Preface

Mathematics is an endlessly fruitful subject. One reason is its ability to make lemons into lemonade. In mathematics, the gap between what we're *hoping* to prove and what is actually *true* can itself become something that we can measure, something we can quantify — the basis for a whole new world of mathematical theory.

Let me give an example. In Calculus II, you learn that every (reasonably smooth) function of one variable is the derivative of another function — the fundamental theorem of calculus says that integration is the reverse of differentiation. In Calculus III, you find out that in higher dimensions there is a necessary condition that must be satisfied if n given functions are to be the partial derivatives of another function. For instance, in dimension 2, if functions u and v are to be the partial derivatives (with respect to x and y) of a function f, then the *integrability condition*

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

must be satisfied.

Is this *necessary* condition always *sufficient*? For functions defined on a disc, the answer is yes ("every irrotational vector field is the gradient of a potential"). On more general domains, though, the answer is no, as is shown by the notorious example

$$u = \frac{y}{x^2 + y^2}, \quad v = \frac{-x}{x^2 + y^2}$$

defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. Most Calculus III courses treat this as a nuisance, an anomaly. What if we instead treated it as a clue, a signpost, the start of a trail that might lead to a new kind of mathematics?

In fact, this trailhead leads us up one of the many routes to the summit of Mount Winding-Number, one of the most beautiful peaks of the Mathematical Range[1]. This book is a sort of hiker's guide to that mountain. Some guides want to get you to the top as quickly as possible so as to move on to "greater" things. I am not one of those. Rather, I want us to take our time, to explore different paths, and to get to know the shape of the mountain from various angles: "winding around" is a description of the book's methodology as well as of its subject matter. Only in the final chapter will we begin to explore the high ridge that connects our mountain to the "greater ranges" of algebraic topology.

The book originates from a course taught in Penn State's MASS program in the fall of 2013. MASS is a unique semester-long intensive experience that brings together a "critical mass" of highly motivated undergraduate students from colleges across the USA and elsewhere. It is a pleasure to record my thanks for the opportunity to share in this program once again and for the energetic participation of all the students in the course. This book is dedicated to you all.

Note to the reader: Our trail in this book will wind through several different parts of mathematics, parts which are often segregated in their own courses with titles like "abstract algebra" or "analysis" or "geometry" or "topology". Probably, you will be more familiar with some of these than with others. *Don't worry!* A series of appendices reviews necessary background (and gives suggestions for further reading if you want to follow up in greater depth) in these various subject areas. As you read through the main text, notes will direct you to the relevant appendix at the first point that its concepts are required. Then you can decide whether to read the appendix for a quick refresher or to continue with the main text and hope for the

[1]For an extended riff on the idea of mathematics as mountaineering, see [**1**].

best. Whichever you decide, be sure to have fun! This is a beautiful mathematical journey and I want you to enjoy it. If you have any comments or suggestions for improving the book, feel free to contact me at `john.roe@psu.edu`.

The website for this book is `www.ams.org/bookpages/stml-76`.

John Roe