\infty}^{x} \left(-\frac{k}{x^2}\right) dx = -\int_{x}^{\infty} \frac{k}{x^2}\, dx = -\frac{k}{x}. \tag{1.9}$$

This agrees with intuition: the work we do when moving the mass from infinity towards the star is negative, meaning that the gravitational force is doing the work for us. Moving "downhill" requires negative amount of work.

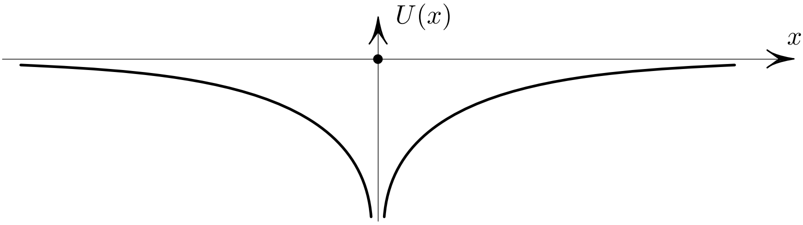


Figure 8. Gravitational potential energy $U = -k/x$. Gravitational force $F(x) = -U'(x) = -k/x^2$.

4. For a bead on a curve as in Figure 3, the force $F(s) = -mg\sin\theta$, where $\theta = \theta(s)$ is the angle between the tangent and the horizontal. Potential energy is $U(s) = -\int_{s_0}^{s} F(\sigma)\, d\sigma$, giving

$$U(s) = \int_{s_0}^{s} mg \underbrace{\sin\theta d\sigma}_{dh} = mg\int_{y_0}^{y} dh = mg(y - y_0). \tag{1.10}$$

We see that the potential energy depends only on the height and not on the horizontal position.

5. The pendulum is a special case of the preceding item. According to (1.10), the potential energy is mgy, where y is the height of the mass above a reference point. We have $y - y_0 = L(1 - \cos\theta) = L(1 - \sin\frac{s}{L})$, where y_0 is the y-coordinate of the lower equilibrium. Using (1.10) we get

$$U(s) = mgL(1 - \cos\theta) = mgL(1 - \cos\frac{s}{L}).$$

4. Kinetic energy

Some texts *define* kinetic energy of a point mass m by the formula:

$$K = \frac{mv^2}{2},$$

where m is the mass and v is the speed. This definition is simple, but it feels unmotivated, and in the end it serves to hide the idea.

Instead, let's define the kinetic energy as the *work required to bring the mass from rest to speed* v, and then prove the formula. We will then see *why* the formula is like it is; for example, why is v squared, and where does 1/2 come from? As an extra benefit, we will understand *why* the total energy is conserved, with no further calculations.

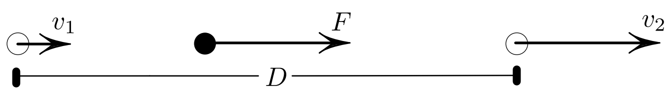


Figure 9. Kinetic energy for one-degree-of-freedom systems.

Proof of $K = mv^2/2$. The work done by the force F is

$$\int F\,dx = \int_0^T F(t)v(t)dt = \int_0^T ma\,vdt = m\int_0^T \frac{d}{dt}\left(\frac{v^2}{2}\right)dt. \tag{1.11}$$

Since $v(0) = 0$ and $v(T) = v$, the fundamental theorem of calculus shows that the integral is $mv^2/2$, as claimed. Note that the work does not depend on how the acceleration varies with time. This is a piece of good luck, since otherwise kinetic energy as we defined it would have depended on the way the particle was accelerated, and thus would have been a meaningless concept. ♢

A calculus-free explanation of $K = mv^2/2$. First, let us accelerate our mass with a constant force F, from speed 0 to v. Let D be the distance the mass travels during its speed-up, and let T be the time of travel. The work done is

$$F \cdot D = ma \cdot D = m\frac{v}{\not{T}} \cdot v_{\text{average}}\not{T} = mv \cdot \frac{v}{2} = \frac{mv^2}{2},$$

as claimed. Note that the size of F "washed out": a larger F would have meant a smaller D to gain the same speed v (as any driver knows); the product FD would have remained the same. We now see where 1/2 in $mv^2/2$ came from (the averaging of the speed to find D), and why v is squared (both F and D depend on v linearly). To see without calculus why a time-varying force F produces the same result $mv^2/2$, let's break up the travel time into many short intervals, so that during each interval we can treat F as nearly constant. If the speed changes from v_k to v_{k+1} during the kth interval, the work done during this interval is F is $\frac{mv_{k+1}^2}{2} - \frac{mv_k^2}{2}$, as a calculation similar to the one above shows. Thus the work done during acceleration from rest to speed v is the telescoping sum

$$\left(\frac{mv_1^2}{2} - \frac{0^2}{2}\right) + \left(\frac{mv_2^2}{2} - \frac{mv_1^2}{2}\right) + \cdots + \left(\frac{mv^2}{2} - \frac{mv_{n-1}^2}{2}\right) = \frac{mv^2}{2}.$$

♢

Conservation of energy. Replacing $t = 0$ and $t = T$ in (1.11) by any two times t_1 and t_2 results in

$$\int_{x_1}^{x_2} F(x)\, dx = \frac{mv_2^2}{2} - \frac{mv_1^2}{2},$$

where $x_1,\ v_1$ and $x_2,\ v_2$ are the corresponding positions and velocities of the particle. Substituting $F(x) = -U'(x)$ (see (1.8)) we get

$$U(x_1) - U(x_2) = \frac{mv_2^2}{2} - \frac{mv_1^2}{2};$$

potential energy lost equals kinetic energy gained. Or, we can rewrite this as

$$\frac{mv_1^2}{2} + U(x_1) = \frac{mv_2^2}{2} + U(x_2) :$$

the total energy of a particle moving in a force field $F(x)$ does not change with time. To summarize, the above physically meaningful definition of K also yields energy conservation as a byproduct. In the next section we give a more streamlined presentation of conservation of energy.

5. Conservation of total energy

Here is an alternative short statement and proof of the law of conservation of energy.

In the same setting as before, we are considering a particle confined to a straight line or a curve and subject to the tangential force $F(x)$, where x denotes the arc length position of the particle.

According to Newton's second law we have

$$m\ddot{x} = F(x). \tag{1.12}$$

It is important to note that the force is assumed to depend on the position x only, and not on the velocity $v = \dot{x}$. This excludes from discussion frictional forces which do depend on the velocity; for such velocity-dependent forces the following theorem fails.

Theorem. *The total energy is constant for any motion* $x = x(t)$ *governed by* (1.12):

$$\boxed{K + U = \frac{m\dot{x}^2}{2} + U(x) = \text{const.}} \tag{1.13}$$

We are not claiming, of course, that different motions have the same energy — only that each motion individually has its own energy which does not change in time.

Proof. It suffices to show that $\frac{d}{dt}(K+U) = 0$, where K and U are evaluated along any solution of (1.12). Differentiating, we get:

$$\frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + U(x)\right) = m\ddot{x}\dot{x} + U'(x)\dot{x} \overset{(1.8)}{=} (m\ddot{x} - F(x))\dot{x} \overset{(1.12)}{=} 0. \tag{1.14}$$

◊

Remark 1.4. It is instructive *to read* (1.14) *backwards*. That is, we could have multiplied both sides of Newton's law $m\ddot{x} - F(x) = 0$ by $\dot{x}$ and then noticed that the left-hand side is just $\frac{d}{dt}(K+U)$. Incidentally, the idea of multiplying by $\dot{x}$ is suggested by the fact that $F\dot{x} = Fv$ is the power expended by the force F applied to a moving particle.[7]

[7] Indeed, $Fv = F\,dx/dt$; since $F dx$ is the work, $F\,dx/dt$ is the work per unit time, i.e., the power.

Mechanics and ODEs. Newton's law (1.12) is an ordinary differential equation (for the unknown function $x = x(t)$); in that sense mechanics is a branch of the theory of ordinary differential equations. A solution is completely specified by a pair $x(0), \dot{x}(0)$ of initial data, according to the theorem on uniqueness and existence for ODEs, provided F is "nice" (continuous differentiability, for instance, qualifies as nice, see [**3**] for details). In the idealized world of classical mechanics the future is completely determined by the present, provided the forces are nice functions of the position.[8]

That mechanics is governed by second order differential equations is an experimental fact. One could imagine a world in which particles' motions were governed by, say, a third order differential equation, or even by an integral equation which involves memory of the past. For instance, we could imagine that a particle held perfectly still and released will, in the absence of external forces, start moving spontaneously in memory to its earlier history, a bit like a raw egg spinning on the table, when stopped momentarily and then released, would start moving again. In fact, Newton's law is only an approximation of fuzzy quantum mechanical objects, and so the "second order" is only an approximate fact and not as intrinsic as might seem, and is probably devoid of deep philosophical meaning.[9]

Examples of energy conservation. In particular examples, conservation of energy (1.13) looks as follows.

1. For a free falling projectile governed by $m\ddot{z} = -mg$:

$$\frac{m\dot{z}^2}{2} + mgz = E.$$

[8] A surprising counterexample to this determinism is a bead on the wire whose graph is $y = -x^{4/3}$. The bead can stay at the equilibrium at $x = 0$ for all time. But the bead can also leave this equilibrium at any time without violating Newton's second law. The same holds for the similar example of a particle in the repelling potential $U = -x^{4/3}$; see Problem 2.33 on page 136.

[9] This descent from the philosophical to technical — or ascent, depending on one's scale of values — happens in science quite often. Fermat's principle is an example. According to Fermat, light rays follow paths of least time. In the early days some saw in this divine intervention. Later, however, a much more prosaic explanation was found: Fermat's principle is the result of phase cancellation or addition. In this case philosophy turned out to be a sophisticated way of admitting, perhaps unknowingly, that one doesn't quite know what's going on.

2. For a mass-spring system governed by $m\ddot{x} = -kx$:

$$\frac{m\dot{x}^2}{2} + \frac{kx^2}{2} = E.$$

3. For the pendulum governed by $m\ddot{s} = -mgL\sin(s/L)$:

$$\frac{m\dot{s}^2}{2} + mgL(1 - \cos(s/L)) = E,$$

or, in terms of the angle $\theta = s/L$:

$$\frac{\dot{\theta}^2}{2} + \frac{g}{L}(1 - \cos\theta) = \text{const.}$$

4. For a particle of unit mass in a two-well potential $U = -x^2 + x^4$ $(\ddot{x} = -U'(x))$:

$$\frac{\dot{x}^2}{2} - x^2 + x^4 = E.$$

6. The phase plane

We continue to discuss Newton's law,

$$m\ddot{x} = F(x),$$

for the motion of a particle on the line. We now describe a fundamental and beautiful way to think of this system geometrically.

Introducing the phase plane. Complete information on the particle's future consists of *two* pieces of data $(x, \dot{x})$ at some t. It is natural to visualize $(x, \dot{x})$ as a point in the plane, thus treating the velocity as the second dimension. The point $(x, \dot{x})$ is referred to as the *phase point*, since it contains full information about the "state" or the "phase" of the system. Thus every point in the plane represents a particle at x with velocity $\dot{x} = y$, and vice versa.

We could call the phase plane the "odometer–speedometer plane", since x is the odometer reading of the particle, and $y = \dot{x}$ is its speedometer reading; see Figure 10.

Seeing the acceleration. As any driver knows, acceleration of the car ahead is much harder to see than its velocity. In the phase plane, however, we can see the acceleration geometrically – namely, as the y-component of phase velocity in the phase plane, Figure 11.

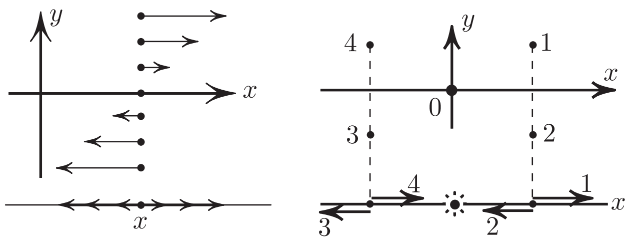


Figure 10. Left: One point on the x-axis can have many velocities as represented by points on the vertical line through x. Right: different velocities for the same position are seen as different y-coordinates in the phase plane.

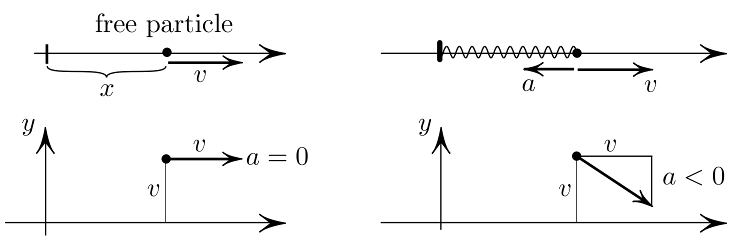


Figure 11. Acceleration seen geometrically, as the vertical component of the velocity vector in the phase plane: $a = \ddot{x} = \dot{y}$. A free particle is shown on the left; its acceleration is zero, and so its velocity vector is horizontal; a particle with negative acceleration is shown on the right.

Phase velocity field. Any mechanical system $m\ddot{x} = F(x)$ is described by a vector field in the plane, as explained in this paragraph. Calling the velocity $\dot{x} = y$, we have $\dot{y} = \ddot{x} = F(x)$ according to Newton's law. Summarizing, we have

$$\begin{cases} \dot{x} = y, \\ \dot{y} = F(x)/m \end{cases} \tag{1.15}$$

or, in vector form,

$$\dot{\mathbf{z}} = \mathbf{V}(\mathbf{z}), \quad \mathbf{z} = (x, y), \quad \mathbf{V} = (y, F(x)/m). \tag{1.16}$$

This is simply a restatement of Newton's law; we traded one second order equation $m\ddot{x} = F(x)$ for two first order equations. But the advantage of the new vector form (1.15) is in this geometrical interpretation: to every point (x, y) in the plane, our system (1.15) assigns the velocity $(\dot{x}, \dot{y})$. Better said, the system (1.15) defines an imaginary fluid flow in the plane, where the velocity of fluid at the point $\mathbf{z} = (x, y)$ is $\mathbf{V}(\mathbf{z}) = (y, m^{-1}F(x))$. Solving the equation amounts to finding the position of any fluid particle at any time.[10]

Interpreting Newton's law $m\ddot{x} = F(x)$ as a fluid flow (1.16) opens a new vista. For example, it becomes easy to view several motions of a mechanical system simultaneously, and to see how they fit together — something that is near-impossible to do by a direct physical observation. Furthermore, we can analyze geometrical features of the flow: is it incompressible? does it stretch in some directions more than in others?, and so on. These features of the "fluid" can then be translated back to mechanics of the particle to give remarkable insights which direct physical intuition does not give. Our goal now is to show how to draw the pattern of the fluid flow given by (1.15). The pattern of flow lines in the phase plane is called the *phase portrait.*

How to draw phase portraits. The key to constructing phase portraits for (1.15) is to use the conservation of energy,

$$\frac{y^2}{2} + U(x) = E = \text{const. for all } \ t, \tag{1.17}$$

where $y = \dot{x}$ and where $x = x(t)$ is any solution of $\ddot{x} = F(x)$. The constant E here depends on the choice of initial data. Geometrically, (1.17) states that the phase trajectories are level curves of the function $y^2/2 + U(x)$, and the question reduces to understanding the pattern of level curves of the energy function. Here are three examples.

1. The harmonic oscillator $\ddot{x} = -x$ corresponds to the system[11]

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x. \end{cases}$$

The trajectories are circles centered at the origin. Indeed, $U = x^2/2$, and the conservation of energy (1.17) gives $\dot{y}^2/2 + \dot{x}^2/2 = c$. The

[10] The solution exists and is unique.

[11] This is the dimensionless version of $m\ddot{x} = -kx$. The rescaling can be achieved by choosing the new time $\tau = at$ in such a way that the period in new units becomes 2π, or $2\pi = a\,2\pi\sqrt{m/k}$, giving $a = \sqrt{k/m}$; see also Problem 1.23 on page 57.

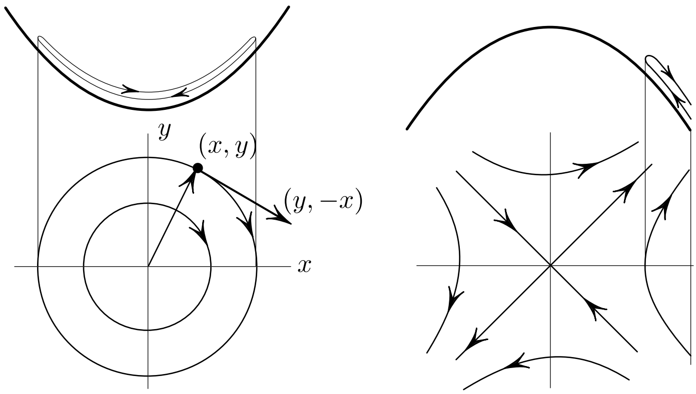


Figure 12. An harmonic oscillator (linear restoring force) and a linear repelling force.

motion on the circles is clockwise, since $\dot{x} = y > 0$ in the upper half-plane. Incidentally, to see that the motion is circular requires no calculation: the velocity vector $(y, -x)$ is perpendicular to the position vector of (x, y) (since their slopes are negative reciprocal).

