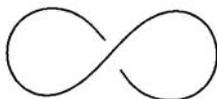


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## Chapter 1



# Knots and Links

## 1. Knots and Links

A *knot* is a simple closed curve, where “simple” means the curve does not intersect itself and “closed” means there are no loose ends. We usually think of knots in three-dimensional space since simple closed curves in the line and plane are pretty boring and, perhaps surprisingly, simple closed curves in 4 or more dimensions are also boring, as we will see.

Two knots  $K_0$  and  $K_1$  have the same *knot type* if we can move  $K_0$  around in space in a continuous way, i.e. without cutting or tearing the knot (or the space in which the knot lives!) to match up  $K_0$  with  $K_1$ . Formally,  $K_0$  is *ambient isotopic* to  $K_1$  if there is a continuous map  $H : \mathbb{R}^3 \times [0, 1] \rightarrow \mathbb{R}^3$  such that  $H(K_0, 0) = K_0$ ,  $H(K_0, 1) = K_1$  and  $H(x, t)$  is injective (one-to-one) for every  $t \in [0, 1]$ . Such a map is called an *ambient isotopy*; if you think of  $t$  as a time variable, then  $H$  is a movie showing how to continuously deform  $K_0$  onto  $K_1$ . If there exists an ambient isotopy  $H$  taking  $K_0$  to  $K_1$  we write  $H : K_0 \xrightarrow{\sim} K_1$ .

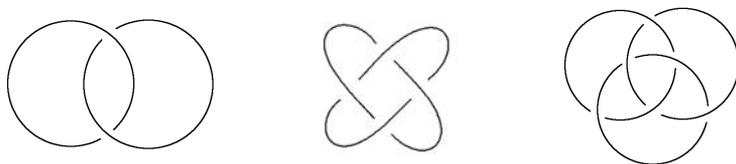
To specify a knot  $K$  we could make a physical model by tying the knot in a rope or cord; a nice trick suggested by Colin Adams in [Ada04] is to use an extension cord, so you can join the ends together by plugging the plug into the outlet end.

To specify knots in a more print-friendly format, we could give a parametric function  $f(t) = (x(t), y(t), z(t))$  where  $0 \leq t \leq 1$  and  $f(0) = f(1)$ . This approach is required in order to study *geometric knot theory*, where the exact positioning of  $K$  in space is important. In *topological knot theory*, however, we only care about the position up to ambient isotopy; thus, a simpler solution is to draw pictures or *knot diagrams*. Formally, a *knot diagram* is a projection or shadow of a knot on a plane where we indicate which strand passes over and which passes under at apparent crossing points by drawing the understrand broken.



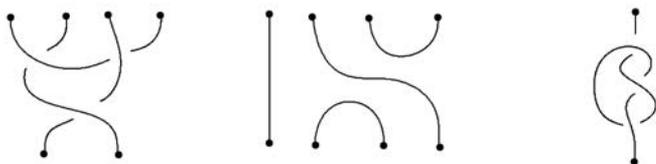
A knot is *tame* if it has a diagram with a finite number of crossing points; knots in which every projection has infinitely many crossing points are called *wild knots*. We will only deal with tame knots in this book.

**Links, Tangles and Braids (oh my!)** There are many kinds of objects related to knots. A *link* consists of several knots possibly linked together; each individual simple closed curve is a *component* of the link. A knot is a link with only one component.

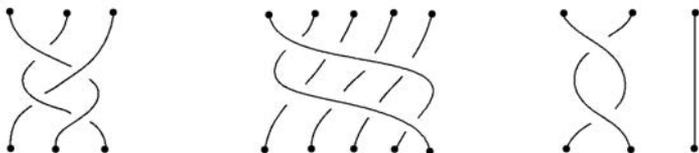


A *tangle* is a portion of a knot or link with fixed endpoints we can think of as inputs and outputs. If there are  $n$  inputs and  $m$  outputs,

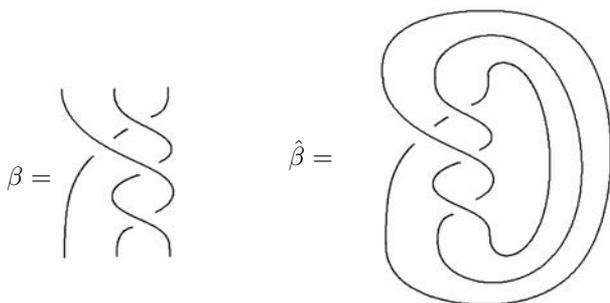
we have an  $(n, m)$ -tangle.



A *braid* is a tangle which has no maxima and no minima in the vertical direction, i.e., a tangle whose strands do not turn around. Note that in any braid, the number of inputs must equal the number of outputs, unlike more general tangles.

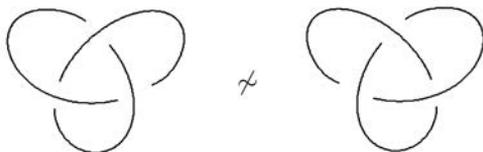


For any braid  $\beta$  there is a knot or link  $\hat{\beta}$  called the *closure* of the braid, obtained by joining the top strands to the bottom strands. The converse is also true – every knot or link can be put into braid form, a fact known as *Alexander’s Theorem*.

