
Chapter 1

Lebesgue Measure

There are different ways one can look at the size of a set. For example, one could look at the set $A = \{0, 2, 3, 5\}$ and say that A has four elements. From a set theoretic standpoint, this is looking at the cardinality of the set A . On the other hand, if one thinks of the members of A as points on the number line, the set A is miniscule in comparison to the real line. Think of coloring these four points blue and the rest of the line purple (or pick your two favorite colors). How much blue would you see when looking at this colored real line? Would you see anything other than purple? A single point takes up no real width on the real line. In fact, if one were asked for the length of A , the natural answer would probably be zero.

Our goal is to generalize the Riemann integral, which has its origins in the notions of length and area. We will be taking the second point of view when looking for the size of a set. Our first task then is to generalize the notion of area (and length, and volume, etc.).

1.1. Lebesgue Outer Measure

We will start our process by considering a very basic set. In \mathbb{R}^n we define a **closed interval** to be a closed rectangle I where

$$I = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \dots, n\}.$$

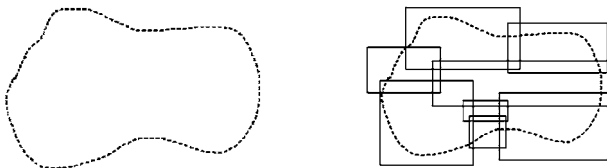


Figure 1.1. Example of a set covered by intervals.

For example,

$$I = \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 \leq x_1 \leq 3, -2 \leq x_2 \leq 4\}$$

is a closed interval in \mathbb{R}^2 , while

$$J = \{x \in \mathbb{R}^1 \mid 2 \leq x \leq 6\}$$

is a closed interval in \mathbb{R}^1 . Notice that in \mathbb{R}^1 these closed intervals are, in fact, our usual closed intervals. The **volume of a closed interval** in \mathbb{R}^n is

$$v(I) = \prod_{i=1}^n (b_i - a_i).$$

Two conventions about closed intervals that will be used throughout this text are that the constants a_i and b_i are all finite, and that $a_i < b_i$ for all i .

Our strategy will be to cover a set $A \subseteq \mathbb{R}^n$ with closed intervals and add the volumes of these intervals. In order to make sense of this, we will cover A with a countable (either a finite or a countably infinite) number of intervals. This should give us an estimate (probably on the large side, but be careful not to assume this) of the volume of A . Then we use this to take the best possible estimate of the volume.

More precisely, let $S = \{I_k\}$ be a countable (finite or countably infinite) collection of closed intervals in \mathbb{R}^n . We say **S is a covering of A by closed intervals** if $A \subseteq \bigcup I_k$. Set $\sigma(S) = \sum v(I_k)$. If the series $\sum v(I_k)$ diverges, set $\sigma(S) = +\infty$. This idea is illustrated in Figure 1.1. In general, if S is a covering of A by intervals, we expect $\sigma(S)$ to be one of our overestimates of the volume of A . Here we have used the notation $\sum v(I_k)$, lacking upper and lower limits in the

summation, to denote a countable (finite or countably infinite) sum. That is, this sum is either a finite sum or a countably infinite series.

Notice that we can always create a cover A by a countably infinite collection of intervals each with volume 1. If so, for such a covering, call it S , $\sigma(S) = +\infty$. So the set

$$\{\sigma(S) \mid S \text{ is a covering of } A \text{ by closed intervals}\}$$

always includes $+\infty$. We can still make sense of the infimum of this set if we use the convention that $s < +\infty$ for every $s \in \mathbb{R}$ so that $s \leq +\infty$ for all $s \in \mathbb{R}$ or $s = +\infty$. With this convention in mind, generalizing our usual definition of the infimum (or greatest lower bound) makes sense. That is,

$$\alpha = \inf\{\sigma(S) \mid S \text{ is a covering of } A \text{ by closed intervals}\}$$

if and only if $\alpha \leq \sigma(S)$ for every such covering S (that is, α is a lower bound) and $\beta \leq \alpha$ for any other lower bound. The difference now is that this infimum might actually equal $+\infty$. This happens when $\sigma(S) = +\infty$ for every covering S of A by closed intervals.

Also, for every S , a covering of A by closed intervals, $\sigma(S) \geq 0$. This makes 0 a lower bound for the set

$$\{\sigma(S) \mid S \text{ is a covering of } A \text{ by closed intervals}\}.$$

We finally officially define the Lebesgue outer measure of a set A .

Definition 1.1.1. Let $A \subseteq \mathbb{R}^n$. The **Lebesgue outer measure** of A is

$$m^*(A) = \inf\{\sigma(S) \mid S \text{ is a covering of } A \text{ by closed intervals}\}.$$

By definition, $m^*(A)$ is always greater than or equal to 0. Also, it follows that if S is any covering of A by closed intervals, then $m^*(A) \leq \sigma(S) \leq +\infty$. In other words, for any set $A \subseteq \mathbb{R}^n$, $0 \leq m^*(A) \leq +\infty$.

Now that we have introduced $+\infty$ as a possible value for the Lebesgue outer measure of a set, it might be worth pointing out a few things about the arithmetic of $\mathbb{R} \cup \{+\infty\}$. We can make sense of addition in that $c + (+\infty) = (+\infty) + c = +\infty$ for any real number c . Also, is it consistent if we define $(+\infty) + (+\infty) = +\infty$. But we will need to avoid any statements involving $+\infty$ and subtraction. (Think about what subtraction means: $5 - 3 = 2$ because $2 + 3 = 5$.)

But $c + (+\infty) = +\infty$ for any real number c , so how does one find $+\infty - (+\infty)$?

Remark 1.1.2. An important feature of the definition of Lebesgue outer measure is that given any set A and any given $\epsilon > 0$, there is a covering S of A by closed intervals such that

$$\sigma(S) \leq m^*(A) + \epsilon.$$

This is easily the case if $m^*(A) = +\infty$. In the case that $m^*(A)$ is finite, this follows by observing that $m^*(A) + \epsilon$ can no longer be a lower bound for

$$\{\sigma(S) \mid S \text{ is a covering of } A \text{ by closed intervals}\}.$$

Moreover, if $m^*(A)$ is finite, we can make the inequality a strict inequality. We will be using this property time and time again.

Example 1.1.3. We will compute the Lebesgue outer measure of $A = \{3\}$.

Let $\epsilon > 0$. Set $S = \{[3 - \epsilon, 3 + \epsilon]\}$. Thus,

$$0 \leq m^*(A) \leq \sigma(S) = 2\epsilon.$$

Since ϵ was arbitrary, it follows that $m^*(A) = 0$.

Example 1.1.4. The Lebesgue outer measure of \emptyset is 0. To see this, let $\epsilon > 0$ be given. Then $S = \{[-\epsilon, \epsilon]\}$ is a covering of \emptyset by closed intervals. Therefore,

$$m^*(\emptyset) \leq \sigma(S) = 2\epsilon.$$

Since ϵ was arbitrary, it follows that $m^*(\emptyset) = 0$.

Example 1.1.5. Let $A = [0, 1]$. The Lebesgue outer measure of A is 1. This should come as no surprise. After all, the length of this interval is 1. In fact, $S = \{[0, 1]\}$ is a covering of A by a single closed interval. Therefore,

$$m^*(A) \leq \sigma(S) = 1.$$

However, it is not an easy matter to prove that if S is a random covering of A by closed intervals, then $\sigma(S) \geq 1$. It is only your intuition that tells you that if we cover A by closed intervals, then the sum of the lengths of the intervals must be greater than the length of A . (Don't get me wrong—I'm not trying to tell you that your

intuition is incorrect. I am only pointing out that this has not been proved. Try writing a rigorous proof. It is not easy.) We will prove in Proposition 1.1.11 that $m^*(A) = 1$.

Example 1.1.6. Let $E = \{(x, 0) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\}$. Let $\epsilon > 0$ be given. Set

$$I_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon \leq y \leq \epsilon\}.$$

Then $S = \{I_\epsilon\}$ is a covering of E by closed intervals. Therefore,

$$m^*(E) \leq \sigma(S) = v(I_\epsilon) = 2\epsilon.$$

Since ϵ was arbitrary, $m^*(E) = 0$.

Compare Example 1.1.5 and Example 1.1.6. We think of both as line segments of length 1. However, the first of these is a subset of \mathbb{R}^1 while the second is a subset of \mathbb{R}^2 . It is this difference in dimension that accounts for the difference in the outer measure of these two seemingly similar sets. More generally, for constants $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n, c \in \mathbb{R}$, and fixed k , the set

$$A = \{x \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \dots, n \text{ for } x \neq k \text{ and } x_k = c\}$$

has Lebesgue outer measure 0.

Example 1.1.7. We will now compute the Lebesgue outer measure of what is known as the Cantor set, or the Cantor middle-third set. Just to make sure we are all thinking of the same set we will start with a description of the Cantor set.