2. Linear repelling force: $\ddot{x} = x$. The equivalent system is

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x. \end{cases}$$

The trajectories are hyperbolas of the form $y^2 - x^2 = c$, Figure 12, as follows from (1.17) or by direct computation:

$$\frac{d}{dt}(y^2 - x^2) = 2(y\dot{y} - x\dot{x}) = 2(yx - xy) = 0.$$

3. The pendulum. The equation of the pendulum in a rescaled form is $\ddot{x} + \sin x = 0$; here x is the angle between the pendulum and the downward vertical (see Problem 1.23 on page 57 for the details of rescaling). Figure 13 shows the phase portrait; all possible motions can be seen at a glance. Closed orbits correspond to oscillatory motions; unbounded orbits correspond to the pendulum rotating. The regions of these two motions are separated by the so-called heteroclinic orbits which asymptotically approach the upside-down equilibrium as $t \to \infty$ and also as $t \to -\infty$.

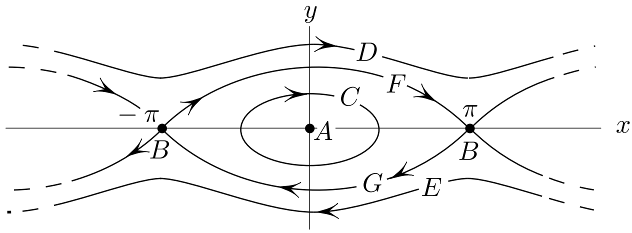


Figure 13. Phase portrait of the pendulum: A – the hanging equilibrium; B – the upside-down equilibrium (represented by $(\pi, 0)$ and its translates by 2π); C – an oscillatory motion; D – clockwise tumbling motion; F and G – motions approaching the unstable equilibrium as $t \to \pm\infty$ (the so-called *heteroclinic motions*).

7. Lagrangian equations of motion

Lagrange's equations were discovered roughly 100 years after Newton's laws. Although Lagrange's equations are equivalent to Newton's laws, they occupy a more central position in mechanics, as Figure 17 on page 287 illustrates. Lagrange's equations also lead to quantum mechanics (as explained on the same page). Since this chapter deals with particles in one dimension, I will formulate Lagrange's equation for this case, postponing the almost identical statement for higher degrees of freedom to Chapter 2.

Lagrange's equation. Consider, as before, a particle moving along the x-axis subject to the force $F(x)$ acting in the direction of the line.

Consider the *difference* of the particle's kinetic and potential energies:

$$L(x, \dot{x}) = \frac{m\dot{x}^2}{2} - U(x); \tag{1.18}$$

this function of $x, \dot{x}$ is called the *Lagrangian*.[12] Let us first treat $x, \dot{x}$ as two independent variables (and not (yet) as functions of t), so at this stage $\dot{x}$ is an independent variable, and not the time-derivative

[12] I know of no *dynamical* interpretation of this difference. However, this difference does have a direct *static* meaning, at the price of reinterpreting t and x. This is described in Section 19.

of some function $x(t)$. Differentiating L first by $\dot{x}$ and then by x, we discover:

$$L_{\dot{x}} = m\dot{x} \quad \text{and} \quad L_x = -U'(x), \tag{1.19}$$

where the subscripts denote partial derivatives: $L_x = \frac{\partial L}{\partial x}$ and $L_{\dot{x}} = \frac{\partial L}{\partial \dot{x}}$. In other words, both the momentum $m\dot{x}$ and the force $-U'(x)$ are partial derivatives of a single function L. Hence Newton's equation $m\ddot{x} = -U'(x)$ can be rewritten as

$$\frac{d}{dt} L_{\dot{x}} = L_x. \tag{1.20}$$

It should be emphasized that $L_{\dot{x}}$, L_x denote partial derivatives when x, $\dot{x}$ are treated as independent variables; however, when taking the derivative $\frac{d}{dt}$ in (1.20), we treat x as the function of time. Equation (1.20) is called the *Euler–Lagrange equation.* The recipe for generating equation of motion from the Lagrangian L applies verbatim in cases much more general than the one just considered.

8. The variational meaning of the Euler–Lagrange equation

Euler–Lagrange equation (1.20) has a remarkable hidden meaning. Loosely speaking, any solution of this equation, i.e., any physical motion, corresponds to the "shortest" path in the (t, x)-plane, in a certain sense which we now make precise.

Functionals and critical functions. Let us fix two points $A_0(t_0, x_0)$ and $A_1(t_1, x_1)$ in the (t, x)-plane. Take any differentiable function $x = x(t)$ whose graph connects connects A_0 and A_1:

$$x(t_0) = x_0, \quad x(t_1) = x_1, \tag{1.21}$$

(see Figure 14), and define the "length" of the graph, called the *action*, as the integral of the difference of kinetic and potential energies:

$$\mathcal{S}[x] = \int_{t_0}^{t_1} \left(\frac{m\dot{x}^2}{2} - U(x) \right) dt = \int_{t_0}^{t_1} L(x, \dot{x})\, dt. \tag{1.22}$$

This integral assigns a real number $\mathcal{S}[x]$ to any given (continuously differentiable) function $x = x(t)$ with fixed ends as in (1.21). Square

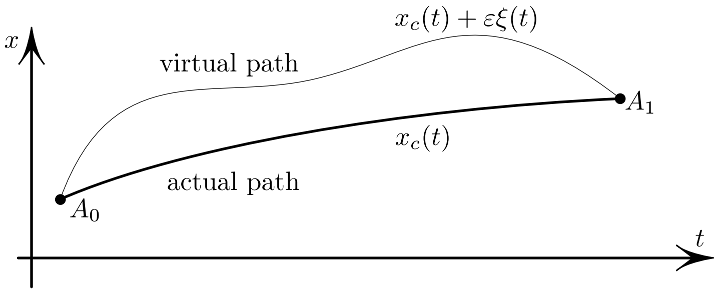


Figure 14. The graph of the actual motion $x = q(t)$ is the "shortest" curve between points $A_0(t_0, x_0)$ and $A_1(t_1, x_1)$ in the sense of the "distance" $\int_{t_0}^{t_1} L\, dt$.

brackets in $\mathcal{S}[x]$ remind us that x is a function. Such scalar-valued functions of a function are called *functionals*.

The minimum, or more generally, a critical function of the functional $\mathcal{S}$ is defined as follows. A function $x_c : [t_0, t_1] \mapsto \mathbb{R}$ is said to be a *critical function* of the functional (1.21)–(1.22) if

$$\frac{d}{d\varepsilon}\mathcal{S}[x_c + \varepsilon\xi]\bigg|_{\varepsilon=0} = 0 \tag{1.23}$$

for any smooth function ξ with $\xi(t_0) = \xi(t_1) = 0$; see Figure 14. Smoothness of ξ must be assumed for the derivative in (1.22) to exist. According to the definition (1.23), the functional changes with zero speed under any deformation of the critical curve.

Variational meaning of dynamical equations. Now we have the most remarkable and fundamental fact in mechanics, which is a culmination of several discoveries and known as *Hamilton's principle*:

The function $x = x(t)$ represents an actual motion, i.e., obeys (1.20)
if and only if
$x(t)$ is a critical function of the action integral (1.22).

Superficially, this is just a mathematical theorem (restated and proven as Theorem 1.1 on page 22). However, more is going on: the fact that actual motions are the "shortest" curves in space-time (t, x) reflects the fact that classical mechanics is the limiting case of quantum mechanics (see pages xviii and 286 for further discussion).

Remark. If $t_1 - t_0$ is sufficiently small, then the actual motion minimizes the action (1.22). This is explained in Chapter 6.

Analogy between functionals and real functions of several variables. To get more intuition on functionals and their critical functions just defined, note that our functional $\mathcal{S}[x]$ is an analog of a real function of many variables. Indeed, imagine discretizing $x(t)$, i.e., replacing it by a sampling of its values at n points in $[t_0, t_1]$. The integral $\mathcal{S}$ would then be replaced by a function of these values — it would become, in other words, a usual function of n variables. Recall the calculus definition of the critical point of such a function (with the goal of finding a familiar analog of (1.23)). A point $\mathbf{x}_c \in \mathbb{R}^n$ is said to be a *critical point* of a function $f(\mathbf{x})$ of n variables $\mathbf{x} = (x_1, \ldots, x_n)$ if the directional derivative at $\mathbf{x}_c$ is zero in every direction, i.e.,

$$\frac{d}{d\varepsilon} f(\mathbf{x}_c + \varepsilon \boldsymbol{\xi}) \bigg|_{\varepsilon=0} = 0 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^n.$$

In other words, f changes with zero instantaneous speed at the critical point $\mathbf{x}_c$. This is precisely the analog of the definition (1.23) of the stationary function. The left-hand side in (1.23) is simply a directional derivative of $\mathcal{S}$ in the function space.

9. Euler–Lagrange equations — general theory

In the preceding section we defined a critical function of a functional

$$\mathcal{S}[x] = \int_{t_0}^{t_1} L(x, \dot{x})\, dt, \quad x(t_0) = x_0, \; x(t_1) = x_1, \tag{1.24}$$

with prescribed endpoints (t_0, x_0), (t_1, x_1). The definition did not rely on the special form $L = K - U$ and we abandon this assumption here; we will only need L to have continuous partial derivatives with respect to its arguments $x, \dot{x}$ up to order two (with x, $\dot{x}$ treated as independent variables).

In this section we derive the “first derivative test” for $\mathcal{S}[x]$, namely, a necessary condition for $x_c(t)$ to be a critical function.

Euler and Lagrange independently found the famous answer described in the following theorem.

Theorem 1.1 (The Euler–Lagrange equation)**.** *Assume that $L(x, \dot{x})$ has two continuous derivatives in its variables (at this stage $\dot{x}$ is treated as an independent variable, and not dx/dt.) If $x = x(t)$ is a critical function (we drop the subscript c from now on) of the functional* (1.24)*, and if x has two continuous derivatives, then x satisfies the differential equation*

$$\boxed{\frac{d}{dt}L_{\dot{x}} - L_x = 0} \tag{1.25}$$

where

$$L_x = \frac{\partial}{\partial x}L(x, \dot{x}), \quad L_{\dot{x}} = \frac{\partial}{\partial v}L(x, v)|_{v=\dot{x}}\,.$$

Note that x and $\dot{x}$ are treated as independent variables when taking these partial derivatives; however, $\frac{d}{dt}$ in (1.25) *treats both x and $\dot{x}$ as functions of t.*

Example. For the harmonic oscillator, $L = m\dot{x}^2/2 - kx^2/2$ we have $L_{\dot{x}} = m\dot{x}$, $L_x = -kx$; substituting these into (1.25), we get $m\ddot{x} + kx = 0$, as expected.

Proof. Let $x = x(t)$ be a critical function of $\mathcal{S}$. By the definition, x satisfies

$$\frac{d}{d\varepsilon}\int_{t_0}^{t_1} L(x + \varepsilon\xi, \dot{x} + \varepsilon\dot{\xi})_{|\varepsilon=0} = 0$$

for all differentiable ξ vanishing at t_0, t_1. Differentiation by ε can be applied to the integrand. Applying the chain rule and then setting $\varepsilon = 0$ we get

$$\int_{t_0}^{t_1} (L_x\xi + L_{\dot{x}}\dot{\xi})dt = 0, \tag{1.26}$$

where $L = L(x, \dot{x})$. Note that in applying the chain rule we had to treat x and $\dot{x}$ as independent variables in taking the partials L_x, $L_{\dot{x}}$. Let us now integrate by parts the second term in (1.26). Using $\xi(t_0) = \xi(t_1) = 0$ to get rid of the boundary term we get

$$\int_{t_0}^{t_1} \left(L_x - \frac{d}{dt}L_{\dot{x}}\right)\xi\ dt = 0. \tag{1.27}$$

Note that $\frac{d}{dt}$ must treat x and $\dot{x}$ as functions of t.

Now since the function ξ is arbitrary (apart from the assumptions mentioned before), we expect the expression $f(t) = L_x - \frac{d}{dt}L_{\dot{x}}$ to be identically zero (as desired). Indeed, assume for a moment the contrary: $f(\bar{t}) \neq 0$, say, $f(\bar{t}) > 0$ for some $\bar{t} \in [t_0, t_1]$. Since f is continuous by our assumptions, we have $f > 0$ on a whole interval I containing $\bar{t}$. Let us then choose $\xi > 0$ on I and $\xi = 0$ elsewhere. But then $\int_{t_0}^{t_1} f(t)\xi(t)\,dt > 0$, in contradiction with (1.27). This completes the proof. ♢

Remark. As stated in the footnote to the theorem, it actually suffices to assume that the critical function $x(t)$ has just one continuous derivative; the existence of the second derivative then follows. The idea of the proof is very nice: instead of integrating by parts the second term in (1.26), integrate the first! This gives

$$\int_{t_0}^{t_1}\left(-\int_{t_0}^{t} L_x d\tau + L_{\dot{x}}\right)\dot{\xi}\,dt = 0;$$

using arbitrariness of ξ, it is easy to show that

$$-\int_{t_0}^{t} L_x d\tau + L_{\dot{x}} = \text{const.}$$

Since the first term is continuously differentiable (C^1), so is $L_{\dot{x}}(x, \dot{x})$. But this implies that $\dot{x}$ itself is continuously differentiable (I omit details which involve using the implicit function theorem). In other words, x is twice continuously differentiable, as claimed.

10. Noether's theorem/Energy conservation

The following is actually a special case of Noether's theorem — the general case is described on pages 267 and 270 when we consider higher degrees of freedom.

Theorem 1.2. *For any solution x of the Euler–Lagrange equation* (1.25) *we have*

(1.28) $$\dot{x}L_{\dot{x}} - L = \text{const.}$$

Proof goes by differentiation: Using the chain rule and the Euler–Lagrange equation, we get

$$\frac{d}{dt}(\dot{x}L_{\dot{x}} - L) = \cancel{\ddot{x}L_{\dot{x}}} + \dot{x}\frac{d}{dt}L_{\dot{x}} - L_x\dot{x} - \cancel{L_{\dot{x}}\ddot{x}} = \dot{x}\Bigg(\underbrace{\frac{d}{dt}L_{\dot{x}} - L_x}_{=0 \text{ by } (1.25)}\Bigg) = 0.$$

♢

Although this proof is short, it does not explain "what is going on." A more illuminating proof, which shows what is happening geometrically, and works for the higher-degree-of-freedom case, is given on page 270.

Example. For the special case: $L = \frac{m\dot{x}^2}{2} - U(x)$, Noether's theorem recovers the conservation of energy:

$$\dot{x}L_{\dot{x}} - L = \frac{m\dot{x}^2}{2} + U(x) = \text{const.},$$

as a direct substitution of this L into (1.28) shows. Note that $\dot{x}L_{\dot{x}} - L$ turned out to be the total energy.

11. Hamiltonian equations of motion

We already saw one remarkable way to reformulate Newton's law $m\ddot{x} = -U'(x)$, as the Euler–Lagrange equation. There is yet another reformulation which combines a beautiful symmetry with a yet additional insight. At this point I describe only *how* to transform Euler–Lagrange's equation into a Hamiltonian system, leaving out the motivation (which can be found in Chapter 8).

The momentum and the Hamiltonian. Let us define the momentum

(1.29) $$m\dot{x} = p$$

and express the total energy $H = m\dot{x}^2/2 + U$ in terms of p by substituting $\dot{x} = p/m$:

(1.30) $$H(x,p) = \frac{p^2}{2m} + U(x).$$

The energy thus expressed in terms of position and momentum is called the *Hamiltonian* of the system. It is instructive to take partial derivatives of H with x, p treated as two independent variables:

$$H_x(x,p) = U'(x) = -F(x), \quad H_p(x,p) = \frac{p}{m}.$$

Using the first equation, Newton's law $m\ddot{x} = F(x)$ becomes $\dot{p} = -H_x$; using the second equation, we can rewrite the definition of p as $\dot{x} = p/m = H_p$. Summarizing, we have

$$\begin{cases} \dot{x} = H_p(x,p), \\ \dot{p} = -H_x(x,p). \end{cases} \tag{1.31}$$

This system is equivalent to Newton's equation. This elegant system looks even more elegant in vector form,

$$\dot{\mathbf{z}} = J\,\nabla H(\mathbf{z}), \tag{1.32}$$

where

$$\mathbf{z} = \begin{pmatrix} x \\ p \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \nabla H = \begin{pmatrix} H_x \\ H_p \end{pmatrix}.$$

Note that J rotates vectors by $\pi/2$ clockwise, and we arrive at a remarkable connection between Hamiltonian and gradient systems: any Hamiltonian vector field (1.31)–(1.32) is obtained from the gradient vector field $\dot{\mathbf{z}} = \nabla H(\mathbf{z})$ by the $\pi/2$ rotation clockwise, Figure 15.

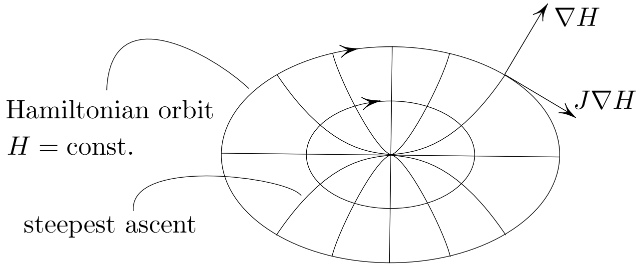


Figure 15. Hamiltonian vector field is orthogonal to the gradient vector field ∇H.

