STUDENT MATHEMATICAL LIBRARY
Volume 8

EXPLORING THE NUMBER JUNGLE: A JOURNEY INTO DIOPHANTINE ANALYSIS

EDWARD B. BURGER

AMERICAN MATHEMATICAL SOCIETY

Table of contents

Opening Thoughts: Welcome to the Jungle

One of the most fundamental notions in mathematics is that of number. Although the idea of number is basic, the numbers themselves possess both nuance and complexity that spark the imagination.

> *Mathematics is the queen of the sciences and number theory is the queen of mathematics.*
>
> — Carl Friedrich Gauss

Welcome to diophantine analysis—an area of number theory in which we attempt to discover hidden treasures and truths within the jungle of numbers by exploring the rational numbers. Diophantine analysis comprises two different but interconnected domains—diophantine approximation and diophantine equations.

The rational numbers are creatures that appear both familiar and understandable. In comparison, the irrational numbers, in some sense, seem completely enigmatic. If we think about the rational numbers as sitting in the real line, then diophantine approximation can be viewed as an exploration of the number line under a microscope. If we view a piece of the real number line under higher and higher magnification we would see the following:

A piece of the real number line

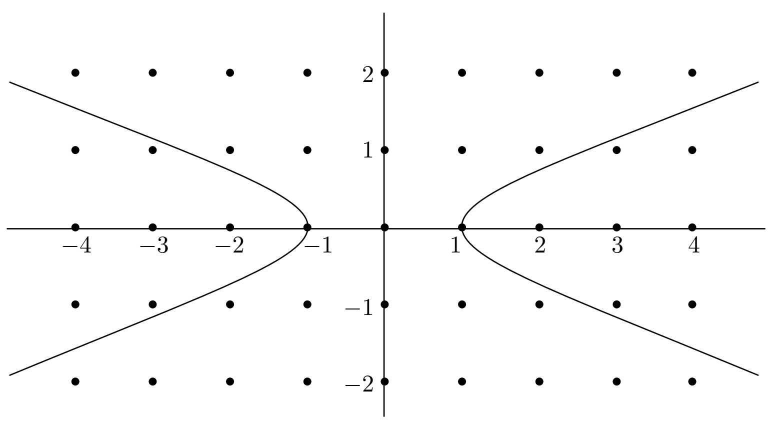
A tiny portion of the previous line segment magnified

Thus we see that the line is such a dense number jungle that higher magnification reveals no greater detail. We'll discover that some of the rich and beautiful structure of numbers can be brought into focus by examining how well irrational numbers can be approximated by rational ones. This strategy is the essence of diophantine approximation.

The other side of diophantine analysis, the study of diophantine equations, revolves around the following basic question: How can an equation be solved if the variables are to take only integer (or rational) values? For example, let's consider the equation

$$x^2 - 7y^2 = 1.$$

Clearly $(\pm 1, 0)$ are solutions, but we'll consider these trivial solutions. Are there any *nontrivial* integer solutions? Are there infinitely many integer solutions? Is there an algorithm to generate them all? The following graph depicts the *real* solutions to the equation.



Does that hyperbola contain integer lattice points other than $(\pm 1, 0)$? Perhaps the curve carefully navigates itself so as to miss all other integer points. A quick search reveals that $(8, 3)$ satisfies the equation and thus there exist nontrivial solutions. Are there others?

Let's perform some algebraic gymnastics with our new-found, nontrivial solution:

$$8^2 - 7(3)^2 = \left(8 - 3\sqrt{7}\right)\left(8 + 3\sqrt{7}\right) = 1,$$

and thus

$$\frac{8}{3} - \sqrt{7} = \frac{1}{3\left(8 + 3\sqrt{7}\right)} = 0.020915\cdots.$$

From the previous identity we discover that $\frac{8}{3}$ is impressively close to the irrational $\sqrt{7}$, and therefore we see that an integer solution to the equation leads to an amazing rational approximation to an associated irrational number. This observation illustrates a surprising connection between diophantine approximation and diophantine equations. This intriguing interplay will be a recurring theme throughout our journey.