Set

$$C_0 = [0, 1].$$

The next set in our construction is formed by deleting the open middle third from C_0 . In other words,

$$C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1].$$

C_1 consists of two intervals of length $\frac{1}{3}$. C_2 is formed by removing the open middle third from each of these intervals. Hence,

$$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1].$$

Continue with this process. In general, for each positive integer n , C_n consists of 2^n intervals of length $1/3^n$. C_{n+1} is formed from C_n by

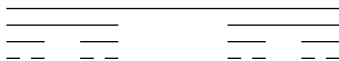


Figure 1.2. The first stages in the construction of the Cantor set.

deleting the open middle third of each of these intervals. The first few stages in this construction are illustrated in Figure 1.2. The Cantor set is what is left in the end. More precisely, the Cantor set is \mathcal{C} , where

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n.$$

The Cantor set has many remarkable features. For example, it is a closed, uncountable set that contains no intervals.

To see that the Cantor set is uncountable, note that every $x \in [0, 1]$ has a binary expansion. That is,

$$x = \sum_{n=1}^{\infty} \frac{b_n}{2^n} = ._{(2)}b_1b_2b_3\dots, \quad \text{where } b_n = 0 \text{ or } 1.$$

For example,

$$\begin{aligned} \frac{1}{5} &= ._{(2)}001100110011\dots & \text{and} \\ \frac{1}{2} &= ._{(2)}10000000\dots = ._{(2)}01111111\dots \end{aligned}$$

Some numbers, such as $\frac{1}{2}$, have more than one binary expansion. For the purpose of this example, when given a choice we will always choose the expansion that does not have a finite number of 1's. For example, we would choose $\frac{1}{2} = ._{(2)}01111\dots$

Similarly, every $x \in [0, 1]$ has a ternary expansion. That is,

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} = ._{(3)}a_1a_2a_3\dots, \quad \text{where } a_n = 0, 1, \text{ or } 2.$$

For example,

$$\begin{aligned}\frac{1}{2} &= \cdot_{(3)}11111111\dots, \\ \frac{1}{3} &= \cdot_{(3)}10000000\dots = \cdot_{(3)}02222222\dots.\end{aligned}$$

Again, given a choice of more than one ternary expansion, we will choose the one that does not have a finite number of nonzero digits. Moreover, x is in the Cantor set if and only if x has a ternary expansion where none of the digits are 1.

Finally, we define a function $f: [0, 1] \rightarrow \mathcal{C}$ as follows. For $x \in [0, 1]$ write x in its binary form,

$$x = \sum_{n=1}^{\infty} \frac{b_n}{2^n},$$

choosing the expansion that does not have a finite number of 1's when given a choice. Set $f(x)$ to be

$$f(x) = \sum_{n=1}^{\infty} \frac{2b_n}{3^n}.$$

Here $f(x)$ will be a ternary expansion where every digit is 0 or 2. Thus, $f(x)$ will be in the Cantor set. The function f is known as the **Cantor function**. The Cantor function is one-to-one, which gives a one-to-one correspondence between the interval $[0, 1]$ and a subset of \mathcal{C} . Hence, the Cantor set is uncountable.

But we are here to talk about Lebesgue outer measure. To compute the outer measure of the Cantor set, note that for every n , C_n provides us with a natural covering of \mathcal{C} by closed intervals. As noted above, C_n consists of 2^n intervals of length $\frac{1}{3^n}$. Thus,

$$m^*(\mathcal{C}) \leq 2^n \frac{1}{3^n} = \left(\frac{2}{3}\right)^n.$$

The only way this can hold for every positive integer n is for $m^*(\mathcal{C}) = 0$.

The Cantor set shows us that it is possible for an uncountable set to have Lebesgue outer measure 0.

Our goal is to generalize the notion of volume (or area, depending on dimension). One of the properties of volume that we want to retain is that if one set is contained in a second set, then the volume of the first should be less than or equal to the volume of the second. The next proposition shows this to be true for Lebesgue outer measure as well.

Proposition 1.1.8. *If $A \subseteq B \subseteq \mathbb{R}^n$, then $m^*(A) \leq m^*(B)$.*

Proof. Let S be a covering of B by closed intervals. It follows that S is also a covering of A by closed intervals. Thus,

$$m^*(A) \leq \sigma(S),$$

where S is any covering of B by closed intervals. Hence,

$$m^*(A) \leq \inf\{\sigma(S) \mid S \text{ is a covering of } B \text{ by closed intervals}\}.$$

Therefore, $m^*(A) \leq m^*(B)$, as claimed. \square

Another feature of volume we wish to retain is that the volume of the union of two sets is less than or equal to the sum of the volumes of the two sets. The next proposition asserts this to be true for outer measure. In addition, the result extends to a countable union of sets.

Proposition 1.1.9. *The following additivity properties hold for Lebesgue outer measure:*

(i) *For any two sets A and B ,*

$$m^*(A \cup B) \leq m^*(A) + m^*(B).$$

(ii) *For any countable collection of sets $\{A_n\}$,*

$$m^*\left(\bigcup A_n\right) \leq \sum m^*(A_n).$$

Since we are working with a countable collection of sets, $\{A_n\}$ may be either a finite collection or a countably infinite collection. To avoid having to make separate cases, we simply write $\bigcup A_n$ to indicate that this is either the union of a finite collection of sets or the union of a countably infinite collection of sets. We use a similar convention with the summation \sum . We just need to be sure that any assertions we make are true in both cases.

One way to proceed with the proof of (i) is to observe that if S is any covering of A by closed intervals and T is any covering of B by closed intervals, then $S \cup T$ forms a covering of $A \cup B$ by closed intervals. It follows that

$$m^*(A \cup B) \leq \sigma(S \cup T).$$

On the other hand, we can take the closed intervals in $S \cup T$ and look at those that came from S and those that came from T . It follows that

$$m^*(A \cup B) \leq \sigma(S \cup T) \leq \sigma(S) + \sigma(T),$$

keeping in mind that some intervals might appear in both S and T . We can then take first the infimum over all possible coverings of A and then the infimum over all possible coverings of B to obtain the desired result. This argument will generalize to any finite union of sets by using the principle of mathematical induction. However, in (ii) we wish to allow a countably infinite union of sets so induction will no longer apply. Hence, we will demonstrate i) in a manner that can be generalized to verify ii).

Proof. (i) If either $m^*(A)$ or $m^*(B)$ is infinite, then $m^*(A \cup B)$ is also infinite by Proposition 1.1.8. Thus, in this case the result is true by the convention that $+\infty \leq +\infty$. So, assume both $m^*(A)$ and $m^*(B)$ are finite.

Let $\epsilon > 0$ be given. By Remark 1.1.2 there exists S , a covering of A by closed intervals, with

$$m^*(A) \leq \sigma(S) < m^*(A) + \frac{\epsilon}{2}.$$

Similarly, there exists T , a covering of B by closed intervals, with

$$m^*(B) \leq \sigma(T) < m^*(B) + \frac{\epsilon}{2}.$$

Thus,

$$m^*(A \cup B) \leq \sigma(S \cup T) \leq \sigma(S) + \sigma(T) \leq m^*(A) + m^*(B) + \epsilon.$$

Since ϵ is arbitrary, it follows that

$$m^*(A \cup B) \leq m^*(A) + m^*(B),$$

as claimed.

- (ii) This is proved in the same spirit as (i). As in the earlier case, if $m^*(A_n)$ is infinite for some n , the result holds. Also, if $\sum m^*(A_n)$ is $+\infty$, the Lebesgue outer measure of any set is less than or equal to $+\infty$, so again the result holds. Therefore, we will assume that all of these quantities are finite.

Let $\epsilon > 0$ be given. For each n take S_n to be a covering of A_n by closed intervals with

$$m^*(A_n) \leq \sigma(S_n) < m^*(A_n) + \frac{\epsilon}{2^n}.$$

Thus,

$$\begin{aligned} m^*\left(\bigcup A_n\right) &\leq \sigma\left(\bigcup S_n\right) \\ &\leq \sum \sigma(S_n) \\ &\leq \sum \left(m^*(A_n) + \frac{\epsilon}{2^n}\right) \\ &= \sum m^*(A_n) + \epsilon. \end{aligned}$$

As before, ϵ is arbitrary. Therefore,

$$m^*\left(\bigcup A_n\right) \leq \sum m^*(A_n),$$

as claimed. \square

For those who are interested in the fine details, in the last part of this proof the reason we can only claim $\sigma\left(\bigcup S_n\right) \leq \sum \sigma(S_n)$ is that for distinct i and j , S_i and S_j might contain the same closed interval. This interval would only be counted once in $\sigma\left(\bigcup S_n\right)$ but would be counted more than once in $\sum \sigma(S_n)$.

Corollary 1.1.10. *If $A \subseteq B \subseteq \mathbb{R}^n$ and $m^*(B)$ is finite, then*

$$m^*(B) - m^*(A) \leq m^*(B \setminus A).$$

Proof. This is Exercise 7. \square

One of the drawbacks of Lebesgue outer measures is that even if sets A and B are disjoint, it is possible (if one assumes the Axiom of Choice) for $m^*(A \cup B) < m^*(A) + m^*(B)$. An example of this will

be discussed later in Example 1.3.6. This is not exactly a desirable result. The remedy for this is to define Lebesgue measure, but that is a topic for a later section. In the meantime, we will establish more properties of Lebesgue outer measure.