Trajectories are level curves of H. Indeed, H is constant along each solution of (1.31):

$$\frac{d}{dt}H(\mathbf{z}(t)) = H_x\dot{x} + H_p\dot{p} = H_xH_p + H_p(-H_x) = 0.$$

This can be seen geometrically from (1.32): $\dot{\mathbf{z}} \perp \nabla H$, and thus $\dot{\mathbf{z}}$ is tangent to a level curve of H; this means that $\mathbf{z}(t)$ stays on the level curve.

Time-dependent Hamiltonians. Consider the motion of a particle in a potential which depends on time:

$$\ddot{x} = -U_x(x,t); \tag{1.33}$$

examples include a pendulum whose pivot undergoes vertical oscillations:

$$\ddot{\theta} = -\frac{1}{L}(g + a(t))\sin\theta,$$

where a is the acceleration of the pivot, or a mass hanging on a spring whose end is oscillating in the vertical direction, etc.

Exactly as before, Newton's law (1.33) can be written as a Hamiltonian system:

$$\begin{cases} \dot{x} = H_p(x,p,t), \\ \dot{p} = -H_x(x,p,t), \end{cases} \qquad H = \frac{p^2}{2m} + U(x,t) \tag{1.34}$$

or, in vector form,

$$\dot{\mathbf{z}} = J\nabla H(\mathbf{z}, t). \tag{1.35}$$

Exercise. Does H remain constant along solutions of (1.35)?

Answer. Denoting partial derivative by subscripts, we get, differentiating H along a solution of the Hamiltonian system:

$$\frac{d}{dt}H(x,p,t) = H_x\dot{x} + H_p\dot{p} + H_t = H_xH_p + H_p(-H_x) + H_t = H_t.$$

We conclude that if H depends on t, it does not remain constant.

12. The phase flow

In this and the next section we introduce two fundamental concepts — the *flow* and the *divergence*, important in their own right, and used in Liouville's theorem stated in Section 16.

Liouville's theorem states that if a vector field has zero divergence (see page 28 for the definition), then the flow generated by the vector field is incompressible, or area-preserving: as a "blob" of initial data is carried by the flow, the area of the blob remains unchanged.[13]

We assume throughout that the vector field $\mathbf{v}(\mathbf{x})$ in $\mathbb{R}^2$ is smooth on $\mathbb{R}^2$, and that for any initial condition in $\mathbb{R}^2$ the solution of the ODE

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}) \tag{1.36}$$

is defined for all t.

The phase flow. For any point $\mathbf{x}_0 \in \mathbb{R}^2$ there is a unique solution $\mathbf{x} = \mathbf{x}(t)$ of (1.36) satisfying the initial condition $\mathbf{x}(0) = \mathbf{x}_0$; this is the statement of the existence/uniqueness theorem for ODEs, applicable by the assumptions we made on $\mathbf{v}$. The standard notation for this solution is $\mathbf{x}(t) = \varphi^t \mathbf{x}_0$. We can view φ^t as the map of $\mathbb{R}^2$ which sends $\mathbf{x}_0$ to $\varphi^t \mathbf{x}_0$. Since t is arbitrary, we have a one-parameter family of these maps, also referred to as *the time t maps* (of the phase space to itself). This family of maps is called the *flow* (associated with the ODE (1.36)); the term "flow" is suggested by thinking of $\varphi^t \mathbf{x}_0$ as the position at time t of a particle of fluid.

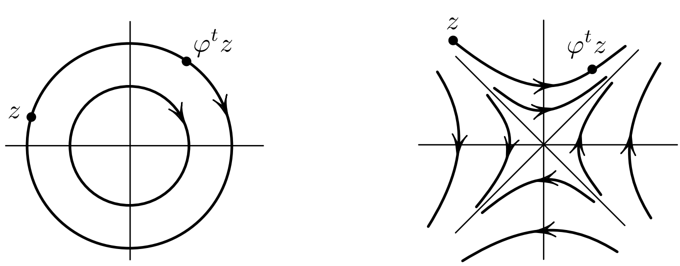


Figure 16. Flows associated with some vector fields.

[13] We are speaking of flows in $\mathbb{R}^2$; in $\mathbb{R}^3$ the area must be replaced by volume, and in $\mathbb{R}^n$ by n-volume.

Theorem 1.3. *The family of maps* $\{\varphi^t : t \in \mathbb{R}\}$ *associated with the vector field* $\mathbf{v} = \mathbf{v}(\mathbf{z})$ *satisfies the following properties:*

(1) $\varphi^t \circ \varphi^\tau = \varphi^{t+\tau}$, *for any real* t, τ.

(2) φ^0 *is an identity map:* $\varphi^0\mathbf{x} = \mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{R}^n$.

In other words, the family forms a group under composition.

This theorem is just a restatement of the existence and uniqueness theorem for ordinary differential equations, coupled with the fact that $\mathbf{v}$ is autonomous (i.e., does not depend on t). Without the latter assumption property (1) would fail.

13. The divergence

The concept of divergence is the second prerequisite for the formulation of Liouville's theorem. We again limit discussion to $\mathbb{R}^2$, although most ideas apply to any dimension almost verbatim.

The definition. The divergence of a vector field $\mathbf{v}$ in $\mathbb{R}^2$ at a point $\mathbf{x}$ is the outward flux per unit area through the boundary of a small region as the region shrinks to the point $\mathbf{x}$:

$$\text{div } \mathbf{v}(\mathbf{x}) = \lim_{D \to \{\mathbf{x}\}} \frac{1}{|D|} \int_{\partial D} \mathbf{v} \cdot \mathbf{n} \, ds, \tag{1.37}$$

where D is a region enclosing $\mathbf{x}$ (Figure 17), $|D| =$ area(D), ∂D is the boundary of D, $\mathbf{n}$ is the unit outward normal vector and s is the arc length.[14]

The limit (1.37) exists and does not depend on the particular choice of D, provided only that $\mathbf{v}$ is a smooth function of x, y, and that D is bounded by a piecewise smooth curve without self-intersections.

[14]In dimension $n > 2$ the definition is the same, except that ds is the $(n-1)$-dimensional surface element of the n-dimensional region D.

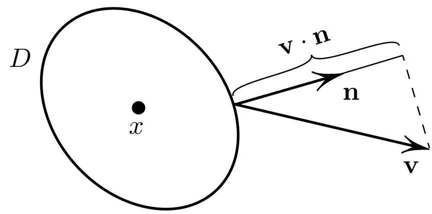


Figure 17. Definition of the divergence of a vector field.

The formula. By choosing D in the definition (1.37) to be a rectangle, $D = [x, x + dx] \times [y, y + dy]$, one arrives at the standard formula,

$$\text{div } \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}, \quad \text{where} \quad \mathbf{v} = (v_1, v_2). \tag{1.38}$$

Intuitively, the formula makes perfect sense: $\frac{\partial v_1}{\partial x}$ detects the dependence of the x-component of the velocity on x; if $\frac{\partial v_1}{\partial x}$ is large, then the horizontal velocity v_1 is greater through the right side of the box than through the left side, contributing to positive net flux out of the box. More details on the topic can be found in [**17**].

Theorem 1.4 (The Divergence Theorem)**.** *If* $\mathbf{v}$ *is a continuously differentiable vector field on a bounded domain* D *in* $\mathbb{R}^2$ *with the piecewise smooth boundary, then*

$$\int_D \text{div } \mathbf{v} \; dx = \int_{\partial D} \mathbf{v} \cdot \mathbf{n} \, ds, \tag{1.39}$$

where dx *is the element of area, and* ds *is the element of arc length of the boundary* ∂D*.*

Remark 1.5. The theorem and the proof apply to the case $\mathbb{R}^n$ for any n almost verbatim.

Proof. This theorem is almost obvious if one uses the definition (1.37), rather than the computational formula (1.38). Indeed, let us divide the domain D into subdomains of small diameters $\leq \delta$, as shown in Figure 18, and denote a typical subdomain by D_i. The first step is to note that

$$\int_{\partial D} \mathbf{v} \cdot \mathbf{n} \, ds = \sum_i \int_{\partial D_i} \mathbf{v} \cdot \mathbf{n} \, ds; \tag{1.40}$$

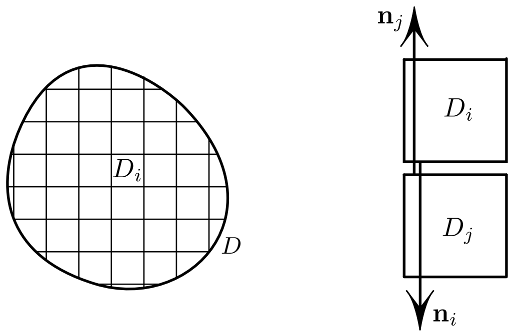


Figure 18. Proof of the divergence theorem. Right: fluxes on shared boundaries cancel.

indeed, the integrals over the shared boundaries cancel, since the outward normals on a shared boundary point in opposing directions; see Figure 18. We now apply the definition of divergence to each D_i: picking a point $\mathbf{x}_i \in D_i$ for each i, we have by (1.37) that

$$\operatorname{div} \mathbf{v}(\mathbf{x}_i) = \frac{1}{|D_i|} \int_{\partial D_i} \mathbf{v} \cdot \mathbf{n} \, ds + r_i,$$

where the remainder r_i is small if $\operatorname{diam}(D_i) \leq \delta$ is small. Actually, the r_i are *uniformly small*: for any ε there exists δ such that $|r_i| < \varepsilon$ for *all* i, provided $\operatorname{diam}(D_i) \leq \delta$; I omit the details of the proof, which uses continuous differentiability of $\mathbf{v}$. Rewriting the above as

$$\int_{D_i} \mathbf{v} \cdot \mathbf{n} \, dx = (\operatorname{div} \mathbf{v}(\mathbf{x}_i) - r_i) \, |D_i|, \tag{1.41}$$

and adding up, we get

$$\underbrace{\sum_i \int_{\partial D_i} \mathbf{v} \cdot \mathbf{n} \, ds}_{\int_{\partial D} \mathbf{v} \cdot \mathbf{n} \, ds \text{ by } (1.40)} = \sum_i \operatorname{div} \mathbf{v}(\mathbf{x}_i) |D_i| - \underbrace{\sum_i r_i |D_i|}_{< \varepsilon |D|}. \tag{1.42}$$

The second term is the Riemann sum for the first integral in (1.39). In the limit $\varepsilon \to 0$, (1.42) becomes (1.39). This proves the divergence theorem. ♢

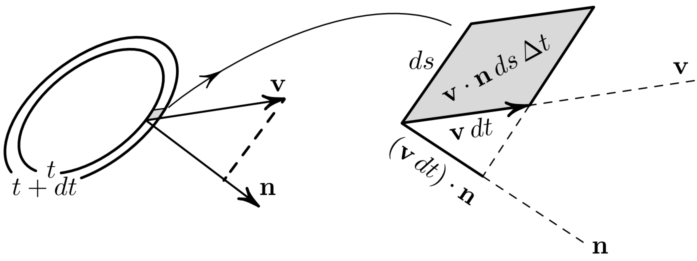


Figure 19. Computing ΔA (Steps 1–4).

14. A lemma on moving domains

The divergence in $\mathbb{R}^2$ admits a very nice interpretation: it is the exponential rate of growth of an infinitesimal area[15] carried by the vector field. In other words, the divergence is the local logarithmic rate of the area's growth. Now to make this precise, we need to express the rate of change of the volume of a moving domain; the lemma in this section does it. Incidentally, this lemma is a generalization of the fundamental theorem of calculus, as explained at the end of the section. The extension to $\mathbb{R}^n$ is verbatim and involves no new ideas; and since our goal is to learn ideas, we stick to $\mathbb{R}^2$.

Lemma 1.1. *Let $D \in \mathbb{R}^2$ be a region with a piecewise smooth boundary; let $\mathbf{v}$ be a vector field in $\mathbb{R}^2$, with the associated flow φ^t, and let $A(t) = \text{area}(\varphi^t D)$. Then*

$$A'(t) = \int_{\partial D_t} \mathbf{v} \cdot \mathbf{n} \, ds. \tag{1.43}$$

A heuristic explanation of the lemma is supplied by Figure 19: an arc ds moving with the flow for time Δt sweeps area approximately $\mathbf{v} \cdot \mathbf{n} \, ds \, \Delta t$ (shown as a shaded parallelogram), and thus the area swept by the entire boundary in time Δt is

$$\Delta A = \int_{\partial D} \mathbf{v} \cdot \mathbf{n} \, ds \, \Delta t + o(\Delta t).$$

[15] In $\mathbb{R}^n$ the n-volume should replace the area.

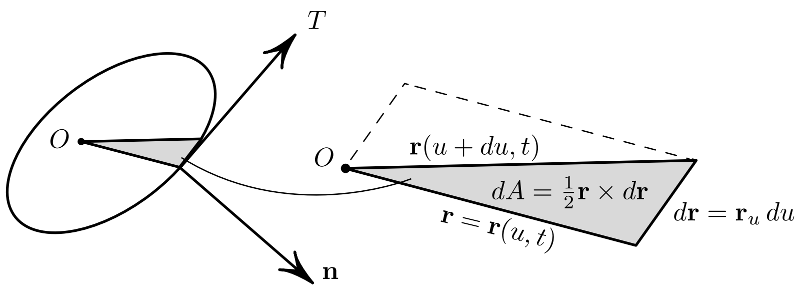


Figure 20. Explaining (1.44) geometrically.

Dividing by Δt and taking $\Delta t \to 0$ we get (1.43). Rather than massaging this heuristic argument into a rigorous proof, I give an alternative one.

Proof of the lemma in $\mathbb{R}^2$. Let us parametrize ∂D by $\mathbf{x} = \mathbf{r}(u)$, $0 \le u \le 1$, $\mathbf{r}(0) = \mathbf{r}(1)$. The curve $\partial(\varphi^t D)$ is then also parametrized by u: $\mathbf{r}(u,t) = \varphi^t \mathbf{r}(u)$. The area of $\varphi^t D$ is then given by

$$A(t) = \operatorname{area}(\varphi^t D) = \frac{1}{2}\int_0^1 \mathbf{r} \times \mathbf{r}_u \, du \tag{1.44}$$

(see Figure 20); here $\mathbf{r} \times \mathbf{r}_u = \det(\mathbf{r}\ \mathbf{r}_u)$ is the signed area of the parallelogram generated by the pair of vectors (also called the scalar cross product). Differentiating by t and integrating the second term by parts, we get:

$$A'(t) = \frac{1}{2}\int_0^1 (\mathbf{r}_t \times \mathbf{r}_u + \mathbf{r}_t \times \mathbf{r}_{ut})\, du = \frac{1}{2}\int_0^1 (\mathbf{r}_t \times \mathbf{r}_u - \mathbf{r}_u \times \mathbf{r}_t)\, du.$$

Since $\mathbf{r}_u \times \mathbf{r}_t = -\mathbf{r}_t \times \mathbf{r}_u$, this reduces to

$$A'(t) = \int_0^1 \mathbf{r}_t \times \mathbf{r}_u \, du.$$

But $\mathbf{r}_t = \frac{d}{dt}\mathbf{r}(u,t) = \frac{d}{dt}\varphi^t \mathbf{r}(u,t) = \mathbf{v}(\mathbf{r}(u,t))$, and $\mathbf{r}_u du = \mathbf{T}\, ds$, where $\mathbf{T}$ is the unit tangent vector to $\partial\varphi^t D$. We therefore have

$$A'(t) = \int_{\partial(\varphi^t D)} \mathbf{v} \times \mathbf{T} ds = \int_{\partial(\varphi^t D)} \mathbf{v} \cdot \mathbf{n}\, ds;$$

the identity $\mathbf{v} \times \mathbf{T} = \mathbf{v} \cdot \mathbf{n}$ holds because the angles $\angle(\mathbf{v}, \mathbf{T})$ and $\angle(\mathbf{v}, \mathbf{n})$ are complementary. ♢

Proof of the lemma in $\mathbb{R}^n$. The above proof can be extended to $\mathbb{R}^n$, by replacing (1.44) with

$$V(t) = \frac{1}{n}\int \det(\mathbf{r}, \mathbf{r}_{u_1}, \ldots, \mathbf{r}_{u_{n-1}})\, du_1 \ldots du_{n-1},$$

the subscripts denoting partial derivatives.