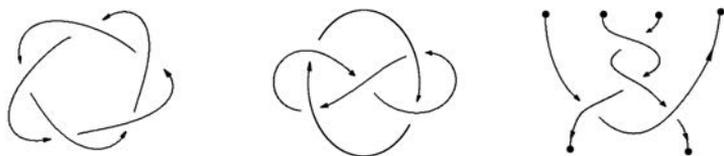


The *obverse* of a knot  $K$  is the mirror image of  $K$ , denoted  $\bar{K}$ . A knot may or may not be equivalent to its obverse – the trefoil knot comes in distinct left- and right-handed varieties, for instance. Knots which are different from their obverses are called *chiral*, while knots

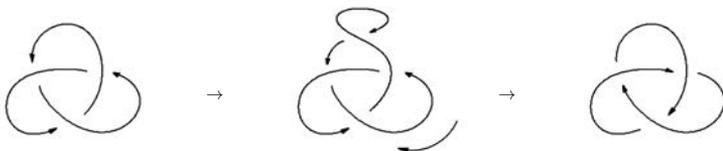
which are ambient isotopic to their obverses are called *amphichiral*.



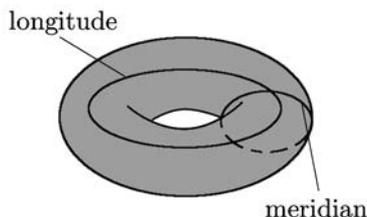
**Oriented Knots.** For each strand in a knot, link, tangle or braid, we can make a choice of *orientation* or preferred direction of travel. Knots described by a parametrization have an implied orientation in the direction of increasing  $t$  value; braids also have an implied orientation of all strands oriented in the same direction (up or down depending on the author's choice of convention). For generic oriented knots, links and tangles, we specify the orientation of each strand with an arrow.



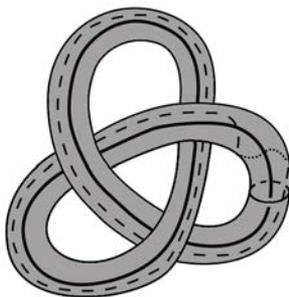
Reversing the orientation of an oriented knot  $K$  yields a possibly different oriented knot called the *inverse* or *reverse* of  $K$ , denoted  $-K$ . For two oriented knots  $K_0$  and  $K_1$  to be equivalent, we need an ambient isotopy  $H : K_0 \xrightarrow{\sim} K_1$  which respects the orientation of  $K_1$ . For example, the trefoil knot  $K$  below is equivalent to its inverse  $-K$  as illustrated:

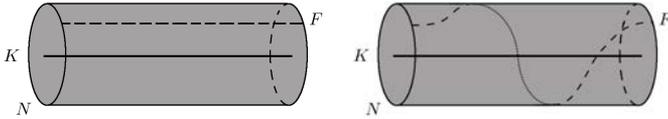


**Framed Knots.** Like a choice of orientation, a *framing* of a knot is a choice of extra structure we can give to a knot which then must be preserved by an ambient isotopy for two framed knots to be equivalent. Start by inflating the knot  $K$  like an inner tube, so we have a knotted solid torus  $N$  with  $K$  as its core. This solid torus is called a *regular neighborhood* of the knot. A circle on the torus which goes around the torus with the knot is called a *longitude*, while a circle going around a disk slice of the solid torus with the knot at its center is called a *meridian*.



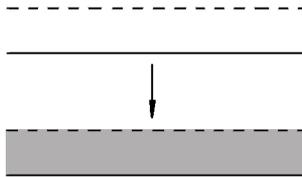
A *framing curve*  $F$  is a simple closed curve on the surface of the torus which projects down onto the original knot  $K$  in an injective (one-to-one) way, i.e., a longitude of the torus. While  $F$  goes around the torus with  $K$  exactly once in the longitudinal direction, it can wrap around the meridional direction of the torus any integer number of times.





Let  $K$  be a knot and  $F$  a framing curve. A *framed isotopy* of  $(K_0, F_0)$  to  $(K_1, F_1)$  is an ambient isotopy which carries  $K_0$  to  $K_1$  and carries  $F_0$  to  $F_1$ . For a given knot  $K$ , with framing curve  $F$ , the number of times  $F$  wraps meridionally around  $K$  (with counterclockwise wraps counted with a positive sign and clockwise twists counted with a minus sign) is called the *framing number* of the framed knot  $(K, F)$ . For a fixed knot  $K$ , two framed knots  $(K, F_0)$  and  $(K, F_1)$  are framed isotopic only if the framing numbers are equal.

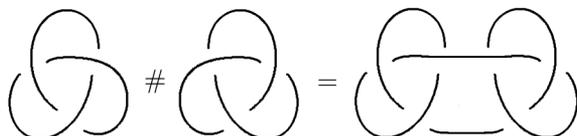
We can think of a framed knot as a 2-component link with the knot and its framing curve forming two sides of a ribbon.



Then, framed isotopy can be understood as movement of the ribbon through space. Similarly a framed link of  $n$  components can be understood as an ordinary link of  $2n$  components where the components come in parallel pairs forming the two sides of  $n$  ribbons, with each of the original  $n$  components having its own framing curve and framing number. Similarly, in a framed braid or framed tangle, each strand has its own framing curve and framing number.