Proposition 1.1.11. *For any closed interval $I \subseteq \mathbb{R}^n$, $m^*(I) = v(I)$.*

As mentioned in Example 1.1.5, at first glance one might not fully appreciate the reason why we need a proof of Proposition 1.1.11. After all, $S = \{I\}$ is a perfectly acceptable covering of I by closed intervals. This guarantees that $m^*(I) \leq v(I)$. It is establishing the reverse inequality that is trickier, that is, we need to show that $v(I) \leq m^*(I)$. We can accomplish this by showing that if S is any covering of I by closed intervals, then $v(I) \leq \sigma(S)$. If I is covered by a countable collection of closed intervals S , it may seem obvious that the volume of I ought to be less than or equal to $\sigma(S)$, the sum of the volumes of the intervals in S ; but have you ever proved this? Remember you must be able to distinguish what you believe ought to be true versus what has or can be established. Think about writing a careful proof of this. In fact, proving this in the case that S is a *finite* collection of closed intervals is not exactly straightforward. The proof for this in \mathbb{R}^2 is outlined in Exercise 26 and Exercise 27. The strategies suggested can also be adapted to higher dimensions. The reason for doing these exercises is to show that the proof of this “obviously” requires careful bookkeeping and that doing them is not really necessary in order to proceed. However, one of the big messages in real analysis is that we cannot take it for granted that what may work in a finite case will also work in an infinite case. But in order to keep from getting too bogged down in the details, we will assume the results of Exercise 26 and Exercise 27. That is, we will proceed by assuming that if I is covered by a *finite* collection of closed intervals S , then $v(I) \leq \sigma(S)$. To use this to recover the intuitively clear result for the case when S is a countably infinite covering of I , we will take advantage of the compactness of I . Here, then, is a proof of Proposition 1.1.11.

Proof. We need to show that $v(I) \leq m^*(I)$. Because of the above discussion, we only need to prove that if $S = \{I_k\}$ is a countably infinite covering of I by closed intervals, then $v(I) \leq \sigma(S)$. Given

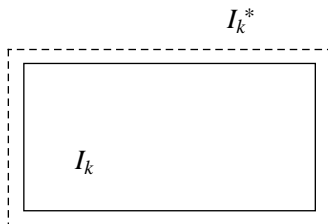


Figure 1.3. I_k^* is an expanded version of I_k .

$\epsilon > 0$, let I_k^* be an expanded version of I_k such that

$$I_k \subseteq \text{int}(I_k^*)$$

and

$$v(I_k^*) \leq (1 + \epsilon)v(I_k) .$$

This is illustrated in Figure 1.3. Then $\{\text{int}(I_k^*)\}$ is an open cover of I . That is,

$$I \subseteq \bigcup_{k=1}^{\infty} \text{int}(I_k^*) .$$

Since I is compact (I is closed and bounded), I can be covered by a finite subcover, say

$$I \subseteq \bigcup_{k=1}^M \text{int}(I_k^*) \subseteq \bigcup_{k=1}^M I_k^* .$$

This means that $S' = \{I_k^*\}_{k=1}^M$ is a covering of I by a finite number of closed intervals. (It is at this point where we will assume the intuitive result concerning covering I by a finite number of closed intervals or

the result of Exercise 27.) Hence,

$$\begin{aligned}
 v(I) &\leq \sum_{k=1}^M v(I_k^*) \\
 &\leq (1 + \epsilon) \sum_{k=1}^M v(I_k) \\
 &\leq (1 + \epsilon) \sum_{k=1}^{\infty} v(I_k) \\
 &= (1 + \epsilon)\sigma(S).
 \end{aligned}$$

As ϵ is arbitrary, $v(I) \leq \sigma(S)$. Consequently,

$$v(I) \leq \inf\{\sigma(S) \mid S \text{ is a covering of } I \text{ by closed intervals}\}.$$

Therefore, we obtain the necessary inequality $v(I) \leq m^*(I)$. \square

Example 1.1.12. We can now compute the Lebesgue outer measure of $B = [-1, 2] \cup \{3\}$. By Proposition 1.1.9,

$$m^*(B) \leq m^*([-1, 2]) + m^*({3}).$$

By Proposition 1.1.11 and Example 1.1.3,

$$m^*([-1, 2]) = 2 - (-1) = 3 \quad \text{and} \quad m^*({3}) = 0.$$

Thus,

$$m^*(B) \leq 3.$$

On the other hand, $[-1, 2] \subseteq B$. Hence,

$$3 = m^*([-1, 2]) \leq m^*(B).$$

Consequently, $m^*(B) = 3$.

The following theorem (note this is a theorem, not just a proposition) states that any set with finite Lebesgue outer measure is contained in some open set with arbitrarily close outer measure. This may not seem like such a great feature right now. But it tells us that instead of dealing with our original set, we can use an open set with almost the same outer measure. The advantage is that we know some useful properties of open sets.

Theorem 1.1.13. *Let $A \subseteq \mathbb{R}^n$ be a set with finite outer measure. For every $\epsilon > 0$ there is an open set G such that $A \subseteq G$ and*

$$m^*(G) < m^*(A) + \epsilon.$$

Proof. Given $\epsilon > 0$ there is a covering of A by closed intervals $S = \{I_k\}$ such that

$$\sigma(S) = \sum v(I_k) < m^*(A) + \frac{\epsilon}{2}.$$

(Here we are using the assumption that $m^*(A)$ is finite to obtain the strict inequality.) Because S can consist of either a finite collection of intervals or a countably infinite collection, we are using the convention mentioned after the statement of Proposition 1.1.9 and are not indicating whether the summation consists of a finite number of terms or an infinite number of terms.

For each k let I_k^* be an expanded version of I_k such that

$$I_k \subseteq \text{int}(I_k^*)$$

and

$$v(I_k^*) \leq v(I_k) + \frac{\epsilon}{2^{k+1}}.$$

Set

$$G = \bigcup \text{int}(I_k^*).$$

By construction G is an open set. Moreover, by Proposition 1.1.9 and Proposition 1.1.11,

$$\begin{aligned} m^*(G) &\leq \sum m^*(I_k^*) \\ &= \sum v(I_k^*) \\ &\leq \sum \left(v(I_k) + \frac{\epsilon}{2^{k+1}} \right) \\ &\leq \sigma(S) + \frac{\epsilon}{2} \\ &< m^*(A) + \epsilon. \end{aligned} \quad \square$$

Corollary 1.1.14. *Let $A \subseteq \mathbb{R}^n$. For every $\epsilon > 0$ there is an open set G such that $A \subseteq G$ and*

$$m^*(G) \leq m^*(A) + \epsilon.$$

Proof. If $m^*(A)$ is finite, we can use the open set G from the previous theorem. In the case that $m^*(A)$ is infinite, use $G = \mathbb{R}^n$. \square

One of the more tempting traps at this time is to believe that Theorem 1.1.13 and Corollary 1.1.14 tell us something about $m^*(G \setminus A)$. For example, Theorem 1.1.13 does tell us that

$$m^*(G) - m^*(A) < \epsilon.$$

Corollary 1.1.10 tells us that

$$m^*(G) - m^*(A) \leq m^*(G \setminus A).$$

Unfortunately, this last inequality goes in the wrong direction. We are unable to make any claims about $m^*(G \setminus A)$. Trust me—in the future it might be very tempting to make such a claim, but it isn't always true.

1.2. Lebesgue Measure

As mentioned before, one of the drawbacks of outer measure is that it may be possible for $m^*(A \cup B) < m^*(A) + m^*(B)$, even when A and B are disjoint sets. This is the idea illustrated by Example 1.3.6. The way we will avoid this is to place a restriction on which subsets of \mathbb{R}^n we will call measurable.

Definition 1.2.1. A set $E \subseteq \mathbb{R}^n$ is **Lebesgue measurable** if for every $\epsilon > 0$ there is an open set G so that $E \subseteq G$ and

$$m^*(G \setminus E) < \epsilon.$$

In this case we define the **Lebesgue measure** of E , denoted $m(E)$, to be

$$m(E) = m^*(E).$$

In Chapter 4 we will encounter other measures and outer measures. Until that point, however, any time we say outer measure and measure, we are referring to Lebesgue outer measure and Lebesgue measure, respectively.

Example 1.2.2. We will show that $E = \{3\}$ is Lebesgue measurable.

Given $\epsilon > 0$ let $G = (3 - \frac{\epsilon}{3}, 3 + \frac{\epsilon}{3})$. By Proposition 1.1.8 and by Proposition 1.1.11,

$$\begin{aligned} m^*(G \setminus E) &= m^*((3 - \frac{\epsilon}{3}, 3) \cup (3, 3 + \frac{\epsilon}{3})) \\ &\leq m^*([3 - \frac{\epsilon}{3}, 3 + \frac{\epsilon}{3}]) \\ &= \frac{2\epsilon}{3} < \epsilon. \end{aligned}$$

Example 1.2.3. If G is an open set, then $m^*(G \setminus G) = m^*(\emptyset) = 0 < \epsilon$ for every $\epsilon > 0$. Consequently, every open set in \mathbb{R}^n is Lebesgue measurable.

Example 1.2.4. Every set with Lebesgue outer measure 0 is measurable. To verify this, suppose $E \subseteq \mathbb{R}^n$ is a set with $m^*(E) = 0$. Given $\epsilon > 0$, by Theorem 1.1.13, there is an open set G containing E with

$$m^*(G) < m^*(E) + \epsilon = \epsilon.$$

By Proposition 1.1.8,

$$m^*(G \setminus E) \leq m^*(G) < \epsilon.$$

Hence, E is Lebesgue measurable.