Here is yet another dimension-independent proof of the Lemma in $\mathbb{R}^n$; the judgment on which proof is better is left to the reader. Using the change of variables $\mathbf{x} = \varphi^t\mathbf{y}$ we get:

$$A(t) = \int_{\varphi^t(D)} d\mathbf{x} = \int_D \det\, \Phi(t, \mathbf{y}) d\mathbf{y},$$

where

$$\Phi(t, \mathbf{y}) = \frac{\partial \varphi^t \mathbf{y}}{\partial \mathbf{y}} \tag{1.45}$$

is the Jacobian derivative matrix of the map φ^t. Differentiating by t, we get

$$A'(t) = \int_D \frac{d}{dt}\det \Phi\; d\mathbf{y}, \tag{1.46}$$

and our goal now is to relate the integrand to the vector field $\mathbf{v}$. This relation can only come from the definition of φ^t, according to which

$$\frac{d}{dt}\varphi^t\mathbf{y} = \mathbf{v}(\varphi^t\mathbf{y}),$$

for all $\mathbf{y} \in D$. Differentiating by the initial condition $\mathbf{y}$ and exchanging the order of differentiation we obtain

$$\frac{d}{dt}\Phi = \frac{\partial \mathbf{v}(\varphi^t\mathbf{y})}{\partial \mathbf{y}}\Phi. \tag{1.47}$$

This implies, by Abel's theorem:

$$\frac{d}{dt}\det \Phi = \operatorname{tr}\, \frac{\partial \mathbf{v}(\varphi^t\mathbf{y})}{\partial \mathbf{y}}\det \Phi. \tag{1.48}$$

But

$$\operatorname{tr}\, \frac{\partial \mathbf{v}(\varphi^t\mathbf{y})}{\partial \mathbf{y}} = \operatorname{div}\, \mathbf{v},$$

and (1.48) becomes

$$\frac{d}{dt}\det\Phi = \operatorname{div}\, \mathbf{v}(\varphi^t\mathbf{y})\, \det\, \Phi.$$

Substituting this into (1.46) gives

$$A'(t) = \int_{D_0} \operatorname{div}\, \mathbf{v}(\varphi^t\mathbf{y})\, \det \Phi\, d\mathbf{y}.$$

Returning to the old variables $\mathbf{x} = \varphi^t \mathbf{y}$ we arrive at the desired statement:

$$A'(t) = \int_{D_t} \operatorname{div} \mathbf{v}(\mathbf{x})\, d\mathbf{x} = \int_{\partial D_t} \mathbf{v}(\mathbf{x}) \cdot \mathbf{n}\, ds,$$

where the last step uses the divergence theorem. ♢

Remark 1.6. The fundamental theorem of calculus (FTC) is a special case of the lemma on moving boundaries: indeed, FTC deals with the rate of change of the area $\int_a^t f(x)\, dx$ under the curve $y = f(x)$, with $a \le x \le t$, where only one piece of the boundary is moving, namely, the "right wall" $x = t$. The speed of this wall is $v = \frac{dt}{dt} = 1$, and its length $L = f(t)$. The boundary integral of the normal speed is therefore $Lv = f(t) \cdot 1 = f(t)$, so that our lemma gives

$$\frac{d}{dt} \int_a^t f(x)\, dx = f(t),$$

showing that FTC is indeed a special case.

15. Divergence as a measure of expansion

This section explains that the divergence in $\mathbb{R}^2$ is the rate of growth of an infinitesimal area per unit area (in $\mathbb{R}^n$ the same statement holds if the area is replaced by n-volume). Here is a precise statement.

Theorem 1.5. *Let $A(t) = \text{area}(\varphi^t D)$ denote the area of a domain D moving with the flow φ^t of the vector field $\mathbf{v}$. We then have:*

$$\operatorname{div} \mathbf{v}(\mathbf{x}) = \lim_{D \to \{\mathbf{x}\}} \frac{A'(0)}{A(0)}. \tag{1.49}$$

Proof. By the lemma on moving domains

$$A'(0) = \int_{D_t} \mathbf{v} \cdot \mathbf{n}\, ds,$$

as follows from (1.43) by setting $t = 0$. Substituting this into the definition of divergence (1.37), we obtain (1.49). ♢

Equation (1.49) confirms the earlier statement that $\operatorname{div} \mathbf{v}(\mathbf{x})$ measures the instantaneous rate of change of the area of a small region surrounding $\mathbf{x}$, per unit area, as the area shrinks to $\mathbf{x}$. In other words, $\operatorname{div} \mathbf{v}(\mathbf{x})$ is the *exponential rate of growth of area* in a small neighborhood of $\mathbf{x}$, also referred to as the *logarithmic derivative* of the small area.

Incidentally, the *interest rate in continuous compounding in finance is an example of divergence* in $\mathbb{R}^1$. Indeed, the interest rate k is defined by

$$k = 100\,A'(t)/A(t),$$

where $A(t)$ is the amount of money in the account at time t, the one-dimensional version of (1.49). The vector field v in compounding is given by $v(x) = kx$, since the money grows according to $\dot{A} = kA$.

We would have used the very intuitive expression (1.49) as the definition of the divergence were it not for the fact that (1.49) requires the mention of the time and of the flow φ^t, which our definition (1.37) does not. The advantage of (1.49), however, is that it better reflects the term "divergence", since it indeed measures the expansion of area. Another advantage of (1.49) is that it suggests that if $\operatorname{div}\mathbf{v}(\mathbf{x}) = 0$ for all $\mathbf{x}$, then the flow is area-preserving. This is indeed the content of Liouville's theorem, which is discussed next.

To summarize, (1.37) and (1.49) reflect two views of the divergence: one by fixing the domain and watching the particles go by, and the other by following a moving domain. The first of these is referred to as the Eulerian approach, the second as the Lagrangian.

16. Liouville's theorem

As before, φ^t is the flow of a smooth vector field $\mathbf{v}$ in $\mathbb{R}^2$ and we assume, as always, that φ^t is well defined for all $t \in \mathbb{R}$.

Theorem 1.6 (Liouville)**.** If $\operatorname{div}\mathbf{v} = 0$, then the flow φ^t of $\mathbf{v}$ is area-preserving, that is, for any planar region D bounded by a piecewise smooth curve without self-intersections

$$A(t) = \operatorname{area}(\varphi^t D) = \text{const.} \tag{1.50}$$

Proof.

$$A'(t) \overset{(1.43))}{=} \int_{\partial(\varphi^t D)} \mathbf{v}\cdot\mathbf{n}\,ds \overset{(1.39)}{=} \int_{\varphi^t D} \operatorname{div}\mathbf{v}\,d\mathbf{x} = 0. \tag{1.51}$$

◇

Remark 1.7 (On nonautonomous vector fields)**.** Liouville's theorem holds even for the time-dependent vector fields $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$. None of the statements or arguments in this section are affected by introducing the time-dependence. It suffices to assume that $\mathbf{v}$ is continuous (or even merely summable) in t, see [**3**].

Remark 1.8. To summarize its gist, Liouville's theorem amounts to little more than redefining divergence dynamically, as (1.49). In fact, this alternative definition of divergence is an infinitesimal version of Liouville's theorem. When viewed in this way, Liouville's theorem is completely transparent.

Exercise 1.2. Show that a divergence-free vector field $\mathbf{v}(\mathbf{x})$ in $\mathbb{R}^2$ is locally Hamiltonian: that is, if div $\mathbf{v} = 0$ in a neighborhood of a point, then there exists a real-valued function H such that $\mathbf{v} = (H_y, -H_x)$. Such H is called a *Hamiltonian.*

Hint: Define $H(z)$ as the flux of $\mathbf{v}$ through a curve connecting z with some chosen point. Show that H does not depend on the choice of the curve (using div $\mathbf{v} = 0$).

Exercise 1.3. Show that the preceding statement is only true locally, i.e., give an example of a divergence-free vector field for which a single–valued Hamiltonian function does not exist.

Answer. $\mathbf{v}(\mathbf{z}) = \frac{1}{|\mathbf{z}|^2}\mathbf{z}$ is a divergence-free flow in the punctured plane (the flow due to a point source at the origin). $H = \arg z$ satisfies $\mathbf{v} = (H_y, -H_x)$ but H is not single-valued. Any other Hamiltonian of $\mathbf{v}$ differs from H by a constant, hence there is no single-valued Hamiltonian of $\mathbf{v}$.

17. The "uncertainty principle" of classical mechanics

In this section we consider time-dependent Hamiltonian systems:

$$\begin{cases} \dot{x} = H_p(x,p,t), \\ \dot{p} = -H_x(x,p,t). \end{cases}$$

Liouville's theorem still applies, since

$$\operatorname{div}(H_p, -H_x) = H_{px} + (-H_{xp}) = 0.$$

We conclude that the flow φ^t of this system is area-preserving: for any domain $D \subset \mathbb{R}^2$ with a piecewise smooth boundary we have $\operatorname{area}(\varphi^t D) = \text{const}$. This implies that if, for some t, the image $\varphi^t D$ is squeezed in, say, the x-direction, then it must be stretched in the

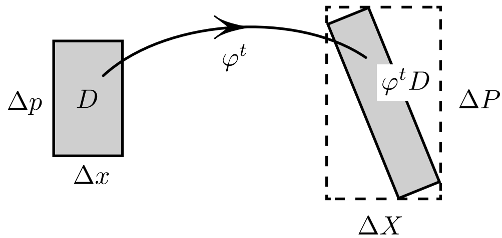


Figure 21. The uncertainty principle in classical mechanics.

p-direction. In other words, *the more we know x, the less we know p* — a kind of uncertainty principle in classical mechanics.

To get some practical conclusion, let D be a rectangle, as shown in Figure 21; as the figure illustrates, ΔX, ΔP are the horizontal and vertical widths of $\varphi^t D$ for some later time t. Since area(D) = area$(\varphi^t D)$, we have

$$\Delta x \Delta p \leq \Delta X \Delta P. \tag{1.52}$$

To suggest the analogy with quantum mechanics, denote area$(D) = \Delta x \Delta p = h$ (recall $\hbar$, the Planck constant); (1.52) becomes

$$\Delta X \Delta P \geq h.$$

An example with particles. The motion of particles in a potential $U(x,t)$ on the line is governed by $\ddot{x} = -U_x(x,t)$; the potential may depend on t arbitrarily. The rectangle D in Figure 21 corresponds to a "cloud" of initial data with the range Δx of positions and with the range Δp of velocities; the view in the (t,x)-plane is shown in Figure 22. Now assume that $\Delta X < \Delta x/100$; that is, assume that all the particles from the initial cloud gathered up into a narrower interval. Squeezing in the x-direction means expansion in the p-direction, according to (1.52):

$$\Delta P \geq \frac{\Delta x}{\Delta X} \Delta p > 100 \Delta p.$$

This is a remarkable conclusion: the more the particles bunch up together, the more disparate their velocities become.

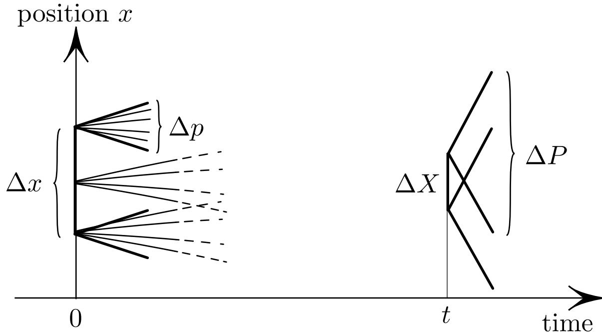


Figure 22. Classical mechanical uncertainty principle applied to trajectories or rays.

The uncertainty principle explaining how telescopes work. The discussion of the preceding paragraph has an optical interpretation. Figure 22 shows the set of solutions of $\ddot{x}+U_x(t,x)=0$. According to the preceding paragraph, if these solutions "squeeze" through a narrow gap ΔX at some t, then ΔP (the range of their slopes at that time t) is large. Now a very similar "uncertainty principle" holds not just for the solutions of $\ddot{x}+U_x(t,x)=0$, but also for rays passing through a binocular, or a telescope. A remarkable consequence is this: *the mere fact that the telescope converts a parallel beam of rays into a narrower parallel beam implies that the telescope magnifies distant objects.* A loose explanation is the following. Because the widths of parallel beams decrease in passing through the telescope, the angles between any two beams increase — Liouville's theorem has its optical counterpart. But this means that distant objects are magnified, since we perceive the size of an object, say the Moon, by the angles between the nearly parallel beams emitted by different parts of the object as these beams enter our eyes. Indeed, a parallel beam focuses on a dot on the retina, and the greater is the angle between the beams, the greater is the distance between two illuminated dots on the retina, and the greater is the perceived distance between the sources of the two beams.

More details on this, including an optical counterpart of Liouville's theorem, can be found in [**14**], page 129.

18. Can one hear the shape of the potential?

The famous (among mathematicians) question of Mark Kac: "can you hear the shape of the drum?" refers to the problem of recovering the shape of a vibrating membrane from the knowledge of all of the frequencies, or overtones, of its vibrational modes. A much simpler analog of this problem deals with one-dimensional oscillations of a classical particle governed by $\ddot{x} = -U'(x)$.

Question. Can one recover the shape of the potential $U(x)$ given the period $T(E)$ of oscillation of a particle as a function of its energy (or the amplitude)?[16]

The remarkable answer, due to Abel, is "yes", under mild symmetry assumptions. Let the potential U be as in Figure 23; more precisely, assume that $U(-x) = U(x)$, with $U(0) = U'(0) = 0$, and that U is monotone increasing for $x > 0$. Our goal is to produce a formula for $U(x)$ given the period as a function of energy. This goal is reached in three steps.

Step 1 is to derive the formula for the period, given the energy:

$$T(E) = 2\sqrt{2}\int_0^{x_{\max}} \frac{dx}{\sqrt{E - U(x)}}, \quad x_{\max} = U^{-1}(E) > 0. \tag{1.53}$$

To derive (1.53), we observe first that the particle oscillates back and forth between two endpoints $\pm x_{\max}$; these are the points at which the particle is instantaneously at rest, so all of its energy is potential:

$$U(x_{\max}) = U(-x_{\max}) = E.$$

The period of the oscillation is twice the time between $-x_{\max}$ and $x_{\max}$:

$$T(E) = 2\int_{-x_{\max}}^{x_{\max}} dt = 2\int_{-x_{\max}}^{x_{\max}} \frac{dx}{|\dot{x}|}$$

Now, $|\dot{x}|$ is found from

$$\frac{\dot{x}^2}{2} + U(x) = E \tag{1.54}$$

[16]Note that T is the reciprocal of the frequency; in this sense this problem is a classical mechanical analog, in one dimension, of the quantum "drum" problem of the preceding paragraph.

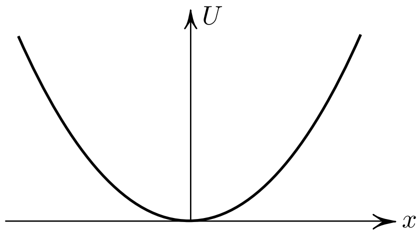


Figure 23. The unknown potential.

as $|\dot{x}| = \sqrt{2(E - U(x))}$. Substituting this into the last integral gives (1.53), after we replace $\int_{-x_{\max}}^{-x_{\max}}$ by $2\int_0^{-x_{\max}}$. We completed the first step.

Step 2: Changing the variable of integration. Our goal is to find U from (1.53); unfortunately, U enters this expression in a nonlinear way. We now eliminate this nonlinearity by choosing $U(x) = u$ as the new variable of integration. Now x becomes a function of u, namely, the inverse of the function $u = U(x)$. We have $dx = x'(u)du$, and our integral equation (1.53) becomes

$$T(E) = 2\sqrt{2}\int_0^E \frac{x'(u)}{\sqrt{E-u}}du, \tag{1.55}$$

where now the unknown is the inverse of U: $x(u) = U^{-1}(u)$.

Step 3: Solving (1.55) for the unknown function $x = x(u)$. Let $F \geq E$ be a parameter. Let us multiply both sides of (1.55) by $1/\sqrt{F-E}$ and integrate by E from 0 to F (this is done to extract $x(u)$ from the right-hand side, as will soon become clear):

$$\int_0^F \frac{T(E)}{\sqrt{F-E}}dE = 2\sqrt{2}\int_0^F \int_0^E \frac{x'(u)}{\sqrt{F-E}\sqrt{E-u}}du\, dE. \tag{1.56}$$

Let us change the order of integration on the right, with the idea of taking $x'(u)$ outside one of the integrals; see Figure 24. This integral on the right simplifies to $\pi x(F)$ (and this is the key):

$$\int_0^F x'(u) \underbrace{\left(\int_u^F \frac{1}{\sqrt{F-E}\sqrt{E-u}}dE\right)}_{=\pi,\ \text{see (1.58) below}} du = \pi \int_0^F x'(v)\, du = \pi\, x(F).$$

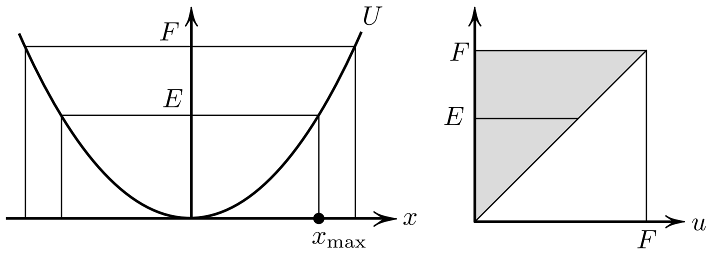


Figure 24. Solving (1.55).

Substituting this into the right-hand side of (1.56) gives

$$x(F) = \frac{1}{2\sqrt{2}\,\pi} \int_0^F \frac{T(E)}{\sqrt{F-E}} dE \tag{1.57}$$

We found an explicit expression $x(F) = U^{-1}(F)$ for the inverse function of U. This describes $U(x)$ completely, and solves the problem.

Proof of the identity

$$\int_u^F \frac{dE}{\sqrt{(F-E)(E-u)}} = \pi : \tag{1.58}$$

a linear substitution reduces the integral to $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} 1 - \sin^{-1}(-1) = \pi$.

Exercise 1.4. A quadratic potential $U = \frac{1}{2}x^2$ is *isochronous* in the sense that all the motions have the same period, namely 2π. Show that the solution (1.57) implies the converse: if a potential is isochronous of period 2π then it is of the form $U = \frac{1}{2}x^2$ (assuming, as we did above, that $U(0) = U'(0) = 0$ and that U is even and convex up).