We can think of framed isotopy as a mathematical model for knotted 3-dimensional ropes or tori, where ambient isotopy is the model for knotted 1-dimensional curves.

**Connected Sums.** Knots and links have an operation known as *connected sum* where two knots are joined into a single knot by cutting the knots and joining the loose ends to form a single knot. We write  $K_0 \# K_1$  for the connected sum of  $K_0$  and  $K_1$ .

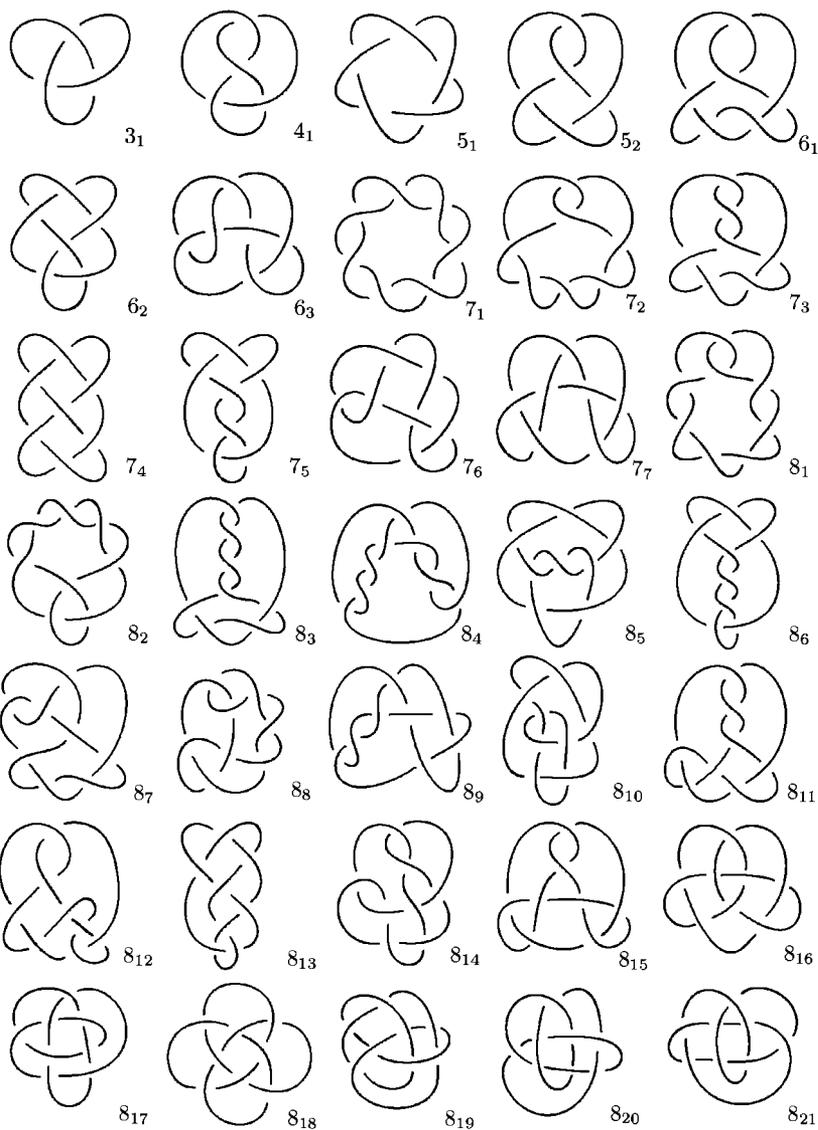


A connected sum  $K_0 \# K_1$  can also be understood as the result of first tying  $K_0$  in a piece of string, then tying  $K_1$  before joining the ends.

A knot  $K$  is *prime* if the only way to decompose  $K$  as a connected sum of two knots is as  $K \# 0_1$  where  $0_1$  is the unknotted circle or *unknot*; that is,  $K$  is prime if  $K$  does not break down as a connected sum of two nontrivial knots. For any knot, to decide whether the knot is prime, we can look at all the ways of intersecting the knot with a circle which meets the knot at exactly two points; this divides the knot into a connected sum of the portion outside the circle (completed by the arc along the circle) and the portion inside the circle (completed with the arc along the circle). If the knot is prime, then every such division will have one side unknotted.



Below is a table of all prime knots with up to eight crossings. Knots are named according to their crossing number with a subscript indicating their position on the table.

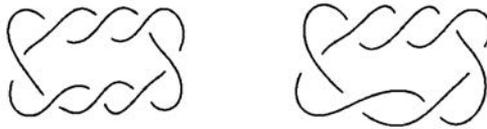


**Exercises.** 1. Take a piece of rope or an extension cord, tie a knot very loosely, and join the ends together. Lay the knot on a flat surface and draw a knot diagram representing the knot. Now, move the knot around to a new position – don't be afraid to add a few twists, just keep the ends joined. Now, draw a diagram of your knot in its new configuration. Repeat a few times; then repeat with a new knot.

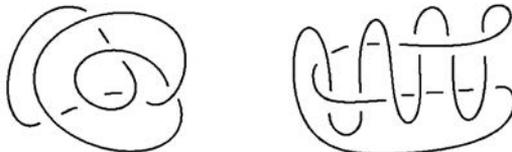
2. Draw all possible knot or link diagrams with exactly two crossings.