Theorem 1.2.5. Let $\{E_k\}$ be a countable collection of Lebesgue measurable sets. Then

$$E = \bigcup E_k$$

is Lebesgue measurable and

$$m(E) \leq \sum m(E_k).$$

Proof. Let $\epsilon > 0$ be given. We must show there exists an open set G containing $E = \bigcup E_k$ such that $m^*(G \setminus E) < \epsilon$.

For each k there exists an open set G_k containing E_k such that

$$m^*(G_k \setminus E_k) < \frac{\epsilon}{2^k}.$$

Set

$$G = \bigcup G_k.$$

It follows that

$$\bigcup G_k \setminus \bigcup E_k \subseteq \bigcup (G_k \setminus E_k).$$

Thus, G is an open set containing E and

$$\begin{aligned} m^*(G \setminus E) &= m^*\left(\bigcup G_k \setminus \bigcup E_k\right) \\ &\leq m^*\left(\bigcup (G_k \setminus E_k)\right) \\ &\leq \sum m^*(G_k \setminus E_k) \\ &< \sum \frac{\epsilon}{2^k} \leq \epsilon. \end{aligned}$$

The assertion that

$$m(E) \leq \sum m(E_k)$$

follows from the definition of Lebesgue measure and Proposition 1.1.9. \square

We will use this to show that some basic sets, namely intervals, are Lebesgue measurable.

Example 1.2.6. Let $I \subseteq \mathbb{R}^n$ be a closed interval in \mathbb{R}^n . Then I is the union of its interior, which is an open set, and its sides. The open interior is measurable by Example 1.2.3. The sides are subsets of hyperplanes, which have Lebesgue outer measure 0. Thus, the sides have Lebesgue outer measure 0 and are Lebesgue measurable by Example 1.2.4. Consequently, I is the countable union of measurable sets. By Theorem 1.2.5, I is Lebesgue measurable. To be a little more precise about this, we write

$$\begin{aligned} I &= \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \dots, n\} \\ &= \{x \in \mathbb{R}^n \mid a_i < x_i < b_i\} \\ &\quad \cup \bigcup_{k=1}^n \{x \in I \mid x_k = a_k\} \cup \bigcup_{k=1}^n \{x \in I \mid x_k = b_k\}. \end{aligned}$$

Furthermore, since I is Lebesgue measurable,

$$m(I) = m^*(I) = v(I)$$

by Proposition 1.1.11.

We are building towards our goal of showing that Lebesgue measure has the feature we desire, that is, the measure of the union of disjoint Lebesgue measurable sets is the sum of the measures. There

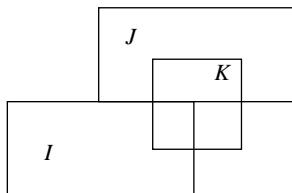


Figure 1.4. I and J are nonoverlapping, while I and K overlap, as do J and K .

are situations close to this. What if two sets abut or are adjacent to each other? To be more precise, we will consider nonoverlapping intervals.

Definition 1.2.7. Let I and J be two closed intervals in \mathbb{R}^n . I and J are said to be **nonoverlapping** if I and J have disjoint interiors.

In other words, two closed intervals I and J are nonoverlapping if $I \cap J$ is either empty or consists of only points that are on the boundary of both I and J .

Example 1.2.8. Let

$$\begin{aligned} I &= \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 2\}, \\ J &= \{(x, y) \mid 2 \leq x \leq 6, 2 \leq y \leq 4\}, \\ K &= \{(x, y) \mid 3 \leq x \leq 5, 1 \leq y \leq 3\}. \end{aligned}$$

Then I and J are nonoverlapping. Neither I and K nor J and K are nonoverlapping.

Lemma 1.2.9. Let $\{I_n\}_{n=1}^M$ be a finite collection of pairwise nonoverlapping closed intervals. Then

$$m\left(\bigcup_{n=1}^M I_n\right) = \sum_{n=1}^M v(I_n).$$

Proof. It follows from Example 1.2.6 and Theorem 1.2.5 that $\bigcup_{n=1}^M I_n$ is measurable and

$$m\left(\bigcup_{n=1}^M I_n\right) \leq \sum_{n=1}^M v(I_n).$$

We need only establish the reverse inequality.

As in Proposition 1.1.11, if $S = \{J_l\}$ is a covering of $\bigcup_{n=1}^M I_n$ by closed intervals, our intuition tells us that

$$\sum_{n=1}^M v(I_n) \leq \sum v(J_l) = \sigma(S).$$

As before, actually writing a proof of this in just the finite case is quite involved. But one can carry this out by making appropriate adjustments to Exercise 26 and Exercise 27.

Therefore, similar to the proof of Proposition 1.1.11, we will assume the desired inequality is true if we cover $\bigcup_{n=1}^M I_n$ by a finite collection of closed intervals and use this to proceed in showing that the inequality remains true if S is a countably infinite collection of intervals. Let $S = \{J_l\}$ be a covering of $\bigcup_{n=1}^M I_n$ by closed intervals. Let $\epsilon > 0$. Let J_l^* be an expanded version of J_l such that

$$J_l \subseteq \text{int}(J_l^*)$$

and

$$v(J_l^*) \leq (1 + \epsilon)v(J_l).$$

Then $\{\text{int}(J_l^*)\}$ is an open cover of the compact set $\bigcup_{n=1}^M I_n$. Thus, for some integer N ,

$$\bigcup_{k=1}^M I_k \subseteq \bigcup_{l=1}^N \text{int}(J_l^*) \subseteq \bigcup_{l=1}^N J_l^*.$$

We have now covered $\bigcup_{n=1}^M I_n$ by a finite collection of closed intervals, so

$$\begin{aligned} \sum_{k=1}^M v(I_k) &\leq \sum_{l=1}^N v(J_l^*) \\ &\leq (1 + \epsilon) \sum_{l=1}^N v(J_l) \\ &\leq (1 + \epsilon) \sigma(S). \end{aligned}$$

Since ϵ was arbitrary, it follows that

$$\sum_{k=1}^M v(I_k) \leq \sigma(S)$$

for any covering S of $\bigcup_{n=1}^M I_n$ by closed intervals. Therefore,

$$\sum_{k=1}^M v(I_k) \leq m \left(\bigcup_{n=1}^M I_n \right),$$

as required. \square

So far we have shown that not only is it the case that the finite union of nonoverlapping intervals is Lebesgue measurable, we actually can find the Lebesgue measure of such a set by adding the volumes of the intervals. We will next show that any nonempty open set is the countably infinite union of nonoverlapping intervals, and we actually can find its measure by adding the volumes of the intervals.

Lemma 1.2.10. *Every nonempty open set $G \subseteq \mathbb{R}^n$ can be written as the countable union of pairwise nonoverlapping closed intervals.*

Proof. Let $G \subseteq \mathbb{R}^n$ be an open set. Divide \mathbb{R}^n into nonoverlapping intervals along the hyperplanes $x_i = k$, where $k \in \mathbb{Z}$, thus creating a countable collection of closed intervals. Set aside those closed intervals which are completely contained in G . This is our first layer; we have set aside a countable number of closed intervals. For the second step, subdivide each remaining closed interval into subintervals along the hyperplanes $x_i = k/2$, where $k \in \mathbb{Z}$. This takes each

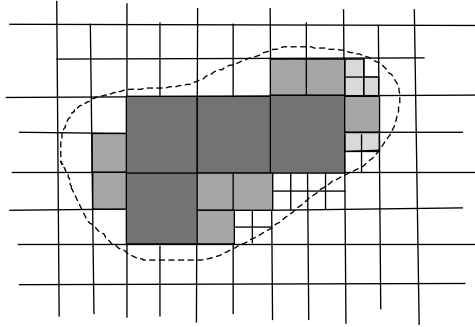


Figure 1.5. Each of the intervals contained in G are retained; the rest are subdivided.

remaining closed interval and creates 2^n nonoverlapping closed subintervals. Once again we have a countable collection of closed intervals. From these, set aside those closed intervals contained in G , again a countable number. Next, subdivide each remaining closed interval into subintervals along the hyperplanes $x_i = k/2^2$, where $k \in \mathbb{Z}$. This takes each remaining closed interval and creates 2^n nonoverlapping closed subintervals. Once again we have a countable collection of closed intervals. From these, set aside those closed intervals contained in G , again a countable number. This is the third step of the process. (See Figure 1.5.) Repeat this process ad infinitum.

We now have set aside a countable collection of nonoverlapping closed intervals $\{I_k\}$ each contained in G . It follows immediately that

$$\bigcup I_k \subseteq G.$$

We will show the reverse inclusion.