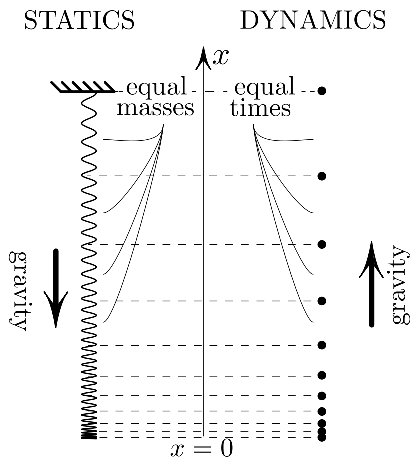


Figure 25. The statics-dynamics equivalence illustrated: hanging spring is mathematically identical to the falling projectile if we reverse the direction of gravity for the projectile.

19. A dynamics-statics equivalence

A simple and yet remarkable fact is that the entire Newtonian dynamics described by $m\ddot{x} = -U'(x)$ is mathematically equivalent to *statics* of Hookean springs. Figure 25 illustrates this equivalence on a simple example. On the left, a "slinky" — a heavy Hookean spring (defined precisely in the next paragraph) — hangs motionless by one end; we choose the origin $x = 0$ to be at the free end of the spring. On the right, the picture of a falling particle is drawn *upside-down*; the mass begins to fall from $x = 0$ starting from rest. These two systems are not just analogous but are in fact mathematically *equivalent*, as explained in this section and as summarized by the dictionary on page 45.

The spring. Figure 26 shows an idealized Hookean spring, viewed as a one-dimensional object laid out on the x-axis (the spirals are only drawn to show the nonuniform stretching and not to suggest any thickness). It is natural to parametrize the particles of the spring by the mass t counted from one of the spring's ends; let $x(t)$ denote the position of the corresponding particle of the spring. At this point

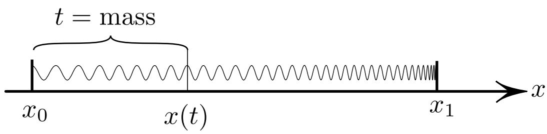


Figure 26. Position $x(t)$ of each particle on the spring is parametrized by the mass t between the particle and the attachment point.

in discussion, $x(t)$ is an arbitrary function, i.e., each particle is held forcibly in its own prescribed position. We can call x a "configuration function" of the spring. The spring is assumed to satisfy Hooke's law; furthermore, the relaxed length of the spring is assumed to be zero.[17]

Theorem 1.7. *Consider a Hookean zero length spring on the x-axis, with a configuration function $x(t)$, $0 \leq t \leq T$, as described in the preceding paragraph. Denote Hooke's constant of a unit mass of a spring by $k_1 = m$. Let $V(x)$ be a potential on the line (that is, $V(x)$ is the potential energy of a unit mass at x). Then the total potential energy of the spring is*

$$E_{\text{total}} = \int_0^T \left(\frac{m\dot{x}}{2} + V(x) \right) dt. \tag{1.59}$$

Proof. Consider a piece of the spring of mass dt. Hooke's constant of this piece is $k_{dt} = k_1/dt = m/dt$ (a shorter spring is stiffer, see Problem 1.1 on page 50). Therefore, the potential energy stored in stretching this piece is

$$dE_{\text{internal}} = \frac{1}{2} k_{dt} (\Delta x)^2 = \frac{m\dot{x}^2}{2} dt,$$

ignoring higher order terms. This proves (1.59) once we also observe that $V(x)\,dt$ is the potential energy of mass dt, since V was defined as the potential energy of a unit mass. ♢

[17]Zero length should not be a concern since this is only a thought experiment; however, a real slinky is not too far from this idealization: its relaxed length is very short compared to its "operating" length.

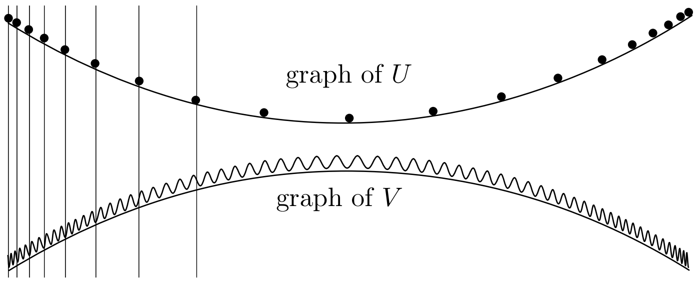


Figure 27. Equal times between positions of the particle in potential U and equal masses for the spring in potential $-U$. The equilibrium state of the spring captures the whole time history of the moving particle.

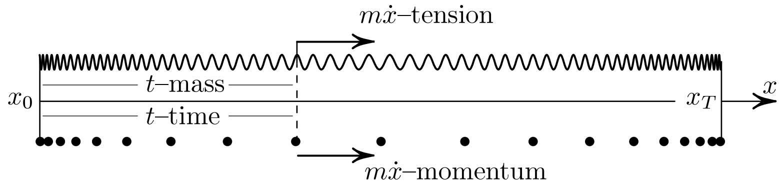


Figure 28. Equivalence between a particle in motion and a spring in equilibrium.

Now the expression (1.59) coincides with

$$\int_0^T \left(\frac{m\dot{x}^2}{2} - U(x) \right) dt,$$

the action of a particle, provided we choose $V = -U$. The two problems are therefore equivalent: the particle moving in the potential U and the spring resting in equilibrium in the potential $V = -U$. The equivalence means that the same critical function $x(t)$ describes the motion of a particle in the potential U and the static equilibrium of the spring in the potential $-U$.

Example. Figure 27 illustrates the equivalence. Note how the stretching of the spring is related to the speed of the particle. Actually, both the spring and the particle are confined to the x-axis rather than to the graphs of the potentials, as represented by the more accurate (but less intuitive) Figure 28. The following table summarizes the equivalence between the particle and the spring; a more detailed explanation is given after the table.

Dynamics of a particle	**Statics of a spring**
t – time	t – mass
m – mass	m – Hooke's constant
$x(t)$ – position of the particle at time t	$x(t)$ – position of the particle corresponding to mass parameter t
$\dot{x}$ – velocity	$\dot{x} = (dt/dx)^{-1} = (\text{density})^{-1}$
$m\dot{x}$ – momentum	$m\dot{x}$ – tension
$\frac{m\dot{x}^2}{2}$ – kinetic energy	$\frac{m\dot{x}^2}{2}\,dt$ "internal", or stretching potential energy of a mass element dt
$U(x)$ – potential energy	$V(x) = -U(x)$ potential energy of a unit mass
$K + U = \text{const.}$	$K - V = \text{const.}$
Action $\int (K - U)$	Potential energy $\int (K + V)$
$\delta \int (K - U) = 0$: $x(t)$ is a solution	$\delta \int (K + V) = 0$: $x(t)$ is an equilibrium
$\delta \int (K - U) = 0 \Leftrightarrow \ddot{x} = -U'(x)$ via Euler–Lagrange's argument	$\delta \int (K + V) = 0 \Leftrightarrow \ddot{x} = V'(x)$ by an elementary equilibrium argument (1.60)
Liouville's theorem $A(0) = A(T)$, i.e., $\int_{\varphi^T\gamma} p\,dx = \int_\gamma p\,dx$ for a closed curve γ	Work done on moving spring's ends in a cyclic fashion is zero: $-\int p_0 dx_0 + \int p_T\,dx_T = 0$

The use of this equivalence is two-fold. First, we get two physical systems analyzed for the price of one, or, putting it differently, we realize that two different physical systems are mathematically identical; and second, this equivalence gives new insights including elementary derivations of the Euler–Lagrange equation and of Liouville's theorem, as described in the next two paragraphs.

An elementary derivation of the Euler–Lagrange equation as a static equilibrium condition. Earlier we established the equivalence

$$\delta \int (K - U) = 0 \Leftrightarrow m\ddot{x} = -U'(x),$$

by showing that the right-hand side is the Euler–Lagrange equation for the functional on the left. The result is nonobvious intuitively as it relies on a formal calculation. But our analogy makes this equivalence obvious, without appealing to the Euler–Lagrange equation. Indeed, let us interpret the action $\int (K - U)\,dt$ for a particle as the potential energy $\int (K + (-U))\,dt$ of a spring in the potential $V = -U$. Then critical action for the particle motion means critical potential energy for the spring. That is, the spring is in equilibrium, and so the sum of all forces on each mass element dt is zero:

$$m\dot{x}(t + dt) + (-m\dot{x}(t)) + (-V'dt) = 0. \tag{1.60}$$

Dividing by dt and taking $dt \to 0$ gives $\ddot{x} = V'(x) = -U'(x)$. We rederived the Euler–Lagrange equation by naive means, in the special case of the Lagrangian $L = K - U$.

A quick proof of Liouville's theorem. The theorem can be restated as

$$\oint_{\gamma_0} pdx = \oint_{\gamma_T} pdx, \tag{1.61}$$

where γ_0 is a closed curve of initial data $(x, m\dot{x} = p)_{t=0}$ and γ_T is the curve formed by $(x, m\dot{x} = p)_{t=T}$ where $x = x(t)$ is a solution of $m\ddot{x} = -U'(x)$. Let us interpret the two integrals in (1.61) in terms of the spring. With the endpoints of the spring held at x_0, x_T, the external forces applied to the ends are equal to the tensions at these locations: $F_0 = -p_0 = -m\dot{x}(0)$ and $F_T = p_T = m\dot{x}(T)$, Figure 29. Let us now move the ends of the spring in a cyclic fashion (slowly, so as not to cause oscillations), bringing them back to the initial location. The net work we do in that case is zero:

$$\oint F_0 dx_0 + \oint F_T dx_T = 0,$$

and since $F_0 = -p_0$, $F_T = p_T$, we get (1.61)! This hand-waving argument takes almost no work (with apologies for two puns) and converts into a rigorous proof; see page 271.

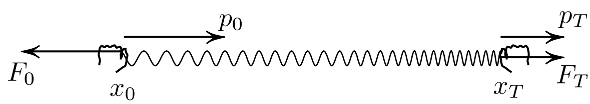


Figure 29. A hand-waving proof of Liouville's theorem: the hands execute a cyclic motion along the line, doing zero work on the spring. In addition to the two forces F_0 and F_T, there is a distributed force on the spring due to the potential V.

Remark 1.9. For any spring in equilibrium in a potential, such as in Figure 25, the difference $K - V$ = const. along the spring. This is surprising and nonobvious, but we know it is true because it is equivalent to $K+U$ = const., the conservation of energy for a particle! In particular, for the hanging spring example in Figure 25, since $V = K = 0$ at the free end, we have $K = V$: the potential energy density, i.e., the energy per unit mass due to stretching equals the potential energy density of elevation!

Exercise 1.5. Consider a Hookean zero length spring described on page 42 in equilibrium, with the points corresponding to t_0, t_1 held fixed at x_0, x_1. Let $S(x_0, x_1, t_0, t_1) = \int_{t_0}^{t_1} (K + V)\, dt$ be the potential energy of the spring; here the equilibrium function $x(t)$ is substituted into the integrand. Find the partial derivatives S_{t_0}, S_{x_0}, S_{t_1}, S_{x_1} and determine their physical interpretation.

Solution. $S_{x_1} = m\dot{x}(t_1)$ is the tension;

$$S_{t_1} = -(K - V)_{t=t_1} \tag{1.62}$$

is the difference between the internal and the external potential energy densities; similarly, $S_{x_0} = -m\dot{x}(t_0)$ and $S_{t_0} = +(K - V)_{t=t_0}$. For the proof, see Theorem 8.1 on page 260; alternatively, here is a naive physical justification. Let $x(t)$ be the equilibrium function with $x(t_0) = x_0$, $x(t_1) = x_1$, Figure 30. Let us find S_{t_1}. With the ends of the spring fixed as described, let us grab the spring at the point $A = x(t_1 - \tau)$ near the right end, Figure 30, and pull this point to the right end $x(t_1)$ and hold it there. In the process the segment of the spring corresponding to $[x_1 - \tau, x_1]$ gets collapsed to a point, while the rest of the spring stretches a little. The new potential energy of the spring is

$$P_{\text{new}} = S(x_0, x_1, t_0, t_1 - \tau) + V(x_1)\tau,$$

the second term being the potential energy of the short segment of the spring which has collapsed to a point. On the other hand, the work we did

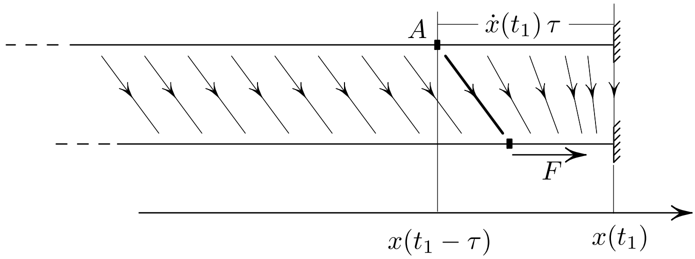


Figure 30. As we pull the point $A = x(t_1 - \tau)$ to the end $x = x_1$, the force F changes linearly with distance, from $F = 0$ to $F = m\dot{x}(t_1) + O(\tau)$.

to move the end of the spring is

$$W = F_{\text{avg}}(x(t_1) - x(t_1 - \tau)) = \frac{1}{2}m\dot{x}(t_1)^2\tau + O(\tau^2).$$

But

$$P_{\text{new}} = P_{\text{old}} + W,$$

i.e.,

$$S(x_0, x_1, t_0, t_1 - \tau) + V(x_1)\tau = S(x_0, x_1, t_0, t_1) + \frac{1}{2}m\dot{x}(t_1)^2\tau + O(\tau^2).$$

Separating the terms with S from the rest, dividing by τ and sending $\tau \to 0$ gives (1.62).[18] ♢

Remark 1.10. The same equivalence between dynamics and statics holds in higher dimension. Figure 31 shows the familiar projectile trajectory side-by-side with a hanging spring; note, however, that the gravity is in the opposite direction for the two cases. The two problems are again equivalent, with the exact same list of analogies as in the table above.

20. Chapter summary

Here are the main points of this chapter.

(1) Newton's equation: $m\ddot{x} = F(x)$.

(2) Potential energy and kinetic energy, defined.

(3) Conservation of energy: $K + U = \frac{1}{2}m\dot{x}^2 + U(x) = \text{const}$.

[18](1.62) is the Hamilton-Jacobi equation (discussed in Chapter 8).

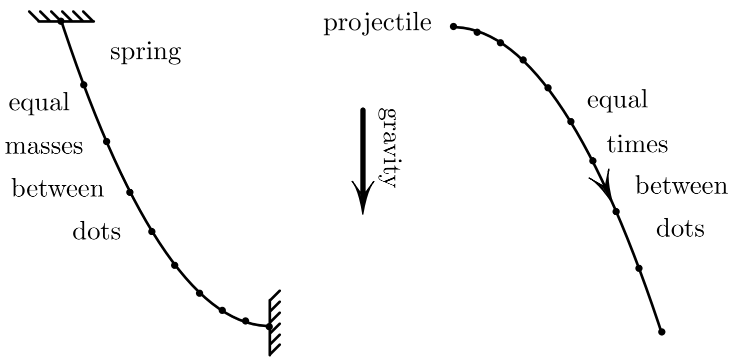


Figure 31. Equivalence between a dynamical problem and a static one in two space dimensions, illustrated on the projectile motion.

(4) The phase plane.

(5) Lagrange's equation: $\frac{d}{dt}L_{\dot{x}} - L_x = 0$, $L = K - U$.

(6) Variational origin of Lagrange's equations: $\frac{d}{dt}L_{\dot{x}} - L_x = 0 \;\Leftrightarrow\; \delta \int_{t_0}^{t_1} L\, dt = 0$.

(7) Recovering the shape of potential from the periods of oscillations.

(8) Hamilton's equations, derived from Lagrange's equation.

(9) Liouville's theorem.

(10) A classical mechanical uncertainty principle.

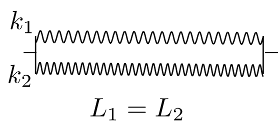
(11) A statics-dynamics equivalence.

(12) Two application of the statics-dynamics equivalence: (i) $\delta \int (K - U) = 0 \;\Leftrightarrow\; m\ddot{x} = -U'(x)$ and (ii) Liouville's theorem.

21. Problems

Hookean Springs.

1.1. Find Hooke's constant of the combination of two Hooke's springs (a) in parallel; (b) in series, given Hooke's constants k_1 and k_2 of the two springs.



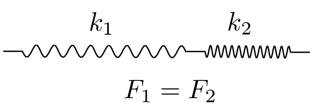


Figure 32. What is the effective Hooke's constant for springs in series and in parallel?

Solution. We show that

$$k_{\text{parallel}} = k_1 + k_2; \quad \frac{1}{k_{\text{series}}} = \frac{1}{k_1} + \frac{1}{k_2}. \tag{1.63}$$

In parallel: The key observation is

$$F = F_1 + F_2, \tag{1.64}$$

where F is the force with which the combined spring was stretched, while F_i is the force with which the ith spring is stretched. By the definition of Hooke's constant, we have $F = k_{\text{parallel}}L$, $F_1 = k_1L$ and $F_2 = k_2L$; note that the elongation L of both springs is the same, the second key point. Substituting this into (1.64) we obtain the first equation in (1.63).