3. Draw all possible knot or link diagrams with exactly three crossings.

4. Is it possible, given what you currently know, that the two diagrams below represent the same knot or link?



5. Is it possible, given what you currently know, that the two diagrams below represent the same knot or link?



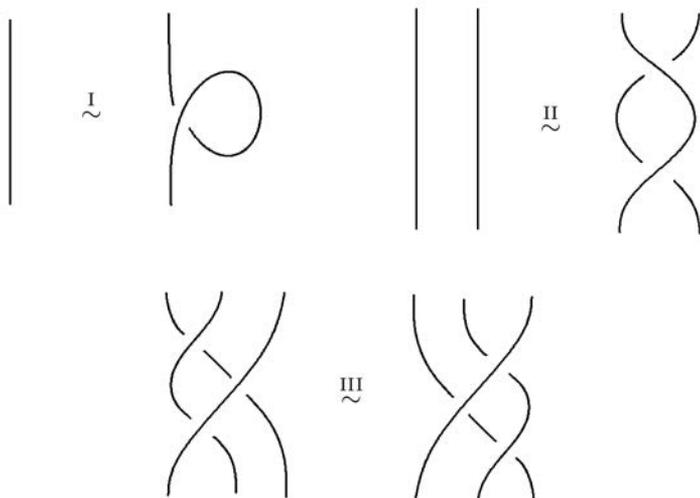
## 2. Combinatorial Knot Theory

The basic question in knot theory is how to tell when two knot diagrams represent the same knot. This is really two questions: (1) given a knot diagram, what are all possible diagrams which represent the same knot type and (2) how can we prove two diagrams represent different knots. The second question we leave to the next section.

The first question was answered in the 1920s by Kurt Reidemeister. A *local move* on a knot diagram involves replacing one portion of the diagram inside a small disk with something else, while the rest of the diagram outside the disk remains unchanged. A *planar isotopy* is a local move which replaces a strand without crossings with another strand without crossings with the same endpoints.

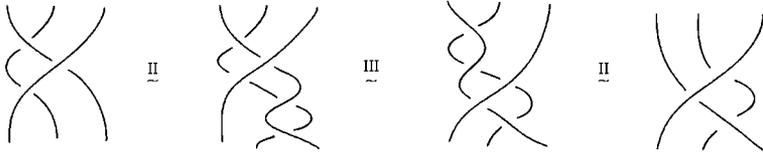


In 1926, Kurt Reidemeister and independently, in 1927, J. W. Alexander and G. B. Briggs proved that two tame knot diagrams,  $K_0$  and  $K_1$ , are ambient isotopic if and only if one can be changed into the other by a finite sequence of planar isotopies and moves of the following three types:



If you look closely, you will find that some other similar moves are implied by the listed moves. For example, move III says you

can move a strand over a crossing where the crossing is the same type (left-over-right) as the other two crossings. We can derive an additional type III move which says you can pass a strand over the other kind of crossing using the given III move and a II move:



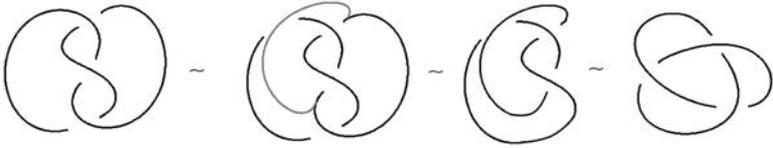
The Reidemeister moves let us translate the topological relationship of ambient isotopy into a combinatorial equivalence relation. That is, we started thinking of knots as geometric objects, simple closed curves in space, but we now have a new way to think of knots: as equivalence classes of knot diagrams under the equivalence relation generated by planar isotopy moves and the Reidemeister moves.

Thus, to prove that two knot diagrams,  $K_0$  and  $K_1$ , represent the same knot type, we can identify an explicit sequence of Reidemeister moves taking  $K_0$  to  $K_1$ . For example, the knot below is secretly the unknot, i.e., an unknotted circle. To prove it, we give a sequence of Reidemeister moves taking it to a circle without crossings.

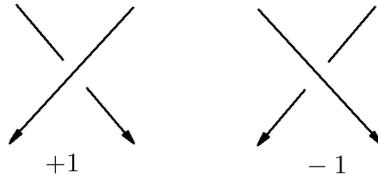


In practice it is often easier to redraw knots using the principle that any portion of a strand with only overcrossings may be replaced with another strand with the same endpoints and all new overcrossings, with the resulting breaks healing. Note that any such “overpass move” can always be broken down into a sequence of Reidemeister

moves and planar isotopies.



**Combinatorial Oriented Knots.** Introducing an orientation on our knot diagrams gives us two kinds of crossings, which we identify as “positive” and “negative” depending on whether the understrand is directed right-to-left or left-to-right when viewed from the overstrand.



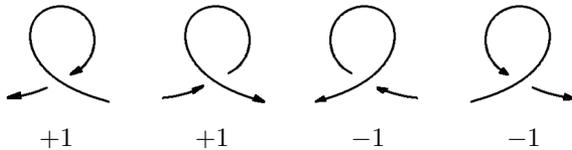
We will denote the sign of a crossing  $C$  as  $\epsilon(C) = \pm 1$ .