Let $x \in G$. Suppose to the contrary $x \notin \bigcup I_k$. Since G is open, there is an open ball B centered at x contained in G . Eventually in our process of subdividing intervals, the closed intervals under consideration will have diameters smaller than the radius of this ball. At the first such stage, x must be in one of the closed intervals which in turn will be contained in B . But B is a subset of G , so this closed interval is now contained in G . Therefore, this interval will now be placed in our collection $\{I_k\}$. But this contradicts the assumption

that $x \notin \bigcup I_k$. Hence, $x \in \bigcup I_k$, so

$$G \subseteq \bigcup I_k.$$

Therefore,

$$G = \bigcup I_k;$$

that is, G can be written as a countable union of nonoverlapping closed intervals. \square

Note that each of the intervals I_k is closed but G is open. If we only required a finite union of nonoverlapping closed intervals, this would mean G is a closed set. The only nonempty subset of \mathbb{R}^n that is both open and closed is \mathbb{R}^n itself. But \mathbb{R}^n cannot be written as a finite union of closed intervals because each interval is bounded. Therefore, it must be the case that the countable union guaranteed by the previous theorem must be a countably infinite union. We can go a little further with this and say something about the measure of the open set G and the volumes of the intervals created by this lemma.

Corollary 1.2.11. *Every open set $G \subseteq \mathbb{R}^n$ can be written as a countably infinite union of nonoverlapping closed intervals $G = \bigcup_{k=1}^{\infty} I_k$ with*

$$m(G) = \sum_{k=1}^{\infty} v(I_k).$$

Proof. By Lemma 1.2.10,

$$G = \bigcup_{k=1}^{\infty} I_k,$$

where $\{I_k\}$ is a countable collection of nonoverlapping closed intervals. By Proposition 1.1.9 and Proposition 1.1.11,

$$m(G) \leq \sum_{k=1}^{\infty} v(I_k).$$

Therefore, it suffices to establish the reverse inequality.

By Lemma 1.2.9, for each integer $M \in \mathbb{N}$,

$$\sum_{k=1}^M v(I_k) = m\left(\bigcup_{k=1}^M I_k\right) \leq m(G).$$

Taking the limit as M approaches infinity establishes the reverse inequality. \square

We are accustomed to thinking about the distance between two points in \mathbb{R}^n . That is, if $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ then the distance between x and y is

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}.$$

We can also define the distance between nonempty subsets of \mathbb{R}^n .

Definition 1.2.12. Let E_1 and E_2 be nonempty subsets of \mathbb{R}^n . The **distance between E_1 and E_2** , denoted $d(E_1, E_2)$, is defined as

$$d(E_1, E_2) = \inf \{d(x, y) \mid x \in E_1, y \in E_2\}.$$

Notice that it is possible for $d(E_1, E_2) = 0$ even if E_1 and E_2 are disjoint.

Example 1.2.13. Consider the following two subsets of \mathbb{R}^1 , $E_1 = [0, 1)$, and $E_2 = (1, 2]$. Then E_1 and E_2 are disjoint, but $d(E_1, E_2) = 0$.

It is also possible for $d(E_1, E_2) = 0$ even if E_1 and E_2 are disjoint closed sets.

Example 1.2.14. Let

$$E_1 = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y \geq \frac{1}{x}\},$$

$$E_2 = \{(x, y) \in \mathbb{R}^2 \mid y \leq 0\}.$$

Then once again E_1 and E_2 are disjoint, but $d(E_1, E_2) = 0$.

Remark 1.2.15. It is a fact that if E_1 and E_2 are disjoint compact sets, then $d(E_1, E_2) > 0$.

The previous remark is actually a theorem resulting from the definition of compactness. Although we will not prove it here, the interested reader may find the proof an interesting exercise. The reason for stating this remark is that we will use it in Theorem 1.2.17.

We will now show that if there is a positive distance between two sets, the outer measure of the union is the sum of the outer measures of the two sets.

Lemma 1.2.16. *If $d(E_1, E_2) > 0$, then*

$$m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2).$$

Proof. If one of $m^*(E_1)$ or $m^*(E_2)$ is infinite, then $m^*(E_1 \cup E_2)$ will also be infinite by Proposition 1.1.8, and the result is true. So we will assume both of these quantities are finite.

By Proposition 1.1.9 we know that

$$m^*(E_1 \cup E_2) \leq m^*(E_1) + m^*(E_2).$$

Hence, we need to verify the reverse inequality.

Let $\epsilon > 0$ be given. There is a covering $S = \{I_k\}$ of $E_1 \cup E_2$ by closed intervals so that

$$\sigma(S) = \sum v(I_k) < m^*(E_1 \cup E_2) + \epsilon.$$

Take these intervals and subdivide them, if necessary, into nonoverlapping closed subintervals with diameter smaller than $\frac{1}{2}d(E_1, E_2)$. Call this new covering of $E_1 \cup E_2$ S^* . By construction, $\sigma(S^*) = \sigma(S)$. All of the intervals in S^* have diameter less than $d(E_1, E_2)$, so none will overlap both E_1 and E_2 . Let

$$\begin{aligned} S_1 &= \{J_l \in S^* \mid J_l \cap E_1 \neq \emptyset\}, \\ S_2 &= \{J_l \in S^* \mid J_l \cap E_2 \neq \emptyset\}, \\ S_3 &= \{J_l \in S^* \mid J_l \cap (E_1 \cup E_2) = \emptyset\}. \end{aligned}$$

In other words, we have taken our new covering of $E_1 \cup E_2$ by closed intervals and sorted the intervals by whether they touch E_1 , or E_2 , or neither. Thus, S_1 is a covering of E_1 by closed intervals, S_2 is a covering of E_2 by closed intervals, and

$$\begin{aligned} m^*(E_1) + m^*(E_2) &\leq \sigma(S_1) + \sigma(S_2) \\ &\leq \sigma(S_1) + \sigma(S_2) + \sigma(S_3) \\ &= \sigma(S^*) \\ &< m^*(E_1 \cup E_2) + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we have

$$m^*(E_1) + m^*(E_2) \leq m^*(E_1 \cup E_2).$$

Therefore,

$$m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2),$$

as claimed. \square

Although our goal is to show that the Lebesgue measure of the union of disjoint sets is the sum of the measures, we still don't have many measurable sets at our disposal. We know that sets with zero outer measure, open sets, intervals, and finite union of the above are measurable. What other sets are measurable? Are closed sets measurable? Given a specific example of a closed set, chances are that one could use what we already know to show that a specific set is measurable. However, we will cover all closed sets with the following theorem.

Theorem 1.2.17. *Every closed subset of \mathbb{R}^n is Lebesgue measurable.*

Proof. Let $F \subseteq \mathbb{R}^n$ be a closed set. We will consider two cases.

- (i) First assume that F is a bounded set. Hence, F is a compact set and $m^*(F)$ is finite. Let $\epsilon > 0$. By Theorem 1.1.13, there is an open set G containing F with

$$m^*(G) < m^*(F) + \epsilon.$$

Remember the caution after Corollary 1.1.14. This alone does not tell us the result we want, that $m^*(G \setminus F) < \epsilon$. It takes a surprising amount of effort to reach this conclusion.

The set $G \setminus F$ is an open set. Thus, by Lemma 1.2.10, $G \setminus F$ can be written as a countable union of nonoverlapping closed intervals, say

$$G \setminus F = \bigcup I_k.$$

For each positive integer N , $\bigcup_{k=1}^N I_k$ is a closed and bounded set, and so is compact. Moreover, $F \cap \bigcup_{k=1}^N I_k = \emptyset$. By Remark 1.2.15,

$$d\left(F, \bigcup_{k=1}^N I_k\right) > 0.$$

For each positive integer N ,

$$\begin{aligned} \sum_{k=1}^N v(I_k) &= m^* \left(\bigcup_{k=1}^N I_k \right) \\ &= m^* \left(F \cup \bigcup_{k=1}^N I_k \right) - m^*(F) \\ &\leq m^*(G) - m^*(F) < \epsilon. \end{aligned}$$

By taking the limit as N goes to ∞ ,

$$\sum_{k=1}^{\infty} v(I_k) \leq \epsilon.$$

Therefore,

$$\begin{aligned} m^*(G \setminus F) &= m^* \left(\bigcup I_k \right) \\ &\leq \sum_{k=1}^{\infty} v(I_k) \leq \epsilon. \end{aligned}$$

Hence, F is measurable.

(ii) Assume F is unbounded. Set

$$\overline{B_R} = \{x \in \mathbb{R}^n \mid |x| \leq R\} \quad \text{and}$$

$$F_N = F \cap \overline{B_N}.$$

Thus, for each integer N , F_N is a closed and bounded set. By (i), F_N is a measurable set for each integer N . Moreover,

$$F = \bigcup_{N=1}^{\infty} F_N$$

is a countable union of measurable sets. Therefore, F is measurable by Theorem 1.2.5. \square

In this last proof we tackled the case where F is unbounded by writing F as the countable union of bounded sets. This is a common strategy; to deal with an unbounded set we simply write it as the countable union of bounded sets. We will see this technique used again.

Theorem 1.2.18. *Let $E \subseteq \mathbb{R}^n$. If E is Lebesgue measurable, then*

$$E^c = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x \notin E\}$$

is measurable.

Proof. Assume $E \subseteq \mathbb{R}^n$ is measurable. Then for every positive integer k there exists an open set G_k containing E such that

$$m^*(G_k \setminus E) < \frac{1}{k}.$$

For every k ,

$$G_k^c \subseteq E^c,$$

hence

$$\bigcup_{k=1}^{\infty} G_k^c \subseteq E^c.$$

Let $Z = E^c \setminus \bigcup_{k=1}^{\infty} G_k^c$ so that

$$E^c = Z \cup \bigcup_{k=1}^{\infty} G_k^c.$$

For each k , G_k^c is a closed set and hence is measurable by Theorem 1.2.17. It remains to show that Z is measurable.