In series: In this case, the elongation is the sum of elongations of the two springs:

$$L = L_1 + L_2, \tag{1.65}$$

Since each spring is stretched by the same force F (the second main point), we have $F = k_iL_i$, $i = 1, 2$. Thus $L_i = k_i/F$, and also $L = k_{\text{series}}/F$. Substituting these into (1.65) and cancelling F we arrive at the second formula in (1.63). ♢

A heuristic explanation of $\frac{1}{k_{\text{series}}} = \frac{1}{k_1} + \frac{1}{k_2}$: Hooke's constant k measures the stiffness: large k, for example, means stiff spring, since k simply measures the force needed to elongate the spring by one unit of length. The reciprocal k^{-1} therefore measures the spring's looseness. Two springs in series makes for a looser spring; the "loosenesses" in fact add, as we proved. By contrast, for the springs are connected in parallel, "stiffnesses" add.

1.2. Produce an equivalence table between the following objects from mechanics on one hand and electricity on the other. Mechanics: Hooke's constant, elongation, force ($k = F/x$). Electricity: Resistance, voltage, current ($R = V/I$) and capacitance, charge, voltage ($C = q/V$). What are the electrical analogues of the formulas (1.63)? Mere substitution of the electric analogs into proofs of (1.63) yields the formulas for resistances and for capacitances connected in parallel and in series:

$$R_{\text{parallel}}^{-1} = R_1^{-1} + R_2^{-1}, \quad R_{\text{series}} = R_1 + R_2;$$
$$C_{\text{parallel}} = C_1 + C_2, \quad C_{\text{series}}^{-1} = C_1^{-1} + C_2^{-1}.$$

A bead on a wire.

1.3. Consider the motion of a particle in a potential: $\ddot{x} = -U'(x)$. Can this equation also describe the arclength parameter of a bead sliding under gravity on an appropriately shaped wire? That is, find the curve $y = V(x)$ such that the arc length parameter s of a bead sliding on this curve under gravity (g = const. pointing down the y-axis) satisfies the same equation: $\ddot{s} = -U'(s)$, and state under what conditions on U this is possible. Find V in the following two cases: (i) $U = \frac{1}{2}x^2$ and (ii) $U = -\cos x$.

Answer. The graph of V is a cycloid in case (i) and a circle in case (ii).

1.4. Figure 33 shows a bead is sliding on a wire given by $y = f(x)$ in the vertical plane subject to gravity pointing down the y-axis. Write the equation of motion for the bead in terms of its x-coordinate, using three different methods:

(1) Using Newton's law $\ddot{s} = -g\sin\theta(s)$ for the arc length $s = \int_0^x \sqrt{1+f'(u)^2}du$.
(2) Using Lagrange's method (i.e., write the Euler–Lagrange equation).
(3) Directly from Newton's second law in $\mathbb{R}^2$. (Hint: for this method, we must consider centripetal acceleration, which has a component in the x-direction.)

1.5. A point mass is glued to the surface of a weightless cylinder which rolls without slipping on the horizontal plane. The point mass is thus traveling in a cycloid. Letting s denote the distance of the particle from the top of its trajectory, measured along the trajectory, show that $\ddot{s} = \frac{g}{D}s$: the motion is the same as for a particle with a linear repelling force. Is this equation valid for all time?

Hint. Regarding the last question, the evolution after the particle hits the floor becomes ambiguous.

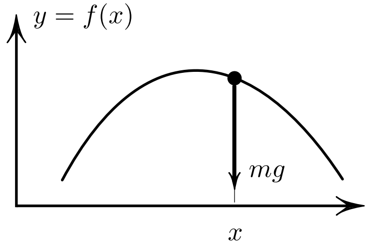


Figure 33. Deriving the equation of motion for a bead on wire with gravity.

1.6. Referring to the bead on the wire in Figure 3(A), page 4, is it true or false that the normal reaction force $N = mg\cos\theta$? That is, does N cancel the normal component of the gravity?

1.7. Consider a bead sliding frictionlessly on the curve $z = f(x)$, in the vertical (x, z)-plane with gravitational field g pointing down in the direction of the negative z-axis. Is it true that the x-coordinate satisfies $\ddot{x} = -gf'(x)$?

1.8. Consider a curve $y = f(x)$ with $f'(0) = 0$. Let s be the distance along the curve from the point with $(0, f(0))$, and let $\theta(s)$ be the angle between the x-axis and the tangent to the curve. Show that

$$\sin\theta(s) = ks + o(s),$$

where $k = f''(0)$ is the curvature of the curve at $x = 0$.[19]

Modeling, hanging chains.

1.9. A heavy homogeneous chain is hanging in equilibrium supported at two ends, Figure 34. Write the differential equation obeyed by the shape $y = f(x)$ of the chain. The chain is to be treated as a perfectly thin, perfectly flexible unstretchable curve with a uniform mass distribution along its length.

1.10. Verify the following remarkable fact: *Regardless of how the chain is suspended (Figure 34), its tension $T = T_0 + \rho gh$*, where T_0 is the tension at the bottom of the chain, and h is the height of the point above the bottom. Prove that the *same holds even if the chain is not freely hanging but rather is resting on a perfectly slippery surface.*

1.11. Referring to the preceding problem, can you explain why the expression $T = T_0 + \rho gh$ is so similar to the expression $p = p_0 + \rho gh$ for the

[19]Here $o(s)$ denotes a quantity which is small compared to s in the sense that $\lim_{s\to 0} o(s)/s = 0$.

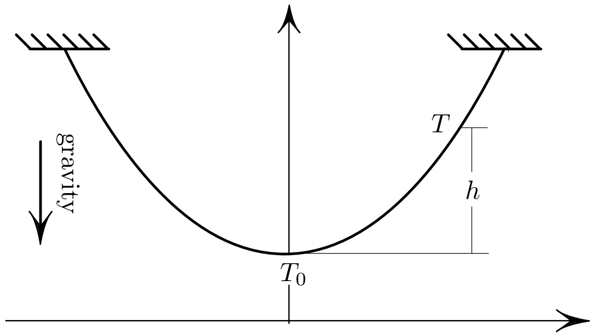


Figure 34. Tension in a hanging chain behaves like hydrostatic pressure: it varies linearly with height!

hydrostatic pressure? (The meaning of ρ in the two expression is different but similar.) Can this similarity be used to solve the preceding problem?

Solution – an outline. Imagine that the chain is a hose filled with water. The hose is perfectly flexible, weightless, unstretchable and very thin, essentially one-dimensional. The hydrostatic pressure of the water in the hose is $p = p_0 + \rho_w gh$, where ρ_w is the density of water. Now the tension of the hose caused by p is $T = pA$, where A is the cross-sectional area of the hose. Thus the tension of the water-filled hose (which in essence is the hanging chain) is $T = T_0 + \rho gh$, where $\rho = \rho_w A$, i.e., the linear density of the hose with water.

1.12. Solve Problem 1.10 by using conservation of energy instead of Newton's first law.

Solution. Focus on a segment AB of the chain; imagine taking up the length ds of the chain at B and feeding in the same length at A. The work done by taking up, i.e., pulling, is $T_B\, ds$; the work involved in feeding in, i.e., in being pulled, is $-T_A\, ds$. The net result is that the mass $dm = \rho g ds$ is lifted by height h from A to B, so that

$$T_B\, ds - T_A\, ds = \rho g\, ds\, h,$$

proving that $T_B - T_A = \rho gh$.

1.13. A thin-walled straight hose of radius r is filled with pressurized gas at pressure p. (i) Find the surface tension of the skin of the hose in the axial direction (i.e., the force required to hold closed a longitudinal slit, per unit length of the slit). (ii) Find the surface tension of the skin in the perpendicular direction.

Solution. Longitudinal tension turns out to be twice the tension across. Indeed, if we cut the hose with a plane perpendicular to the axis, the force required to hold the cut together is $\pi r^2 p$. Dividing by the circumference $2\pi r$, we get the transversal surface tension $\sigma_1 = \pi r^2 p / 2\pi r = rp/2$. To find the longitudinal tension, let us cut a length L of the hose by a plane containing the axis; the force $(2r) \cdot L \cdot p$ required to hold the cut together; divided by the length $2L$ of the cut, this yields the surface tension $rp/2$, half that of the longitudinal tension. This explains why frozen pipes, and boiled sausages, always burst lengthwise.

1.14. Explain the mechanism by which wringing a towel expels water. Estimate the pressure created inside the towel, making some reasonable assumptions (and, in particular, explain what exactly could one mean by pressure in this problem). How does the pitch of a certain helix affect the squeezing efficiency? Why is it harder to squeeze out a thicker roll than a thinner one? Why is wringing much better than squeezing?

1.15. Show that the cable supporting the vertical cables in a suspension bridge is parabolic, assuming the vertical cables are very closely and equally spaced and are under equal tensions, and that the entire weight is contained in the horizontal walkway.

1.16. A long cylindrical hose with perfectly flexible walls is filled with water and is placed on the horizontal surface. Write down a differential equation that describes the shape of the (non-flat) part of the cross-section of the hose.

1.17. A chain hanging in equilibrium is shaped as the graph of a function $y = f(x)$. For any segment of this chain, the horizontal components of tensions on the two ends of the segment are in balance. Express this balance as an equality involving $f(x)$. (The resulting equation is a first integral of the second order differential equation of the catenary.)

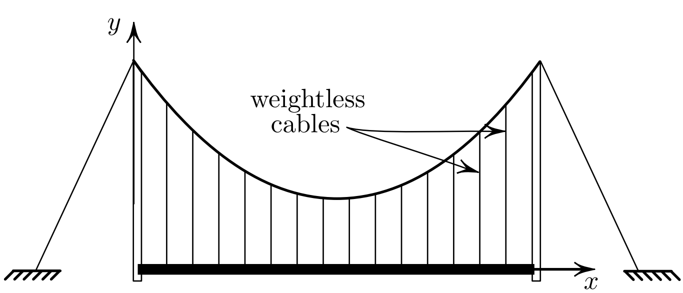


Figure 35. Suspension cable is a parabola.

Huygens's pendulum, evolutes and bike tracks.

1.18. Show that the cycloid is a tautochrone, as follows. Consider a bead sliding without friction under the influence of gravity g on a cycloid generated by a circle of diameter D, as in Figure 4, page 5. Show that $\ddot{s} = -ks$, where s is the arc length measured from the lowest point of the cycloid.

Proof (an outline). Referring to Figure 36, we show that $da = -k\,ds$, where a is the tangential acceleration of the bead. But $ds = D\sin\theta d\theta$, as the figure shows, and $a = g\cos\theta$, so that $da = -g\sin\theta d\theta$. We conclude that $da = -\frac{g}{D}ds$, implying that $a = -ks + \text{const}$. But $a = 0$ when $s = 0$, which shows that $a = -ks$, i.e., $\ddot{s} = -ks$. The period of s therefore does not depend on the amplitude, and in fact is equal to $2\pi/\sqrt{k} = 2\pi\sqrt{D/g}$.

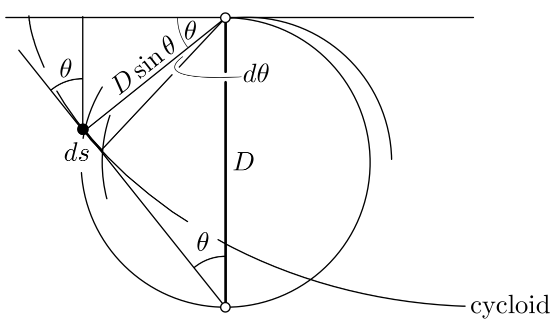


Figure 36. Brachistochrone is a tautochrone: the proof.

1.19. Consider Huygens's pendulum, or, more generally, a string fixed at point C (Figure 5, page 5), and wrapped partly around an obstacle. The free end P of the string is moved so as to keep the string taut. Prove that the velocity of P is perpendicular to the string. This would then show that the *obstacle is the evolute* (i.e., the envelope of the family of normals) of the trajectory of P.

1.20. Prove that the evolute of a curve (defined as the envelope of the family of normals to the given curve) is also the locus of the centers of curvature of the curve; see Figure 37.

Hint. Here is a kinematic argument which can be converted into a formal proof, for a small fee. If a point travels in a circle with speed v, then the radius of the circle is

$$R = v/\omega, \tag{1.66}$$

where v is the speed of P and where ω is the angular velocity of the normal to the trajectory. Now the same definition applies verbatim if the trajectory

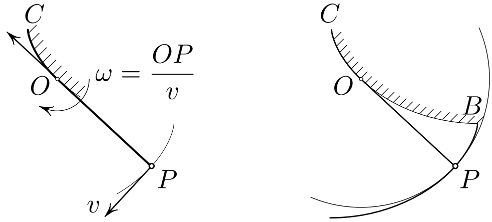


Figure 37. Proving that every point on the evolute is a center of curvature of the evolvent.

is an arbitrary curve. Now what is the angular velocity of the moving segment OP? Velocity of O aligns with OP, i.e., it has zero component normal to OP. Therefore, the *motion of O contributes nothing to the angular velocity of OP*, and we can treat O as fixed to find $\omega = v/OP$, or $OP = v/\omega$. Comparison with (1.66) shows that $R = OP$.

1.21 (Finding center of curvature using a bike)**.** Referring to Figure 38, show that *the center of curvature of the bike's rear track is at the intersection of the lines of the axles*, i.e., of the two normals to the tracks. The bike is idealized: it is a segment RF (for "rear" and "front", the points where the wheels are in contact with the ground); assume that the segment RF moves in the plane so that the length $|RF| = \text{const.}$, and that the velocity of R is aligned with RF.

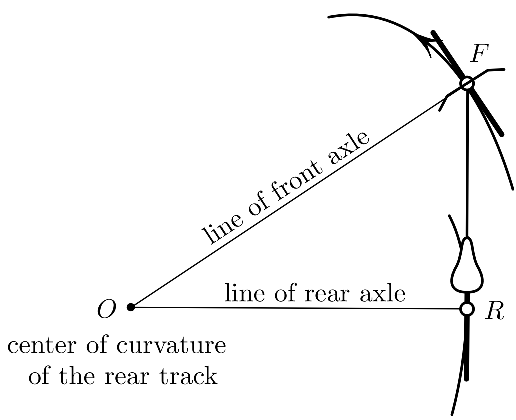


Figure 38. Proving that the point of intersection of the bike's axles is the center of curvature of the rear track.

Vibrations.

1.22. 1. Derive the equations of motion for a point mass m in Figure 39. The spring satisfies Hooke's law: the tension is $k(L - L_0)$, where L_0 is the relaxed length.

2. How many equilibria can the particle have, depending on the relationship between D and L_0?

3. Find the frequency of small vibrations near a stable equilibrium. Assume the amplitude of oscillations to be very small.

4. Show that for the zero length spring (the one for which the relaxed length $L_0 = 0$) the oscillations are harmonic.

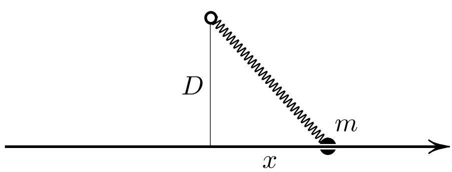


Figure 39. The mass m is constrained to the line, with no friction.

1.23. Consider the equation governing the angle θ of the pendulum:

$$\frac{d^2\theta}{dt^2} + \frac{G}{L}\sin\theta = 0.$$

Show that the introduction of the rescaled time

$$\tau = \sqrt{\frac{g}{L}}t, \tag{1.67}$$

i.e., setting $\varphi(\tau) = \theta(t) = \theta(\sqrt{\frac{g}{L}}\tau)$ turns this equation into

$$\frac{d^2\varphi}{d\tau^2} + \sin\varphi = 0. \tag{1.68}$$

Note that $T = 2\pi\sqrt{\frac{L}{g}}$ is the period of linearized oscillations near the equilibrium, and thus our rescaling (1.67) amounts to measuring time in the more natural units of the period.

1.24. (Based on a Bond movie I saw.) Figure 40 shows idealized James Bond attached by a long rope to the top of the cliff. A villain pushes Bond off the top. Once the rope becomes taut it acts as a spring, softening what would otherwise have been a fatal jerk. Bond survives the maximal tension of the rope (see his strained position at the bottom) and resumes climbing. Show that the *maximal tension in the rope does not depend on* L_0, the

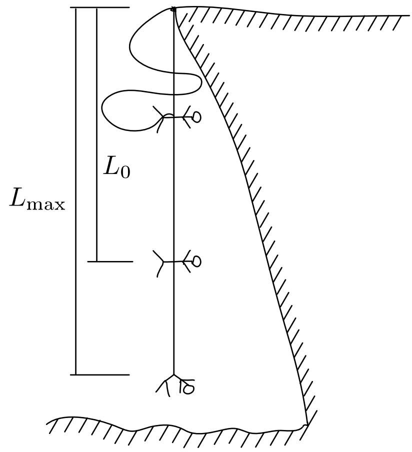


Figure 40. The maximal tension does not depend on the rope's length.

length of the unstretched rope, assuming that the rope satisfies Hooke's law: $F = k(L - L_0)$ for $L > L_0$, where $k > 0$ is a constant. Ignore the air resistance and other possible complications (such as hitting the ground too early, or the villain cutting the rope).

Hint. L_0 affects both the maximal speed of fall and Hooke's constant k. Show that the two effects cancel out.

An inverse problem.

1.25. Given

$$T(E) = \int_0^{E^{1/4}} \frac{dx}{\sqrt{E - x^4}},$$

write $T'(E)$ as an integral. (Note: there is a complication when differentiating with respect to the upper limit of the integral.)