Including orientations means we now have more Reidemeister moves than we did before. Instead of two type I moves, we now have four; one type II becomes four – two *direct* moves where the strands are oriented in the same direction, and *reverse* moves where the strands are oriented in opposite directions, and there are eight oriented type III moves.

In practice, many of the moves are implied by the other moves. Indeed, it is an interesting exercise to find a minimal generating set of moves, i.e., a subset containing as few moves as possible from which all of the other oriented moves can be recovered.

The sum of all the crossing signs is a quantity known as the *writhe* of the diagram; writhe is a property of diagrams, not of knots, since starting with a given knot diagram we can adjust the writhe to whatever we want using type I moves. For links, each component has its own writhe determined by counting only crossings where the component crosses itself; multi-component crossings do not contribute to the component writhes.

Note that for any single-component crossing both possible choices of orientation determine the same sign for each crossing since switching the orientation of a component reverses the directions of both strands in the crossing. In particular, writhe is well defined even for unoriented diagrams, and kinks have well-defined signs regardless of orientation choice.



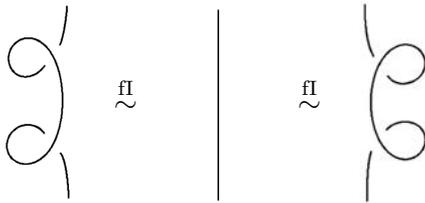
**Combinatorial Framed Knots.** Given a knot diagram  $K$ , there is an easy standard way to choose a framing curve – simply “push off” a copy of  $K$ , i.e., draw a framing curve parallel to  $K$ . This is traditionally called the *blackboard framing* since it is the easiest framing to draw on a blackboard. More precisely, let  $F$  be the knot traced by a normal vector to the knot  $K$ . If  $F$  is endowed with an orientation parallel to that of  $K$ , then the *framing number* of  $K$  is the sum of the crossing signs at crossings where  $F$  crosses over  $K$ . Conversely, if we assign an integer  $j$  to a knot  $K$ , then we can construct a normal vector to  $K$  and a knot  $F$  such that  $j$  is the framing number. The blackboard framing is then the natural framing with framing number equal to the writhe of  $K$ . In particular, every knot or link diagram can be considered as a framed knot or link by using the blackboard framing.

Geometrically, the *framing number* of a framing curve  $F$  is the number of times  $F$  wraps around the solid torus with  $K$  as its core. The framing number of a blackboard-framed knot is equal to its writhe. A little thought reveals that Reidemeister II and III moves do not change the writhe of a diagram, while Reidemeister I moves do. Thus, to preserve the blackboard framing, we must modify the type I move to preserve writhe. In particular, to cancel the +1 to writhe from adding a positive kink, we must also add a negative kink. Kinks of both signs come in two versions, clockwise and counterclockwise, also known as kinks of *winding number*  $-1$  and  $+1$ , respectively.

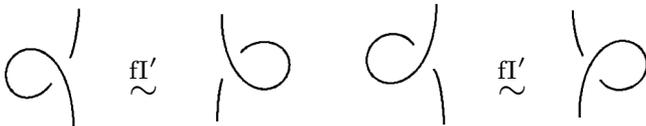
As observed in [FR92], it turns out that if both the winding numbers and crossing signs of the kinks are opposite, we can cancel the kinks using only II and III moves (the following illustration is the simplest “Whitney trick” [Whi44]):



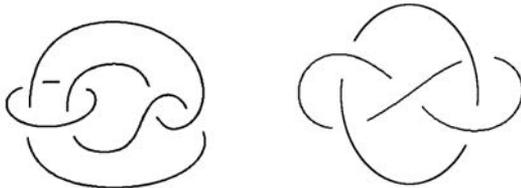
In the case of kinks with equal winding number and opposite writhe, we need an explicit move. These are the *blackboard framed type I moves*:



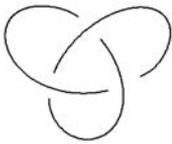
These moves are equivalent to the alternate framed type I moves:



**Exercises.** 1. Using Reidemeister moves, show that the diagrams below represent the same link. This link is known as the *Whitehead link*.



2. Using Reidemeister moves, determine whether the knot  $K$  below is equivalent to the *trefoil* or the *Figure 8*.



Trefoil



$K$

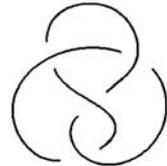
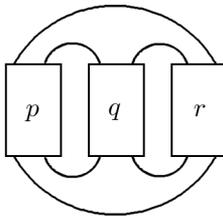


Figure 8

3. Let  $p, q, r$  be three integers. A  $(p, q, r)$ -pretzel link is a knot or link of the form



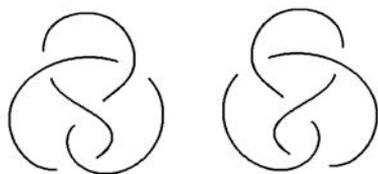
where the boxes are replaced with stacks of  $p, q$  and  $r$  oriented crossings respectively (a negative value means use negative crossings – also note that the orientation of the crossings in the boxes may not extend to the whole link!) For example, the  $(2, 1, -3)$  pretzel link is



How many components are possible in a pretzel link? What conditions on  $p, q$  and  $r$  ensure that we have a knot? A 2-component link?