We will show that Z has Lebesgue outer measure 0. Then by Example 1.2.4, Z will be Lebesgue measurable. For each integer k ,

$$Z \subseteq E^c \setminus G_k^c = G_k \setminus E.$$

Thus,

$$m^*(Z) \leq m^*(G_k \setminus E) < \frac{1}{k}$$

for each positive integer k . It follows that $m^*(Z) = 0$.

Therefore, E^c may be written as

$$E^c = Z \cup \bigcup_{k=1}^{\infty} G_k^c,$$

the union of measurable sets. Thus, E^c is measurable. \square

Now we have shown that open sets, closed sets, countable unions of measurable sets, and complements of measurable sets are measurable. One might wonder if the intersection of measurable sets is also measurable. This is indeed the case. We state the following proposition and leave the proof to the reader as an exercise.

Proposition 1.2.19. *Let $\{A_j\}$ be a countable collection of Lebesgue measurable subsets of \mathbb{R}^n . Then the set*

$$A = \bigcap A_j$$

is Lebesgue measurable.

Proof. This is Exercise 11. \square

We have now shown that the collection of Lebesgue measurable sets contains the empty set, is closed under set complement, and is closed under countable unions. Such a collection of sets is known as a σ -algebra. We will encounter σ -algebras in Chapter 4. For now, we make the observation that since all open sets are measurable and the collection of measurable sets is closed under countable intersections, a set that is the intersection of a countable collection of open sets must be measurable. Similarly, all closed sets are measurable. Thus, a set that is the union of a countable collection of closed sets is also measurable.

Definition 1.2.20. A set H is of *type* G_δ if H is the intersection of a countable collection of open sets. A set H is of *type* F_σ if H is the union of a countable collection of closed sets.

We have already shown that all sets of type G_δ or of type F_σ are Lebesgue measurable. But to get a better feel for these sets we will look at some examples.

Example 1.2.21. The half open interval $(0, 1]$ in \mathbb{R}^1 is of type G_δ since

$$(0, 1] = \bigcap_{n=1}^{\infty} \left(0, 1 + \frac{1}{n}\right).$$

Example 1.2.22. The set

$$A = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x < 2 \text{ and } 3 < y \leq 5\}$$

in \mathbb{R}^2 is of type F_σ since

$$A = \bigcup_{n=1}^{\infty} \left\{ (x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2 - \frac{1}{n} \text{ and } 3 + \frac{1}{n} \leq y \leq 5 \right\}.$$

Your next step should be to write down an example of a set and determine if it is of type G_δ or of type F_σ . Once you do so, you will discover that there are many examples of sets that are of type G_δ or of type F_σ . In fact, it is hard to imagine a set that is not one of these two types. In Chapter 4 we will show (assuming the Axiom of Choice) that there are indeed sets that are not one of these two types.

Our definition for a set to be Lebesgue measurable required that the set be contained in an open set where the excess has arbitrarily small outer measure. One can also require that the set contain a closed set where the excess has arbitrarily small measure.

Proposition 1.2.23. *Let $E \subseteq \mathbb{R}^n$ be a set. E is Lebesgue measurable if and only if for every $\epsilon > 0$ there is a closed set F with $F \subseteq E$ and $m^*(E \setminus F) < \epsilon$.*

Proof. This is Exercise 15. □

Finally, we arrive at the advantage that Lebesgue measure has over outer measure. That is, the measure of the union of a countable collection of disjoint measurable sets is the sum of the measures.

Theorem 1.2.24. *Let $\{E_k\}$ be a countable collection of pairwise disjoint Lebesgue measurable subsets of \mathbb{R}^n . Then*

$$m\left(\bigcup E_k\right) = \sum m(E_k).$$

Proof. First we observe that this result clearly holds if for some $k \in \mathbb{N}$, $m(E_k)$ is infinite. Thus, we may assume that for every $k \in \mathbb{N}$, $m(E_k)$ is finite.

- (i) First consider the case where each E_k is a bounded set. By Proposition 1.1.9 we have the inequality

$$m\left(\bigcup E_k\right) \leq \sum m(E_k).$$

We need to show the reverse inequality.

Let $\epsilon > 0$ be given. By Proposition 1.2.23 (proved in Exercise 15), for each $k \in \mathbb{N}$ there is a closed set $F_k \subseteq E_k$ with

$$m^*(E_k \setminus F_k) < \frac{\epsilon}{2^k}.$$

In this case, $m(E_k) = m^*(E_k)$ is finite for each k . By Corollary 1.1.10,

$$m^*(E_k) - m^*(F_k) \leq m^*(E_k \setminus F_k) < \frac{\epsilon}{2^k}$$

for each integer k . Since $\{E_k\}$ is a countable collection of pairwise disjoint bounded sets, $\{F_k\}$ is a collection of pairwise disjoint closed sets. Moreover, since each E_k is bounded, each F_k is bounded as well. Hence, $\{F_k\}$ is a collection of pairwise disjoint compact sets. Consequently, pairwise there is a positive distance between these sets. By induction and Lemma 1.2.16, for every positive integer M ,

$$m\left(\bigcup_{k=1}^M F_k\right) = \sum_{k=1}^M m(F_k).$$

Therefore, for every positive integer M ,

$$\sum_{k=1}^M m(F_k) \leq m\left(\bigcup E_k\right).$$

Thus, for every M ,

$$\sum_{k=1}^M \left(m(E_k) - \frac{\epsilon}{2^k} \right) \leq m \left(\bigcup E_k \right),$$

which implies

$$\sum_{k=1}^M m(E_k) \leq m \left(\bigcup E_k \right) + \epsilon.$$

Taking the limit as M approaches infinity yields

$$\sum m(E_k) \leq m \left(\bigcup E_k \right) + \epsilon.$$

Since ϵ was arbitrary, we have established the desired inequality.

- (ii) We now consider the situation where it is not the case that each E_k is bounded. As in Theorem 1.2.17, we will reduce this case to our earlier bounded case. Set

$$E_{k,1} = E_k \cap \overline{B_1}$$

and

$$E_{k,j} = E_k \cap (\overline{B_j} \setminus \overline{B_{j-1}})$$

for $j = 2, 3, \dots$. Now $\{E_{k,j}\}$ is a countable collection of pairwise disjoint, bounded, measurable sets. By part (i),

$$\sum_j m(E_{k,j}) = m \left(\bigcup_j E_{k,j} \right) = m(E_k).$$

Also by part (i),

$$m \left(\bigcup_{k,j} E_{k,j} \right) = \sum_{k,j} m(E_{k,j}).$$

Therefore,

$$\begin{aligned}
 m\left(\bigcup_k E_k\right) &= m\left(\bigcup_k \bigcup_j E_{k,j}\right) \\
 &= \sum_k \sum_j m(E_{k,j}) \\
 &= \sum_k \left(\sum_j m(E_{k,j})\right) \\
 &= \sum_k m(E_k).
 \end{aligned}$$

Thus,

$$m\left(\bigcup E_k\right) = \sum m(E_k),$$

as claimed. \square

1.3. A Nonmeasurable Set

In this section we will show the existence of a nonmeasurable set in \mathbb{R}^1 . The proof relies on the Axiom of Choice and can be generalized to \mathbb{R}^n . The only place in later chapters that we will use the material from this section is in Remark 4.1.14 so this section can be omitted. On the other hand, to see that we needed to make the seemingly awkward definition of measurability in order to prove something like Theorem 1.2.24 is interesting in its own right.

The main tool to show the existence of a nonmeasurable set is the following lemma.

Lemma 1.3.1. *Let $E \subseteq \mathbb{R}^1$ be a measurable set. If $m(E) > 0$, including infinite measure, then the set of all arithmetic differences*

$$D_E = \{x - y \mid x, y \in E\}$$

contains an interval centered at 0.

The proof of this lemma is somewhat long and fairly technical. Our real goal is to show the existence of a nonmeasurable set. In order to keep from getting bogged down in the details of the lemma, we will defer its proof until later. First, we will illustrate this lemma.

Example 1.3.2. Let $E = \{-1\} \cup [2, 3) \cup (4, 6]$. Then $m(E) = 3$ and

$$D_E = [-7, -5] \cup [-4, 4] \cup (5, 7],$$

which contains an interval centered at 0.

Example 1.3.3. Although the Cantor set C has measure 0, as we will show, the corresponding set of arithmetic differences is

$$D_C = [-1, 1].$$

Since $C \subseteq [0, 1]$, it must be the case that $D_C \subseteq [-1, 1]$. We will show that the reverse inclusion holds as well.