1.26. Find the derivative $T'(E)$ of the period

$$T = \int_{-x_{\max}}^{x_{\max}} \frac{dx}{\sqrt{2(E - U(x))}}, \quad x_{\max} = U^{-1}(E).$$

Assume that $U(0) = U'(0) = 0$, $U'' > 0$ and that U is even. Note: direct differentiation will not work since the integrand is infinite at the endpoints.

1.27. Show that any tautochrone symmetric with respect to a vertical axis is a cycloid. (Recall that the tautochrone is a curve in the vertical plane such that a bead sliding on this curve without friction has the period of oscillations independent of the amplitude.)

Hint. Let s be the arc length measured from the lowest point of the curve. Use Exercise 1.4 (page 41) to conclude that for a particle sliding on a tautochrone one has $\ddot{s} = -ks$ (for some $k =$ const.). Then show that the latter relation implies that the curve is a cycloid.

Hamiltonian systems.

1.28. Show that the Hamiltonian of a planar Hamiltonian system has the following interpretation: *for any two points A and B in the plane, $H(B) - H(A)$ is the flux of the Hamiltonian vector field across any curve connecting A and B.*

1.29. Let $\mathbf{v}(\mathbf{z})$ be a smooth vector field defined in the entire plane $\mathbb{R}^2$. Prove that if div $\mathbf{v} = 0$ then $\mathbf{v}$ is a Hamiltonian vector field.

Hint. Written backwards: .melborp suoiverp eht esU

1.30. Find the Hamiltonians of these systems:

$$(1.69) \qquad (1): \begin{cases} \dot{x} = \frac{x}{x^2+y^2}, \\ \dot{y} = \frac{y}{x^2+y^2}, \end{cases} \qquad (2): \begin{cases} \dot{x} = \frac{y}{x^2+y^2}, \\ \dot{y} = \frac{-x}{x^2+y^2} \end{cases}$$

One of these Hamiltonians is multiple-valued. Can you explain this using the interpretation of H as flux given by Problem 1.28?

Answer. (1) $H = \arg(x + iy)$ is multiple-valued; the flow corresponds to a point source at the origin. Geometrically, the flux through a curve connecting two points A and B picks up 2π each time the curve winds around the origin. (2) corresponds to a point vortex; $H = \ln|z|$ is single-valued.

1.31. Show that any planar vector field can be converted into a Hamiltonian vector field by adjusting the speeds, but keeping the directions at every point. In other words, show that any ODE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in $\mathbb{R}^2$ can be converted into a Hamiltonian system $\dot{\mathbf{x}} = \rho(\mathbf{x})\mathbf{f}(\mathbf{x})$ by an appropriate choice of $\rho : \mathbb{R}^2 \to \mathbb{R}$.

1.32 (Cows and Hamiltonian systems)**.** Consider a herd of grazing cows on a hill; we will treat this herd as a collection of particles on a surface $z = H(x, y)$. Assume that each cow avoids going up the hill — too much work — or down the hill — too hard on the front legs and too hard to graze at an angle to the horizontal (intelligence can be mistaken for laziness). In short, our cows abhor gradients. This leaves the cow with only one option: to follow a level curve on the topographic map. Assume also that the cows dislike gradients in one more way: they walk faster where the hill is steep — specifically, with speed equal to the slope of the hill at the cow's location.

(1) Show that under these assumption the (x, y)-position of each cow on the topographic map satisfies the Hamiltonian system where the Hamiltonian is the height $H(x, y)$ of the hill at (x, y). Assume that H is smooth.

(2) How does the herd's density change with time? The density is measured by the proportion of the total number of cows per unit area in the (x, y)-plane *(rather than per unit surface area of the hill).*

Answer. Each cow follows a trajectory of the Hamiltonian system $\dot{x} = H_y(x, y)$, $\dot{y} = -H_x(x, y)$, and the density remains constant along the trajectories, i.e., each cow feels equally crowded at all times: if $\rho(\mathbf{x}, t)$ is the herd's density function at time t, then $\frac{d}{dt}\rho(\mathbf{x}(t), t) = \nabla\rho \cdot \dot{x} + \rho_t = 0$.

1.33. Sketch the phase portrait for the particle in the potentials: (i) $U(x) = x - \frac{x^3}{3}$, (ii) $U(x) = x^4/4 - x^2/2$, (iii) $U(x) = -x^4/4 + x^2/2$.

Can a particle have a variable mass without exchanging matter with the outside world? The following problem gives a way to realize such a particle.

1.34. Figure 41 shows a dumbbell, free to slide along a given rigid curve. The sleeve around the curve keeps the dumbbell always perpendicular to the curve, and slides without friction. Write down the differential equation for the arclength parameter s of the sleeve. All the mass is concentrated in the two balls, each of mass $m/2$, and the length of each arm is taken to be 1 (one can always achieve this by a choice of units of length).

Solution. The velocities of the two balls are $v \pm kv$, where v is the speed of the sleeve and where $k = k(s)$ is the curvature of the track. The kinetic energy of the system is therefore

$$\frac{m/2(v + kv)^2)}{2} + \frac{m/2(v - kv)^2)}{2} = \frac{m(1 + k^2)v^2}{2}.$$

The interesting conclusion is that *the apparent mass of the sleeve is variable*: $M = m(1+k^2(s))$, i.e., it depends on the location s! Since the kinetic energy is constant (there are no external forces acting on our system):

$$\frac{m(1 + k^2)v^2}{2} = E = \text{const.},$$

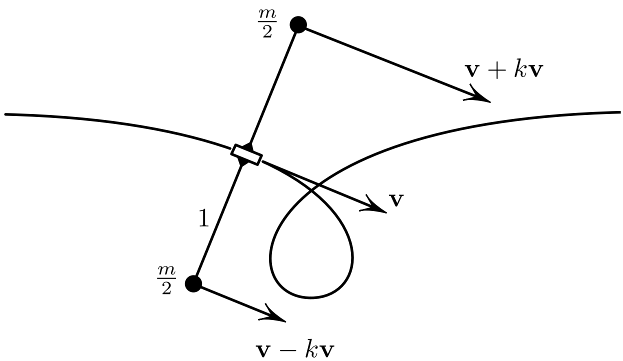


Figure 41. This dumbbell behaves as if the sleeve were a particle with variable mass! In addition, it behaves as if it were subject to a potential force, despite the absence of external forces.

we obtain $v = v(s)$ as a function of position s:

$$v = \frac{\sqrt{2E}}{\sqrt{1+k^2}} = \frac{v_0}{\sqrt{1+k^2}}, \quad \text{where} \quad v_0 = \sqrt{2E}. \tag{1.70}$$

Note that v_0 is the speed the sleeve would have on the straight section of the track. Since v changes along the track, it appears as if a tangential force were acting on the sleeve. What is the magnitude of this force? Here are the answers: differentiating v by time we obtain the acceleration

$$\ddot{s} = \dot{v} = \frac{d}{dt}\frac{v_0}{\sqrt{1+k^2}} = -v_0\frac{kk'\dot{s}}{(1+k^2)^{3/2}} \overset{(1.70)}{=} -v_0^2\frac{kk'}{(1+k^2)^2}; \tag{1.71}$$

by substituting $v_0 = v\sqrt{1+k^2}$ in the last term we finally get

$$a = \dot{v} = -v^2kk'. \tag{1.72}$$

Can this apparent force reverse the direction of motion of the sleeve? No, since $v \neq 0$, according to (1.70). Interestingly, kv^2 in (1.72) is the centripetal acceleration. We conclude: the sleeve's tangential acceleration equals its centripetal acceleration times k'.

Equilibrium; stability.

1.35. Find the tensions of each segment of the cable in Figure 42, given the angles α and β.

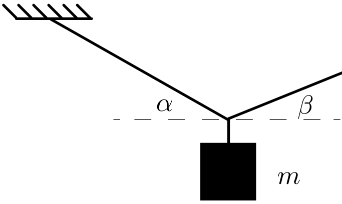


Figure 42. Find the tension of the cable.

1.36. Figure 43 shows an asymmetric dumbbell whose masses m_1 and m_2 rest on the legs of the right triangle. Find the tension of the rod and the angle α, given m_1, m_2 and θ. There is no friction; the rod is weightless.

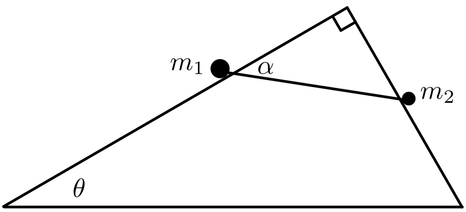


Figure 43. Towards Problem 1.36

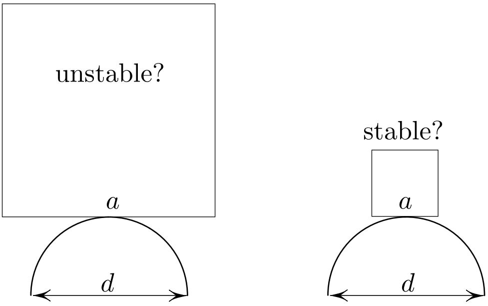


Figure 44. What size cube will be in stable balance?

1.37. Figure 44 shows a cube with side of length a resting on the top of a sphere so that the base of the cube is horizontal. There is no sliding between the cube and the sphere. Under what condition on a and d is the equilibrium stable?

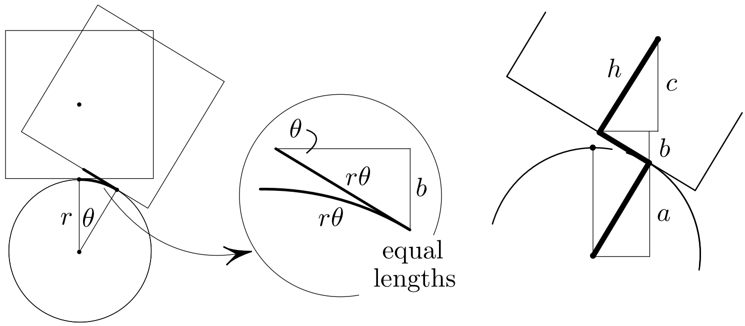


Figure 45. Towards the solution of Problem 1.37.

Solution. I present two methods: one very quick, the other general.

1. A quick method. Imagine rolling the cube past the top equilibrium as in Figure 45. Both the center of mass and the contact point move; *stability will happen if the contact moves faster than the center of mass* at the moment when the contact point is on top of the cylinder. Let ω be the angular velocity of the cube at that moment. Since the cube rotates instantaneously around the contact point, the center of mass moves with speed $\omega\frac{a}{2}$ (when the cube is on top). The contact point, on the other hand, moves with the speed $\omega\frac{d}{2}$, since the radius of the contact point rotates with the same angular velocity ω as the cube. So the contact point moves faster iff (i.e., if and only if) $\omega\frac{d}{2} > \omega\frac{a}{2}$, i.e., iff

$$d > a.$$

This is a beautiful answer: The cube is stable if and only if it does not overhang the sphere.

2. A general method. Let us roll the cube through an angle θ, Figure 45, and find the resulting potential energy $U(\theta)$ of the cube. The equilibrium corresponding to $\theta = 0$ is stable if $U(0)$ is a local minimum of U, i.e., if $U''(0) > 0$ .As the figure shows, the potential energy of the cube is, with the center of the cylinder chosen as ground level,

$$U(\theta) = mg(a + b + c) = mg(r\cos\theta + r\theta\sin\theta + h\cos\theta),$$

where $r = d/2$, $h = a/2$. Simple algebra shows that $U''(0) > 0$ iff $d > a$. ♢

Small vibrations.

1.38. A dumbbell balances on the cylinder in the horizontal position; see Figure 46. Find the frequency of small oscillations of the dumbbell, given the length L of the dumbbell and the radius R of the cylinder. The contact is nonsliding.

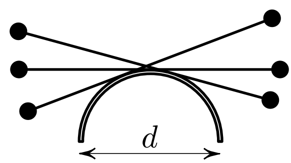


Figure 46. What is the frequency of small oscillations? Problems 1.38 and 1.39.

1.39. A dumbbell balances on a convex object, not necessarily circular, as in Figure 46, in the horizontal position. Find the frequency of small oscillations near the equilibrium, given the curvature k of the object at the topmost point and the length L of the dumbbell.

1.40. Figure 47 shows a rod with a point mass m at the end of it, attached to the cylinder; the cylinder rolls without slipping on the horizontal plane. The radius of the cylinder is R, its mass is M, and the length of the rod is L.

(1) Write the Lagrangian in terms of the angle θ.

(2) Find the period of small oscillations when $R \neq L$. Describe small oscillations in the case of $R = L$. Is this a realistic problem?

Hint. Here is a nice shortcut: the system is undergoing instantaneous rotation around C; this should allow for a quick expression of the kinetic energy.

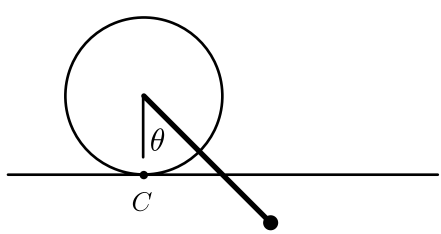


Figure 47. A rolling pendulum, Problem 1.40.

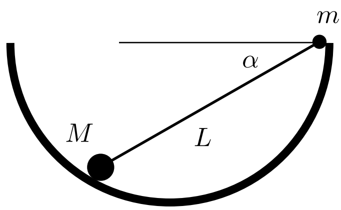


Figure 48. What is the relationship between m, M, L and α? (Problem 1.42).

1.41. Find the approximate period of small-amplitude oscillations of the pendulum (1.6), page 4.

Solution. For small θ, we have $\sin\theta = \theta + O(\theta^3)$, and we replace $\sin\theta$ with θ, obtaining an approximating equation $\ddot{\theta} + (g/R)\theta = 0$. All solutions of this equation have period $2\pi\sqrt{R/g}$.

1.42. A dumbbell in Figure 48 with masses m and M is resting in the hemispherical bowl as shown: the smaller mass is at the height of the rim, and the dumbbell forms angle α with the horizontal. (i) Find the relationship between m, M, L and α. (ii) Find the period of small vibrations near an equilibrium (assuming the bowl continues a little above the smaller mass).

Momentum, Energy.

1.43. A string of bullets strikes a box of sand resting on a frictionless horizontal plane, Figure 49. Each bullet stays buried in the sand. Find the velocity of the block after the nth impact. All the bullets have the same masses m and the same velocities v. The mass of the box is M.

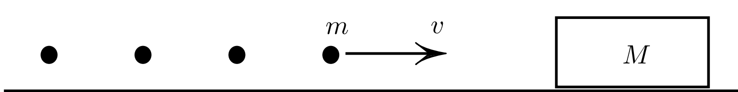


Figure 49. For Problem 1.43.

1.44. A bullet strikes a bag of sand hanging on a rope, causing the bag to swing. The bag deflects by maximal angle θ. Find the bullet's velocity, given that the masses of the bullet and of the bag are m, M, respectively, and that the length of the rope is L.

1.45. A wheeled platform of mass m is rolling without friction. Heavy rain is coming vertically down, with water constantly spilling over the edge of the platform. Does the speed of the platform change, and if so, how does it depend on time? The area of the platform is A; the intensity of the rain ρ (mass of water falling per second on a square meter of surface) is given.

Hint. The platform slows down exponentially since every second it sheds a fixed proportion of its linear momentum through the water that pours off of the platform.

1.46. A car driving in the rain gives kinetic energy to raindrops it hits. Estimate the extra power a car must expend due to this effect, given the area A of the car's frontal profile, the car's speed v, and the fact that rain is coming down at $1cm$ per hour.

1.47. A piece of bird dropping splatters against the windshield of a fast–moving car. As the result, some energy goes into heat, and some goes into the kinetic energy of the motion relative to the ground (there is also energy going into sound, etc., but let us ignore it). What is the ratio of these two energies?

Answer. The ratio equals 1 (under the simplifying assumptions stated in the problem).

1.48. A chain is lying in a pile on the ground; I am pulling one end of the chain up with a constant speed v. Would it take any force to maintain this speed if there was no gravity? In other words, does Newton's first law ("zero acceleration requires zero force") apply — why or why not? If not, what is this force? Linear density (mass per unit length) of the chain is ρ.

The preceding problem hides an interesting paradox which the next problem is asking to resolve.

1.49. Explain the following paradox. Consider the problem of the force required to pull the end of the chain lying on the floor with constant speed v, as described in Problem 1.48. Gravity is to be ignored. On the one hand, an element of mass Δm of the chain accelerates from rest to speed v in time Δt; its average acceleration is thus $a = v/\Delta t$. The average force is then $F = \Delta m a = \Delta m v/\Delta t = \rho v$, according to Newton's second law. On the other hand, the energy I am giving to the chain in time t is $mv^2/2$, where $m = \rho v t$ is the mass of the leaving the floor in time t. The power I am expending is the work I do per unit time:

$$P = \frac{mv^2/2}{t} = \rho v^3/2.$$

Thus the force I am applying is $F = P/v = \rho v^2/2$. This is half of the preceding answer. Which argument hides a mistake? What is the physical significance of the difference between the two results?

Answer. Both answers are wrong. The first answer makes a false presumption that all energy is spent on lifting; in fact, some energy ends up in the form of waves in the chain, for instance. The second answer overlooks a (surprising) possibility that the chain, when pulled up, may push against the floor. Indeed, consider, for instance, the chain made of rods linked by hinges. Consider one such rod/link lying on the ground (gravity plays no role). As we start pulling one end up, the other end will press down on the floor! The true answer to the problem lies somewhere between the given extremes, and depends on the specifics of the problem.