4. Show that the figure eight knot  $4_1$  is ambient isotopic to its mirror image by changing the diagram on the left to the one on the right

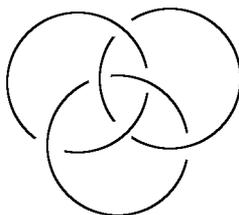
using Reidemeister moves.



5. Using framed Reidemeister moves, show that the knot below is framed isotopic to the unknot with writhe  $-2$ .



6. A link is called *Brunnian* if it is nontrivial, but deleting any component makes the remaining link trivial. Given that the *Borromean rings*



form a nontrivial link, show that the link is Brunnian.

7. Show that the  $\text{fl}$  and  $\text{fl}'$  moves are equivalent in the presence of the type II and III moves by deriving the  $\text{fl}'$  move using only type  $\text{fl}$ , II, and III moves and then deriving the  $\text{fl}$  move using only type  $\text{fl}'$ , II, and III moves.

### 3. Knot and Link Invariants

Changing  $K$  into  $K'$  with Reidemeister moves proves that the two diagrams represent the same knot or link. What if we cannot see

a way to change  $K$  into  $K'$ ? Our inability to change  $K$  into  $K'$  with Reidemeister moves does not say that  $K$  and  $K'$  are different; it might be that there is a way, but it's very complicated, perhaps involving hundreds of moves and requiring introducing and removing many crossings. In order to prove that two diagrams represent different knots, we must be more clever.

A *knot invariant* is a function  $f : \mathcal{K} \rightarrow X$  from the set of all knot diagrams to a set  $X$  such that for each Reidemeister move, we have

$$f(K_1) = f(K_2)$$

where  $K_1$  is the knot diagram before the move and  $K_2$  is the same diagram after the move. If  $f$  is a knot invariant, then any two diagrams related by Reidemeister moves must give the same value when we evaluate  $f$ .

Knot theory might be described as the search for and the study of knot invariants. Many knot and link invariants have been discovered and studied, mostly in the 20th and 21st centuries. For the remainder of this section we will explore a few well-known knot and link invariants.

**Geometric Invariants.** One way to define a knot or link invariant is to identify some geometric or topological quantity determined by a knot or link diagram and take the minimum over all diagrams of  $K$ . Many examples of this style of invariant have been defined and studied, from basic to more esoteric:

- *Crossing Number* – The minimal number of crossings in any diagram of  $K$ .
- *Braid Index* – The smallest number of strands of any braid whose closure is a diagram of  $K$ .
- *Bridge Number* – The smallest number of maxima in any diagram of  $K$ .
- *Stick Number* – The smallest number of straight line segments needed to form  $K$  in  $\mathbb{R}^3$ .
- *Rope Length* – The minimal length of a rope of radius 1 needed to tie  $K$ .

- *Genus* – The minimal number of holes in a surface whose edge is  $K$ .
- *Unknotting Number* – The minimal number of crossing changes needed to unknot  $K$ .

These invariants are easy to define but generally hard to compute. From a particular diagram of  $K$  we can compute an upper bound on the actual value of the invariant for  $K$ , but finding a diagram of  $K$  which realizes the minimal value is not always easy to do.

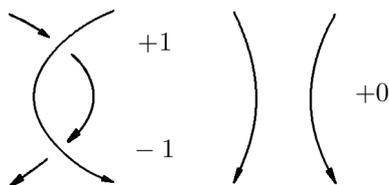
A knot or link invariant  $f$  is *computable* if the actual value of  $f(K)$  can be determined, not just bounded, using any diagram of  $K$ . We will now see three examples of computable knot and link invariants.

**Linking Number.** Perhaps the easiest example of a computable link invariant is the *linking number*. Let  $L = L_1 \cup L_2$  be an oriented link with two components. Let  $\mathcal{M}$  be the set of crossings in  $L$  with one strand from each component. Then the linking number of  $L$  is the sum of the crossing signs of the crossings in  $\mathcal{M}$  divided by 2 since this sum is always even:

$$\text{lk}(L) = \frac{1}{2} \sum_{C \in \mathcal{M}} \epsilon(C).$$

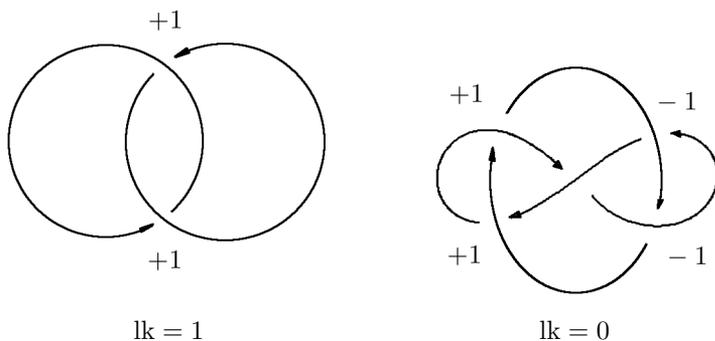
We can verify that the linking number is a link invariant by checking that the contributions match before and after each move. In a type I move, the crossing being introduced or removed is single-component, so it contributes 0 to the linking number, which matches the contribution from the straight strand.

