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# Preface

Quandles and their kin (kei, racks, biquandles and biracks) are algebraic structures whose axioms encode the movements of knots in space in the same way that groups encode symmetry and orthogonal transformations encode rigid motion. Quandle theory thus brings together aspects of topology, abstract algebra and combinatorics in a way that is easily accessible using pictures and diagrams.

The term “quandle” was coined by David Joyce in his PhD dissertation, written in 1980 and published in 1982 [Joy82]. Previous work had been done as far back as 1942 by Mituhisa Takasaki [Tak42], who used the term “kei” for what Joyce would later call “involutory quandles”. In the 1950s Conway and Wraith [CW] informally discussed a similar structure they called “wracks” from the phrase “wrack and ruin”. At the same time Joyce was writing about quandles, Sergey V. Matveev [Mat82] was writing behind the iron curtain about the same algebraic structure, using the more descriptive term “distributive groupoids”. Louis Kauffman [Kau91] used the term “crystals” for a form of the quandle structure. In the mid 1980s a generalized form of the quandle idea was independently discovered by Brieskorn [Bri88], who chose the descriptive term “automorphic sets”.

In 1992 Roger Fenn and Colin Rourke [FR92] wrote a seminal work reintroducing the quandle idea and a generalization; they chose to use the Conway/Wraith term “wracks” while dropping the “w”

to obtain the term “racks”, canceling the “w” along with the writhe independence. In subsequent work [FRS95] they suggested a further generalization known as “biracks” with a special case known as “biquandles”. Biquandles were explored in detail in 2002 by Louis Kauffman and David Radford [KR03], with later work by others [CES04, FRS95, NV06].

Fenn, Rourke and Sanderson introduced in [FRS95] a cohomology theory for racks and quandles, analogous to group homology. This ultimately led to the current popularity of quandles, since it allowed Scott Carter, Daniel Jelsovsky, Seiichi Kamada, Laurel Langford and Masahico Saito in [CJK<sup>+</sup>03] to define an enhancement of the quandle counting invariant using quandle cocycles, leading to new results about knotted surfaces and more. It was this and subsequent work that led the present authors to study quandles, and ultimately led to this book.

If one restricts oneself to the most important quandle axiom, namely self-distributivity, then one can trace this back to 1880 in the work of Pierce [Pei80] where one can read the following comments: “*These are other cases of the distributive principle .... These formulae, which have hitherto escaped notice, are not without interest.*” Another early work fully devoted to self-distributivity appeared in 1929 by Burstin and Mayer [BM29] dealing with distributive quasigroups: binary algebraic structures in which both right multiplication and left multiplication are bijections, and with the extra property that the operation is left and right distributive on itself (called also Latin quandles).

As quandle theorists, we have found quandle theory not only intrinsically interesting but also very approachable for undergraduates due to its unique mix of geometric pictures and abstract algebra. This book is intended to serve as a text for a one-semester course on quandle theory which might be an upper division math elective or as preparation for a senior thesis in knot theory.

This book assumes that the reader is comfortable with linear algebra and basic set theory but does not assume any previous knowledge of abstract algebra, knot theory or topology. The reader should be

familiar with sets, unions, intersections, Cartesian products, functions between sets, injective/surjective/bijective maps as well as vector spaces over fields, linear transformations between vector spaces, and matrix algebra in general. Readers should also be familiar with the integers  $\mathbb{Z}$ , rationals  $\mathbb{Q}$ , reals  $\mathbb{R}$  and complex numbers  $\mathbb{C}$ .

The book is organized as follows.

Chapter 1 introduces the basics of knot theory; advanced readers may opt to skip directly to Chapter 2. Chapter 2 introduces important ideas from abstract algebra which are needed for the rest of the book, including introductions to groups, modules, and cohomology assuming only a linear algebra background. Chapter 3 gives a systematic development of the algebraic structures (quandles and kei) arising from oriented and unoriented knots and links, including both theory and practical computations. Chapter 4 looks at important connections between quandles and groups and introduces the basics of algebraic topology, including the fundamental group and the geometric meaning of the fundamental quandle of a knot. In Chapter 5 we look at generalizations of the quandle idea, including racks, bikei, biquandles and biracks. Chapter 6 introduces enhancements of representational knot and link invariants defined from quandles and their generalizations. In Chapter 7 we conclude with applications to generalizations of knots including tangles, knotted surfaces in  $\mathbb{R}^4$ , and virtual knots.

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