Let $\alpha \in [-1, 1]$. Then $\frac{1}{2}(\alpha + 1) \in [0, 1]$ has a ternary expansion, say

$$\frac{1}{2}(\alpha + 1) = \cdot_{(3)}c_1c_2c_3\dots, \quad \text{where } c_i = 0, 1, \text{ or } 2.$$

Set

$$\begin{aligned} x &= \cdot_{(3)}a_1a_2a_3\dots, \\ y &= \cdot_{(3)}b_1b_2b_3\dots, \end{aligned}$$

where

$$a_i = \begin{cases} 0 & \text{if } c_i = 0 \text{ or } 1, \\ 2 & \text{if } c_i = 2 \end{cases} \quad \text{and} \quad b_i = \begin{cases} 0 & \text{if } c_i = 0, \\ 2 & \text{if } c_i = 1 \text{ or } 2. \end{cases}$$

Thus, x and y are both in the Cantor set (they each have a ternary expansion consisting of only 0's and 2's). By symmetry, $(1 - y)$ is also in the Cantor set. Also, $a_i + b_i = 2c_i$ for each i . Therefore,

$$x + y = 2 \left(\frac{1}{2}(\alpha + 1) \right) = \alpha + 1.$$

Consequently,

$$\alpha = x - (1 - y).$$

We have now shown that α is the difference of two members of the Cantor set.

Observe that in this case, Lemma 1.3.1 does not apply because $m(C) = 0$. Even so, the corresponding set of arithmetic differences does contain an interval centered at the origin. Take a moment to think about what this means. At first glance the Cantor set seems almost sparse. Yet the corresponding set of differences is an interval of length 2!

Example 1.3.4. Let $A = \{2, 6\}$. The corresponding set of arithmetic differences is

$$D_A = \{-4, 0, 4\}.$$

This set does not contain an interval. However, this does not contradict Lemma 1.3.1 since $m(A) = 0$.

The theorem that gives us a nonmeasurable set is due to Vitali.

Theorem 1.3.5. *There exists a nonmeasurable subset of \mathbb{R}^1 .*

Proof. Define the equivalence relation \sim on \mathbb{R} by

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathbb{Q}.$$

This partitions \mathbb{R} into equivalence classes. For example, the equivalence class of 3, denoted $[3]_{\sim}$, is

$$\begin{aligned} [3]_{\sim} &= \{x \in \mathbb{R} \mid x \sim 3\} \\ &= \{x \in \mathbb{R} \mid x - 3 \in \mathbb{Q}\} \\ &= \mathbb{Q}, \end{aligned}$$

while

$$\begin{aligned} [\pi]_{\sim} &= \{x \in \mathbb{R} \mid x \sim \pi\} \\ &= \{\pi + q \mid q \in \mathbb{Q}\}. \end{aligned}$$

Two of these equivalence classes are either the same or disjoint. In fact,

$$\begin{aligned} [x]_{\sim} = [y]_{\sim} & \quad \text{if and only if} \quad x \sim y, \\ [x]_{\sim} \cap [y]_{\sim} = \emptyset & \quad \text{if and only if} \quad x \not\sim y. \end{aligned}$$

For example, $[\sqrt{2} + \frac{2}{3}]_{\sim} = [\sqrt{2}]_{\sim}$, while $[\pi]_{\sim} \cap [\frac{\pi}{2}]_{\sim} = \emptyset$. Moreover, there are an uncountable number of these equivalence classes.

It is here that we employ the Axiom of Choice. Form a set A by picking exactly one element from each equivalence class. We will show that A must be nonmeasurable. To the contrary, assume that A is measurable. Then either (i) $m(A) > 0$ or (ii) $m(A) = 0$.

- (i) Assume A is measurable and $m(A) > 0$. By Lemma 1.3.1, the set of arithmetic differences D_A contains an interval centered at 0. However, if x and y are in different equivalence

classes, then $x - y \notin \mathbb{Q}$. Hence, the only rational number in D_A is 0, contradicting Lemma 1.3.1. Therefore, it cannot be the case that $m(A) > 0$.

- (ii) Assume A is measurable and $m(A) = 0$. The set of rational numbers is countable. So there exists $\{r_k\}_{k=1}^{\infty}$, a counting of \mathbb{Q} . That is, $\mathbb{Q} = \{r_k\}_{k=1}^{\infty}$. For each $k \in \mathbb{N}$ let

$$A_k = \{a + r_k \mid a \in A\}.$$

By Exercise 8, $m(A_k) = m(A) = 0$.

If $x \in \mathbb{R}$, then $x \sim a$ for some $a \in A$. After all, $x \in [x]_{\sim}$ and A contains exactly one element from $[x]_{\sim}$. Thus, $x = a + q$ for some $q \in \mathbb{Q}$. Therefore,

$$\bigcup_{k=1}^{\infty} A_k = \mathbb{R}.$$

On the other hand, if $k \neq j$, then $A_k \neq A_j$, so $\{A_k\}$ is a countable collection of pairwise disjoint measurable sets. Therefore,

$$m(\mathbb{R}) = m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k) = 0,$$

a contradiction.

Therefore, A must be a nonmeasurable set. □

Now that we have seen how Lemma 1.3.1 is used to show the existence of a nonmeasurable set, we will turn to its proof.

Proof. Assume E is a measurable subset of \mathbb{R} with positive measure. Our goal is to show that the set of arithmetic differences D_E contains an interval centered at 0. If E is not bounded, for $n \in \mathbb{N}$ set $E_n = E \cap [-n, n]$. Then $E = \bigcup E_n$ and

$$m(E) \leq \sum_{n=1}^{\infty} m(E_n).$$

Thus, $m(E_n) > 0$ for some n . Also,

$$D_{E_n} \subseteq D_E.$$

If D_{E_n} contains an interval centered at 0, then D_E will as well. Hence, without loss of generality, we may assume that E is bounded, for if not, we simply work with the E_n with positive measure.

By Theorem 1.1.13, given any $\epsilon > 0$ there is an open set G containing E with

$$m(G) < m(E) + \epsilon.$$

In particular, this is the case for $\epsilon = \frac{1}{3}m(E) > 0$. That is, there is an open set G containing E with

$$m(G) < \frac{4}{3}m(E).$$

By Lemma 1.2.10, the open set G is the union of countably many nonoverlapping closed intervals, say

$$G = \bigcup_{k=1}^{\infty} I_k.$$

Also, since $E \subseteq G$,

$$E = \bigcup_{k=1}^{\infty} (E \cap I_k).$$

Moreover, by Corollary 1.2.11,

$$m(G) = \sum_{k=1}^{\infty} m(I_k).$$

Next, we claim that for some k , $m(I_k) \leq \frac{4}{3}m(E \cap I_k)$. If, to the contrary, $m(I_k) > \frac{4}{3}m(E \cap I_k)$ for every k , then

$$\begin{aligned} \frac{4}{3}m(E) &= \frac{4}{3}m\left(\bigcup_{k=1}^{\infty} (E \cap I_k)\right) \\ &\leq \frac{4}{3}\sum m(E \cap I_k) \\ &< \sum m(I_k) = m(G), \end{aligned}$$

contradicting our choice of G .

We now know that $m(I_k) \leq \frac{4}{3}m(E \cap I_k)$ for some k . Denote $\mathcal{I} = I_k$ and $\mathcal{E} = E \cap I_k$. Thus $m(\mathcal{I}) \leq \frac{4}{3}m(\mathcal{E})$. Note that $D_{\mathcal{E}} \subseteq D_E$ since $\mathcal{E} \subseteq E$. We will show that $D_{\mathcal{E}}$ contains an interval centered at the origin.

We showed the existence of the interval \mathcal{I} with $m(\mathcal{I}) \leq \frac{4}{3}m(\mathcal{E})$ because we know what the intervals look like, whereas we have no such knowledge about E or \mathcal{E} . Let $d \in \mathbb{R}$ with $|d| < \frac{1}{2}v(\mathcal{I})$. Set

$$\mathcal{I} + d = \{x + d \mid x \in \mathcal{I}\},$$

$$\mathcal{E} + d = \{x + d \mid x \in \mathcal{E}\}.$$

$\mathcal{I} + d$ is merely the interval \mathcal{I} shifted by less than half of the length of \mathcal{I} and will overlap \mathcal{I} . In fact, by our choice of d , $m((\mathcal{I} + d) \cup \mathcal{I}) < \frac{3}{2}m(\mathcal{I})$.

We will show $(\mathcal{E} + d) \cap \mathcal{E} \neq \emptyset$ for $d \in \mathbb{R}$. To see this, assume the contrary. By Exercise 8, $m(\mathcal{E} + d) = m(\mathcal{E})$ and $m(\mathcal{I} + d) = m(\mathcal{I})$. If $(\mathcal{E} + d) \cap \mathcal{E} = \emptyset$, then by Theorem 1.2.24,

$$\begin{aligned} 2m(\mathcal{E}) &= m(\mathcal{E} + d) + m(\mathcal{E}) \\ &= m((\mathcal{E} + d) \cup \mathcal{E}) \\ &\leq m((\mathcal{I} + d) \cup \mathcal{I}) \\ &< \frac{3}{2}m(\mathcal{I}). \end{aligned}$$

This leads to $\frac{4}{3}m(\mathcal{E}) < m(\mathcal{I})$, a contradiction.

We have established that if $d \in \mathbb{R}$ with $|d| < \frac{1}{2}v(\mathcal{I})$, then $(\mathcal{E} + d) \cap \mathcal{E} \neq \emptyset$. In other words, for some real number x , $x \in (\mathcal{E} + d) \cap \mathcal{E}$. In particular, $x \in \mathcal{E}$ and

$$x = y + d$$

for some $y \in \mathcal{E}$. Hence, $d = x - y$, where both x and y are in \mathcal{E} . Thus, $d \in D_{\mathcal{E}}$.