In a type II move, there are two possibilities: either both crossings are multicomponent or both are not. As before, if both crossings are single-component, the contribution of zero matches the zero contribution of the two uncrossed strands. In the multicomponent case, there is always one positive crossing and one negative crossing, so the contribution is  $+1 - 1 = 0$ .



Verifying that  $f(L_B) = f(L_A)$  for type III moves is left to the reader as an exercise; see problem 1.

The fact that the linking number is a link invariant lets us distinguish some links from others. For example, the Hopf link below has linking number 1 while the Whitehead link has linking number 0.



**The Jones Polynomial.** In 1984 knot theory was reinvigorated by the discovery by Vaughan Jones [Jon85] of a powerful knot and link invariant now known as the *Jones polynomial*. The simplest way to define the Jones polynomial is a recursive definition due to Louis Kauffman using the *Kauffman bracket skein relation*. There are several versions of this invariant related by variable substitution; the version we'll use is from [BN02].

Let  $K$  be a knot or link diagram. The skein relation can be understood as a way of interpreting a crossing as a linear combination of *smoothings*:

$$\langle \text{crossing} \rangle = \langle \text{positive smoothing} \rangle - q \langle \text{negative smoothing} \rangle.$$

Recursively applying this relation to each crossing lets us replace a knot or link diagram with  $n$  crossings with a sum of polynomials in  $q$  times diagrams without crossings. We need a rule for evaluating the bracket of a diagram without crossings. Thus, for a disjoint union of  $n$  copies of the diagram of the unknot with no crossings, we define

$$\langle \bigcirc \bigcirc \dots \bigcirc \rangle = (q + q^{-1})^{n-1}.$$

In particular, we can erase a closed curve without crossings at the cost of multiplying by  $(q + q^{-1})$ .

The bracket function is unchanged by Reidemeister III moves: The reader is encouraged to verify that both  $\langle \text{diagram 1} \rangle$  and  $\langle \text{diagram 2} \rangle$  are equal to

$$-q \left( \langle \text{diagram 3} \rangle + \langle \text{diagram 4} \rangle \right) + q^2 \left( \langle \text{diagram 5} \rangle + \langle \text{diagram 6} \rangle \right) - q^3 \langle \text{diagram 7} \rangle.$$

However, Reidemeister I and II moves do change the value of  $\langle K \rangle$ , but in a predictable way. More precisely, removing a positive crossing multiplies  $\langle K \rangle$  by  $q^{-1}$  and removing a negative crossing multiplies  $\langle K \rangle$  by  $-q^2$ :

$$\begin{aligned} \langle \text{diagram 8} \rangle &= \langle \text{diagram 9} \rangle - q \langle \text{diagram 10} \rangle \\ &= (q + q^{-1} - q) \langle \text{diagram 10} \rangle = q^{-1} \langle \text{diagram 10} \rangle \end{aligned}$$

and

$$\begin{aligned} \langle \text{diagram 11} \rangle &= \langle \text{diagram 12} \rangle - q \langle \text{diagram 13} \rangle \\ &= (1 - q(q + q^{-1})) \langle \text{diagram 12} \rangle = -q^2 \langle \text{diagram 12} \rangle. \end{aligned}$$

Likewise, a crossing-removing type II move multiplies  $\langle K \rangle$  by a factor of  $(-q^2)q^{-1} = -q$ :

$$\begin{aligned} \langle \text{crossing} \rangle &= \langle \text{no crossing} \rangle - q \langle \text{other crossing} \rangle - q \langle \text{empty} \rangle + q^2 \langle \text{cup} \rangle \\ &= (1 - q(q + q^{-1}) + q^2) \langle \text{no crossing} \rangle - q \langle \text{empty} \rangle \\ &= -q \langle \text{empty} \rangle. \end{aligned}$$

Thus, to cancel the effects of type I and II moves, we need to multiply by  $(-1)^n q^{p-2n}$  where  $p$  is the number of positive crossings and  $n$  is the number of negative crossings. The *Jones polynomial* of a link  $L$  is

$$J(L) = (-1)^n q^{p-2n} \langle L \rangle.$$

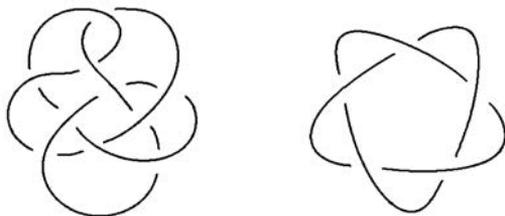
**Example 1.** Let us compute the Jones polynomial of the Hopf link:

$$\begin{aligned} \langle \text{Hopf link} \rangle &= \langle \text{positive crossing} \rangle - q \langle \text{negative crossing} \rangle \\ &= \langle \text{positive crossing} \rangle - q \langle \text{negative crossing} \rangle \\ &= -q \langle \text{positive crossing} \rangle + q^2 \langle \text{negative crossing} \rangle \\ &= q + q^{-1} - q - q + q^2(q + q^{-1}) = q^{-1} + q^3. \end{aligned}$$

If we orient the components so that both crossings are positive, we then have  $q^2(q^{-1} + q^3) = q + q^5$ ; if we reverse the orientation of one component while fixing the other, we have two negative crossings and the Jones polynomial becomes  $q^{-4}(q^{-1} + q^3) = q^{-1} + q^{-5}$ . Thus, the two possible oriented Hopf links have different Jones polynomials and cannot be ambient isotopic to each other.