The following problem may seem like a fluid dynamics problem, and although technically it is, it is essentially a one-dimensional mechanics problem.

1.50. I am ejecting water from a syringe, moving the piston at a constant speed v. Water is perfectly nonviscous, and the piston is perfectly frictionless. What force, if any, must I apply to the piston? Does Newton's first law imply that this force is zero? If not, find that force, given the ratio of the diameters of the piston and of the exit hole.

The next problem shows an interesting fact: To launch a satellite to infinity takes exactly twice the amount of work of lifting it from the center of the planet to the surface, assuming that the planet has a constant density. Recall that the gravitational force inside such a homogeneous ball is a linear function of the distance to the center, Figure 50.

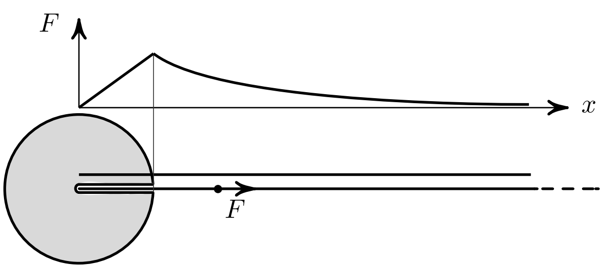


Figure 50. How much work does it take to lift a mass from the center of the asteroid to infinity? The work equals the area under the graph.

1.51. An asteroid is a perfectly round solid homogeneous sphere, Figure 50. A tunnel is drilled to the center of the asteroid, and a sample is lifted from the center first to the surface and then from the surface to infinity. What is the ratio of energies spent on the two stages? In other words, how much more (or less) work does it take to lift from the surface to infinity than from the center to the surface?

Answer. Interestingly, the second stage takes twice the work of the first. Does a similar result hold for nonconstant densities?

Miscellaneous Problems.

1.52. Consider a potential $V(x) \geq 0$ on the line $\mathbb{R}$, Figure 51, with $V(x) \to 0$ as $x \to \infty$, and with a maximum $V(x_{\max}) = H > 0$.

(1) What speeds at $x = -\infty$ would enable the particle to pass over the hill?

(2) One particle $x_1(t)$ moves freely: $x_1(t) = vt$; the other particle $x_2(t)$ moves in the potential V. Find the lag suffered by the second particle $x_2(t)$, i.e., find $\lim_{t\to\infty}(x_1(t) - x_2(t))$, given that the two have the same initial data at $t = -\infty$, i.e., that $\lim_{t\to-\infty}(vt - x_2(t)) = \lim_{t\to-\infty}(v - \dot{x}_2(t)) = 0$.

(3) Can one recover the shape of V by shooting the particle at different speeds and measuring the travel time?

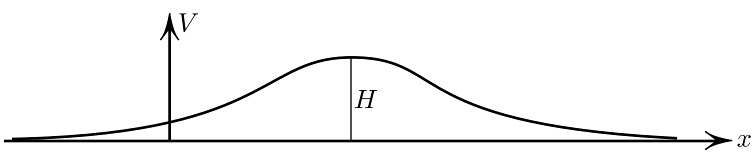


Figure 51. By how much distance does the hump delay particles? (refer to Problem 1.52).

1.53. A 4th degree polynomial $U(x) = x^4 + ax^3 + bx^2 + cx + d$ has two distinct minima. Prove that the periods of any two oscillations of a particle in this potential with the same energy are equal, Figure 52.

Hint. (*This solution assumes some knowledge of the theory of functions of complex variables.*) Consider two oscillations in the different wells with the same energy E, so that $U(x) < E$ on two intervals (x_1, x_2) and (x_3, x_4), and $U(x_i) = E$ for $i = 1, 2, 3, 4$, Figure 52. Kinetic energy $K(x) = E - U(x)$ is also a polynomial of 4th degree, with the roots at x_i. The problem amounts to proving that

(1.73)
$$\int_{x_1}^{x_2} \frac{dx}{\sqrt{K(x)}} = \int_{x_3}^{x_4} \frac{dx}{\sqrt{K(x)}},$$

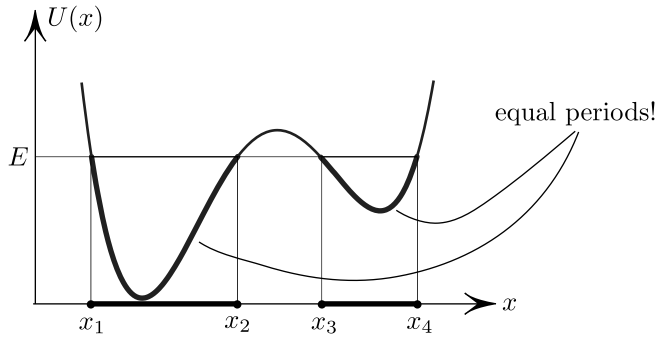


Figure 52. In a quartic potential, equality of energies implies equality of periods.

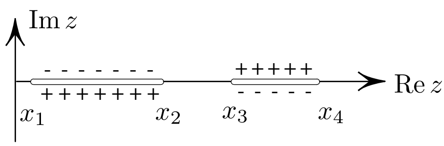


Figure 53. Signs of the chosen branch of $\sqrt{K(z)}$ on the slits' edges.

where

$$K(x) = (x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

To prove (1.73), let us allow x to be complex, denoting it now by $z = x + iy$, and consider $\sqrt{K(z)}$ for complex values of z. Now K is a multiple-valued function because of the square root: Indeed, if z executes a loop around one root, say, x_1, then *arg* K changes by 2π, so that *arg* $\sqrt{K}$ changes by π, i.e. $\sqrt{K}$ changes sign. But if the loop encloses *two* roots then $\sqrt{K}$ does not change as z executes a round trip around the loop. So we cut the slits as in Figure 53 and forbid z to cross them. We thus turn $\sqrt{K(z)}$ into a single-valued function, provided we choose a particular value of the root at some fixed point and then extend to the entire plane minus the slits by continuity. In other words, $\sqrt{K(z)}$ has a single-valued branch in the plane with slits removed. Let us choose the positive sign on the upper edge of the slit $[x_3, x_4]$ as in Figure 53; the signs of our branch of $\sqrt{K}$ on the other edges of slits are determined automatically and are shown in Figure 53.

Consider now two loops A and B as in Figure 54. Loop A can be deformed into loop B by going through infinity (note that $z = \infty$ is a regular point of $\frac{1}{\sqrt{K(z)}}$). This can be seen by mapping the plane to the sphere by stereographic projection and then deforming the loops on the

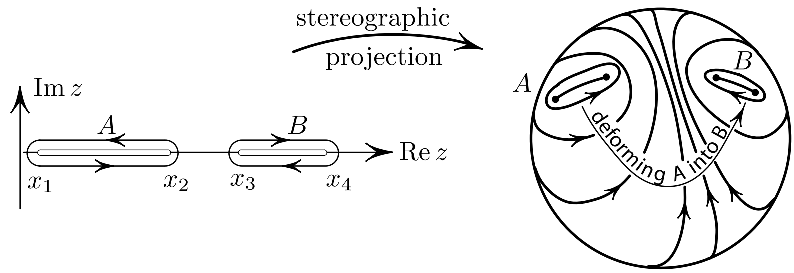


Figure 54. In a quartic potential, equal energies mean equal periods.

sphere as shown in Figure 54. The integral does not change as the contour of integration is deformed, and we conclude that

$$\int_A \frac{dx}{\sqrt{K(x)}} = \int_B \frac{dx}{\sqrt{K(x)}}.$$

But this already proves our claim (1.73), since

$$\int_A \frac{dx}{\sqrt{K(x)}} = 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{K(x)}}, \quad \int_B \frac{dx}{\sqrt{K(x)}} = 2\int_{x_3}^{x_4} \frac{dx}{\sqrt{K(x)}},$$

as is clear from the sign patterns in Figure 53.

The following is a generalization of the preceding problem:

1.54. What is the analogous statement for the cubic potential with a well? What is the relationship between periods of oscillations with the same energy in a polynomial potential of degree $n > 4$?

1.55. Consider the motion of a particle of mass m in a potential $U(x)$ in the presence of linear drag force directly proportional to the particle's speed. Write down the ODE governing the motion of the particle.

Solution. The resultant force on the particle is $F = F_{\text{drag}} + F_{\text{potential}} = -k\dot{x} - U'(x)$. Newton's law then gives

$$m\ddot{x} + k\dot{x} + U'(x) = 0. \tag{1.74}$$

1.56. Prove that no periodic motions exist for the system with drag governed by (1.74) with $k \neq 0$.

Solution. *Method 1.* Assume the contrary: There exists a nonconstant periodic solution, i.e., a solution for which $(x(T), \dot{x}(T)) = (x(0), \dot{x}(0))$ with some $T > 0$. By showing that the energy $E(t) = \frac{\dot{x}^2}{2} + U(x)$ decreases: $E(T) < E(0)$, we will arrive at a contradiction. We have

$$\frac{d}{dt}E(t) = \frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + U(x)\right) = \dot{x}(m\ddot{x} + U'(x)) \overset{(1.74)}{=} -k\dot{x}^2.$$

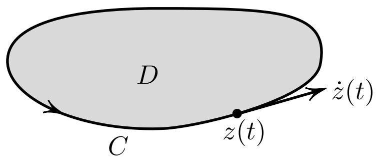


Figure 55. Area enclosed by a periodic orbit must remain constant as it flows; negative divergence implies that this area must decrease, leading to a contradiction.

Now $\dot{x} \neq 0$, with possible exception of isolated values of t, since x is a nonequilibrium solution. Therefore, $E(T) < E(0)$. ♢

Method 2. Assume the contrary: For some $T > 0$ we have $x(t) = x(t + T)$ for all t, Figure 55. Then the phase point $(x(t), \dot{x}(t))$ describes a closed curve C in the $(x, \dot{x})$-plane, Figure 55. Now on the one hand, the region enclosed by the trajectory is invariant under the flow, Figure 55. But on the other hand, the area of the region must decrease when carried by the flow since the divergence of the vector field is negative. Indeed, we have for the area A enclosed by C (see (1.51) on page 35)),

$$A'(t) = \int_D \operatorname{div} \mathbf{f}\, d\mathbf{x},$$

where D is the region enclosed by C. But this is a contradiction with $A = \text{const.}$, since

$$\operatorname{div} \mathbf{f} = \operatorname{div}(y, -ky - U'(x)) = -k < 0.$$

The following problem deals with a time-dependent potential.

1.57. Consider a conservative time-dependent (also called nonautonomous) system

$$\ddot{x} + U_x(x, t) = 0, \quad U_x \equiv \frac{\partial}{\partial x} U(t, x). \tag{1.75}$$

There is no frictional force here. Is the energy conserved?

1.58. The pivot of the pendulum of length L is oscillating in the vertical direction with acceleration $a(t)$. Explain why the ODE for the angle θ with the downward vertical is $L\ddot{\theta} + (g + a(t)) \sin\theta = 0$.

Does the energy conservation go hand-in-hand with the area preservation? The following problem illustrates the answer.

1.59. Consider the motion of a particle in a time-dependent potential: $\ddot{x} + U_x(t, x) = 0$ (the pendulum in the preceding problem is an example). Consider the energy $E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + U(t, x)$.

(1) Is the energy conserved during the particle's motion?

(2) Is the phase flow associated with the vector field

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -U_x(t, x). \end{cases} \tag{1.76}$$

area-preserving? In other words, does Liouville's theorem apply to time-dependent ODEs?

1.60. What are the accelerations of the mass M in Figure 56 (the pulleys are massless)? One of the answers may surprise you.

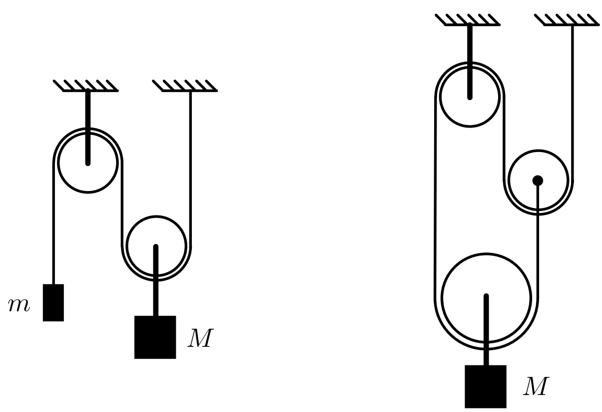


Figure 56. Some pulley problems.

1.61. 1. What force is required to hold the rope in Figure 57(A)? The shaded pulley's mass is M; the other pulleys are massless.

2. What force is required to hold the rope in Figure 57(B)?

1.62. Given that the system in Figure 57(C) is in equilibrium, what is the ratio of masses M/m?

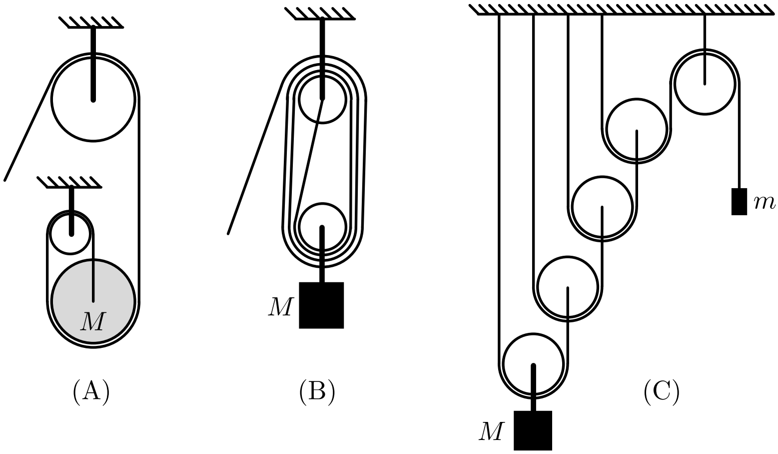


Figure 57. More pulley problems.

1.63. Figure 58 shows two identical monkeys on the rope thrown over a perfectly frictionless pulley. Initially, the two animals are at rest. Then the right one starts climbing. Describe the relative position of the two monkeys. Suddenly, one of the monkeys lets go of the rope. What happens to their relative position then?

Answer. (Written backwards: .thgieh emas eht ta niamer yehT)

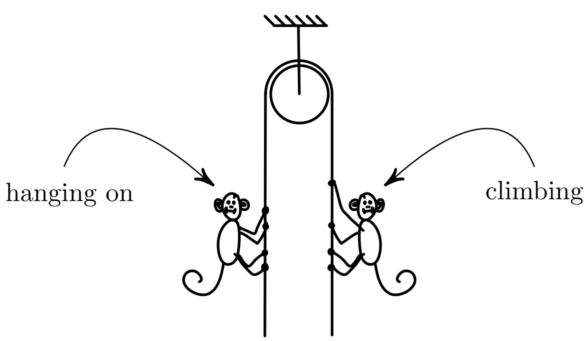


Figure 58. The right monkey is climbing; the left one is just holding on. What will happen?

The dynamics-statics equivalence.

1.64. Consider a heavy Hookean spring hanging by one end in equilibrium; see Figure 25 on page 42. Consider the energy density (per unit mass) of stretching E_s and the gravitational energy E_g per unit mass. Show that $E_s = E_g + \text{const.}$ along the spring. In particular, if the lower end of the spring is on the ground level, then $E_s = E_g$! Does this effect depend on the linearity of Hooke's law?

In the next three problems we consider a zero length Hookean spring of mass M and Hooke's constant k. The spring is to be thought of as a negligibly thin line. Each particle of the spring is labeled by the mass t between the particle and one end of the spring. The spring is laid out along the x-axis *and is kept longitudinally deformed* so that the particle t is located at $x(t)$, where $x(\cdot)$ is a given smooth function. Each particle of the spring is thus forcibly held by some external force at a prescribed location. In addition, a force field with potential $U(x)$ is defined on the line; that is, a unit mass located at x has potential energy $U(x)$, and is therefore subject to the force $F(x) = -U'(x)$.

1.65. Given that Hooke's constant of a homogeneous linear spring is k, what is Hooke's constant of a piece of this spring, given that the mass of the piece forms a proportion $p < 1$ of the total mass of the spring?

1.66. Referring to the setting just described, do the following.

1. Write the total energy $E[x]$ of the spring in terms of the "configuration function" $x = x(t)$.

2. Write the Euler–Lagrange equation for the critical function of $E[x]$.

3. Assume that the external force holding the spring is removed, and that the spring is in equilibrium. That is, each infinitesimal element of the spring has zero net force acting on it in the direction of the x-axis. Express this equilibrium condition as a differential equation for $x(t)$, thereby rederiving the Euler–Lagrange equation from item 2 above.

4. Consider a unit mass moving on the x-axis in the force of the potential energy $-U(x)$, i.e., $F(x) = +U'(x)$. Write down the action integral $A[x] = \int(\text{kinetic} - \text{potential})\, dt$ in terms of $x(t)$, the position of a particle at time t, to show that $A[x] = E[x]$ for any function $x(t)$, and observe that the equilibrium condition of the spring is the same as Newton's second law for the particle.

1.67. A linear spring of mass M and of Hooke's constant k is hanging on its end in the gravitational field. By how much will this spring stretch under the influence of gravity? Assume the relaxed length of the spring to be zero. Note that the stretching is not uniform: the higher up the spring you go, the more it is stretched, as it carries more weight.