Let $\delta = \frac{1}{2}v(\mathcal{I})$. Whenever $|d| < \delta$, then $d \in D_{\mathcal{E}}$. Therefore,

$$(-\delta, \delta) \subseteq D_{\mathcal{E}} \subseteq D_E.$$

Consequently, D_E contains an interval centered at 0. \square

Assuming the Axiom of Choice and the existence of a nonmeasurable set, we will show that there are disjoint sets where the outer measure of the union is strictly less than the sum of the outer measures.

Example 1.3.6. By Exercise 25, there is a nonmeasurable subset A of $[0, 1]$. If $m^*(A) = 0$, then A would be a measurable set by Example 1.2.4. Therefore

$$0 < m^*(A) \leq 1.$$

Let $\delta = m^*(A)$. The set of rational numbers in the interval $[0, 1]$ is a countable set, say $\mathbb{Q} \cap [0, 1] = \{r_k\}$. Hence $\{A + r_k\}$ is a countable collection of pairwise disjoint sets with $A + r_k \subseteq [0, 2]$ for each k . Thus, for every N ,

$$m^*\left(\bigcup_{k=1}^N (A + r_k)\right) \leq m^*([0, 2]) = 2.$$

If it were the case that the outer measure of this union equalled the sum of the outer measures, then

$$N\delta = \sum_{k=1}^N m^*(A + r_k) = m^*\left(\bigcup_{k=1}^N (A + r_k)\right) \leq 2,$$

a contradiction when N is large.

1.4. Exercises

- (1) Let A be a finite set of real numbers. Use the definition of outer measure to show $m^*(A) = 0$.
- (2) Let A be a countable set of real numbers. Use the definition of outer measure to show $m^*(A) = 0$.
- (3) Let S and T be coverings of a set A by intervals.
 - a) Explain why $S \cup T$ is also a covering of A by intervals.
 - b) Show that $\sigma(S \cup T) \leq \sigma(S) + \sigma(T)$.
- (4) Show that for $c \in \mathbb{R}$ and fixed k , the set (known as a hyperplane in \mathbb{R}^n)

$$A = \{x = (x_1, x_2, \dots, x_k, \dots, x_n) \in \mathbb{R}^n \mid x_k = c\}$$

has Lebesgue outer measure 0.

- (5) Suppose A and B are both Lebesgue measurable. Prove that if both A and B have measure zero, then $A \cup B$ is Lebesgue measurable and $m(A \cup B) = 0$.
 - a) Do this directly from Definition 1.2.1.

- b) Give a shorter proof by using Theorem 1.2.5.
- (6) Suppose A has Lebesgue measure zero and $B \subseteq A$. Prove B is Lebesgue measurable and $m(B) = 0$.
- (7) Prove Corollary 1.1.10. Give an example to show that the result does not necessarily hold if $m^*(B)$ is not finite.
- (8) Let A be a subset of \mathbb{R} and $c \in \mathbb{R}$. Define $A + c$ to be the set

$$A + c = \{x + c \mid x \in A\}.$$

- a) Prove $m^*(A + c) = m^*(A)$.
- b) Prove that $A + c$ is Lebesgue measurable if and only if A is Lebesgue measurable.
- (9) Generalize the previous exercise to \mathbb{R}^n .
- (10) Let $c > 0$. For a set $A \subseteq \mathbb{R}$, define cA by

$$cA = \{y \in \mathbb{R} \mid y = cx \text{ for some } x \in A\}.$$

Prove that $m^*(cA) = cm^*(A)$. What happens in \mathbb{R}^n ?

- (11) Prove Proposition 1.2.19.
- (12) Let $Z \subseteq \mathbb{R}$ be a set with $m(Z) = 0$. Let $I = [0, 1]$. Show that $Z \times I$ is a measurable subset of \mathbb{R}^2 with Lebesgue measure 0.
- (13) Let $Z \subseteq \mathbb{R}$ with $m(Z) = 0$. Set

$$E = \{x^2 \mid x \in Z\}.$$

- a) Suppose Z is bounded, that is, $Z \subseteq [-n, n]$ for some integer n . Show that E is Lebesgue measurable and $m(E) = 0$.
- b) What if Z is not bounded? Hint:

$$Z = \bigcup_{n=1}^{\infty} (Z \cap [-n, n]).$$

- (14) Show that if $m^*(A) = 0$, then for any set B ,

$$m^*(A \cup B) = m^*(B).$$

- (15) Prove Proposition 1.2.23.

- (16) Let E be a measurable subset of \mathbb{R}^n . Show that given $\epsilon > 0$ there is a closed set F and an open set G with $F \subseteq E \subseteq G$ and $m(G \setminus E) < \epsilon$.
- (17) A measurable set $A \subseteq \mathbb{R}$ is said to have *density* d at x if the limit

$$\lim_{h \rightarrow 0^+} \frac{m(A \cap [x - h, x + h])}{2h}$$

exists and is equal to d . If $d = 1$, then x is called a *point of density* of A , and if $d = 0$, then x is called a *point of dispersion* of A . Find, with justification, the points of density and the points of dispersion of $A = (-1, 0) \cup (0, 1) \cup \{2\}$. What is the density at other points? Again, justify your answers. Note: x need not be an element of A .

- (18) Let $\mathbb{Q}_1 = \mathbb{Q} \cap [0, 1] = \{x \in [0, 1] \mid x \text{ is rational}\}$.
- What is $m^*(\mathbb{Q}_1)$? Is \mathbb{Q}_1 Lebesgue measurable?
 - Let $A = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q}_1, 0 \leq y \leq 1\}$. What is $m^*(A)$? Is A Lebesgue measurable?
- (19) Let $\{E_k\}$ be a sequence of Lebesgue measurable sets with

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$$

Define the set E to be

$$E = \bigcap_{k=1}^{\infty} E_k.$$

If $m(E_1) < \infty$, show that

$$m(E) = \lim_{k \rightarrow \infty} m(E_k).$$

Show by example that this need not be the case if we remove the assumption that $m(E_1) < \infty$.

- (20) Let $\{E_k\}$ be a sequence of Lebesgue measurable sets for which the series $\sum_{k=1}^{\infty} m(E_k)$ converges. Show that

$$m\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = 0.$$

- (21) Use the previous exercise to prove the **Borel-Cantelli Lemma**: Let $\{E_k\}$ be a sequence of Lebesgue measurable

subsets of \mathbb{R} such that $\sum_{k=1}^{\infty} m(E_k)$ converges. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k 's.

- (22) Construct a subset of $[0, 1]$ in the same manner as the Cantor set, except that at the k th stage each open interval removed has length $\delta 3^{-k}$, where δ is a fixed number strictly between 0 and 1. Show that the resulting set is Lebesgue measurable. Find, with justification, the Lebesgue measure of this “fat” Cantor set by computing the measure of its complement in $[0, 1]$.
- (23) Construct a 2-dimensional Cantor set in the unit square $[0, 1] \times [0, 1]$ as follows: Subdivide the square into nine congruent subsquares and keep only the four closed corner squares, removing the cross-shaped region. Repeat this process on the four corner squares, etc. Show that the remaining set is $C \times C$, where

$$C \times C = \{(x, y) \in \mathbb{R}^2 \mid x \in C \text{ and } y \in C\}.$$

Here C is the usual Cantor set. Find, with justification, the measure of this 2-dimensional Cantor set.

- (24) Let A be a subset of \mathbb{R}^n . Show that there is a set H of type G_δ so that

$$A \subseteq H \text{ and } m^*(A) = m^*(H).$$

- (25) Use a process similar to the proof of Theorem 1.3.5 to show (assuming the Axiom of Choice) there exists a nonmeasurable subset of $[0, 1]$.
- (26) Let

$$I = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq b\}$$

be a closed interval in \mathbb{R}^2 . Let

$$\begin{aligned} a &= a_0 < a_1 < \dots < a_m = b && \text{and} \\ c &= c_0 < c_1 < \dots < c_n = d. \end{aligned}$$

For $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, define the rectangle I_{ij} by

$$I_{ij} = \{(x, y) \in \mathbb{R}^2 \mid a_{i-1} \leq x \leq a_i, c_{j-1} \leq y \leq c_j\}.$$

(This can be thought of as subdividing I into subrectangles along the vertical lines $x = a_1, x = a_2, \dots, x = a_{m-1}$ and the horizontal lines $y = c_1, y = c_2, \dots, y = c_{n-1}$.) Using the definition of volume, prove

$$\sum_{i=1}^m \sum_{j=1}^n v(I_{ij}) = v(I).$$

(The ambitious reader can generalize this to higher dimensions.)

(27) Let

$$I = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq b\}$$

be a closed interval in \mathbb{R}^2 . Let J_1, J_2, \dots, J_n be a finite collection of closed intervals that cover I . That is,

$$I \subseteq \bigcup_{k=1}^n J_k.$$

By carefully subdividing I and the J_k 's into subrectangles, use the previous exercise to show that

$$v(I) \leq \sum_{k=1}^n v(J_k).$$