It turns out that the Jones polynomial of the mirror image of a knot or link  $K$  can be obtained from the Jones polynomial of  $K$  by

replacing  $q$  with  $q^{-1}$ . The Jones polynomial is a very powerful invariant, but it is not a complete invariant – there are known examples of pairs of different knots which have the same Jones polynomial, such as the knots below:

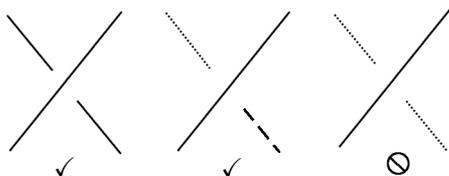


Indeed, one of the more famous unsolved (as of this writing) problems in knot theory is whether the Jones polynomial detects the unknot; that is, is there a nontrivial knot  $K$  with Jones polynomial  $J(K) = 1$ ? For links with multiple components, the answer is yes, there are nontrivial links with trivial Jones polynomial. For knots, though, the problem is currently unsolved. Direct computations have shown that no nontrivial knot with fewer than 16 crossings has trivial Jones polynomial.

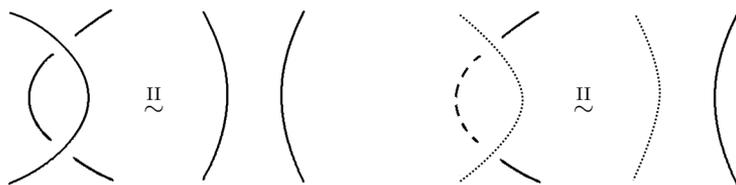
The Jones polynomial is a powerful knot invariant, but computationally it is very intense. The recursive algorithm described above is an *exponential time* algorithm, meaning each additional crossing doubles the number of computations needed to compute  $J(K)$ .

**Tricoloring.** For our final example of a computable knot invariant, we will define *Fox tricoloring*, introduced by Ralph Fox in the 1950s. A *tricoloring* of a knot or link diagram is a choice of color for each arc in the diagram from a set of three colors – we'll use solid, dotted and dashed, but you can use whatever colors you like. A tricoloring is *valid* if at every crossing we either have all three colors the same or all three colors different. A valid tricoloring is *nontrivial* if it uses

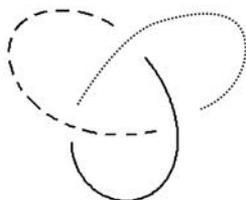
all three colors.



To use tricoloring as a knot invariant, we notice that if we start with a valid tricoloring of a diagram  $K$  before doing a Reidemeister move, there is a unique valid tricoloring of the diagram after the move which agrees with the original coloring outside the move area. For example, all strands in a type I move must be the same color. There are two cases for type II moves: both strands the same color before crossing and two different colors before crossing:

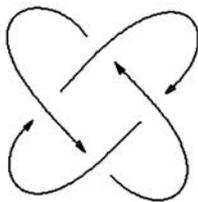


For the type III moves, there are various cases which the reader is encouraged to check. Moreover, if a tricolored diagram is monochrome before a move, the corresponding diagram after the move is also monochrome. Hence, the existence of a nontrivial tricoloring of a knot or link diagram is an invariant of knots and links. For example, the only valid tricolorings of an unknotted diagram are monochrome colorings, while the trefoil has a nontrivial tricoloring; thus there can be no sequence of Reidemeister moves taking the trefoil to the unknot.



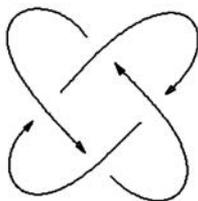
**Exercises.** 1. Verify that Reidemeister III moves do not change linking number. (Hint: Choose one oriented type III move and consider all cases depending on which strands are from the same component).

2. A link is *split* if it is possible to separate the components so that one component lies entirely on one side of a line and the other component lies entirely on the other side of the line. Prove that the  $(4, 2)$ -torus link below is not split.

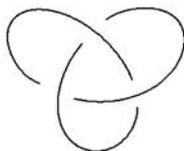
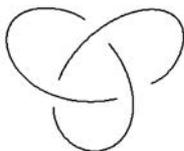


3. Prove that the framing number of a blackboard framed oriented knot is the linking number of the framed knot considered as a 2-component link.

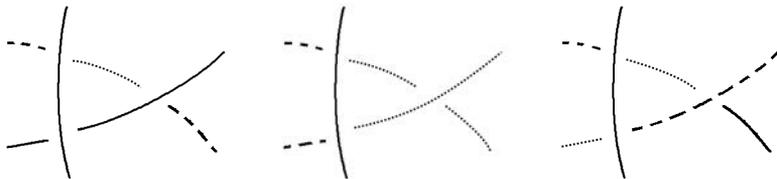
4. Compute the Jones polynomial of the  $(4, 2)$ -torus link below.



5. Use the Jones polynomial to prove that the right-handed and left-handed trefoils below are not equivalent.



6. For each of the tricolored tangle diagrams below, find the unique corresponding tricolored tangle diagram after doing a type III move:



7. Show that there is no nontrivial tricoloring of the figure 8 knot below.



8. How many valid tricolorings of the trefoil knot are possible? How many are nontrivial?