The name "diophantine" honors Diophantus, a mathematician who lived in Alexandria sometime between 150 and 350 A.D. Diophantus is known for his passion for finding integer solutions to certain equations, and he was the author of the seminal work *Arithmetica*. In fact it was while Pierre de Fermat was reading *Arithmetica* that he was inspired to jot a note in the small margin which only later became known as Fermat's Last Theorem. Very few details are known about the personal life of Diophantus outside of the following conundrum that appeared in *Greek Anthology* from 600 A.D.:

> *God granted him to be a boy for the sixth part of his life, and adding a twelfth part to this, He clothed his cheeks with down. He lit him the light of wedlock after a seventh part, and five years after his marriage He granted him a son. Alas! late-born wretched child; after attaining the measure of half his father's life, chill Fate took him. After consoling his grief by this science of numbers for four years he ended his life.*

Given this brief life history, your first challenge now awaits:

How old was Diophantus when he died?

Little background is required for the journey ahead, and a familiarity with number theory is not expected. An understanding of calculus and basic linear algebra together with the desire and ability to prove theorems are all that is needed for most of the material. Toward the end of this book, a few modules require some familiarity with beginning abstract algebra. Each mathematical theme is presented in a self-contained manner and is motived by very basic notions. However beware: There are a few major jolts along the way—some extremely challenging questions and even some open problems are thrown in!

In this book, you as the reader will be an active participant in the explorations that lie ahead. In fact, the cover art illustrates this theme—hidden in the front and back covers are "magic" stereogram images. The secret to seeing the hidden images is to gaze at the cover but focus (converge your eyes) on a position twice as far away as the cover. After a few minutes the 3-D image should appear (helpful viewing hints are included on the inside of the front cover). Thus by actively participating, you will be able to discover the otherwise invisible structure and richness in the number jungle.

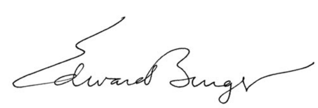
Each module comprises a sequence of numbered questions that you are asked to answer and statements that you are asked to verify. Many hints and remarks are included and should be freely utilized and enjoyed. The hints do not reveal any answers; instead they are designed to give a gentle push in a particular direction. They are placed at the end of the book only to allow readers, if they wish, to think about the issues first without being distracted. There is no wrong way of using the hints, and readers are welcome to immediately visit them if they wish. There are three types of (star-struck) hints, and throughout the text you will see the symbols [⋆ HINT], [⋆⋆ HINT], and [⋆ ⋆ ⋆ HINT]. A one-starred hint provides a tiny nudge to help get you started. A two-starred hint contains more extensive remarks and suggestions, and a three-starred hint is more significant and perhaps includes an outline or overview of a plan of attack. Each module closes with a Big Picture Question that invites you to step back from all the technical details and take a panoramic view of how the ideas at hand fit into the larger mathematical landscape. Commentaries for selected Big Picture Questions are also included at the end of the book. With the text as your guide and your own creative contributions as the points of interest, I hope you find your journey to be a truly interactive experience.

> *...the primary question was not What we know, but How do we know it.*
>
> — Aristotle

Although this book is small, its aims are not. The book is an invitation to develop the ideas of diophantine analysis and to discover the fundamental results on your own. I believe that this subject is one of the most beautiful and rich areas of mathematics, and I hope you will share my point of view and enjoy the sights and challenges that lie ahead. This book also provides a setting where different areas and ideas of mathematics come together. Here you will see basic notions from algebra, analysis, geometry, and topology join together as one collective whole. Most importantly however, I hope that through the process of actively generating the ideas behind the subject, you will continue to hone the habits of thought required to analyze issues and reason in a more effective and creative manner.

Uncovering the mysterious structure of number has been one of the great intellectual challenges of human history. I hope you enjoy your journey through the beautiful mathematics that lies ahead and enjoy the challenge of conquering this area of number theory for yourself. *You can do it!*

With all good wishes,



Edward Burger
January 1, 2000

Module 9

Liouville's work on numbers algebraic and not

In Module 7 we saw that there are numbers α, namely badly approximable numbers, for which the inequality $|\alpha - p/q| < 1/q^2$ is best possible in that the exponent on q cannot be improved for infinitely many rationals. We've also seen that every quadratic irrational is badly approximable.

In this module we consider the case when α is an algebraic number, but not quadratic. Will such an irrational number be badly approximable? Will such a number not be badly approximable? These questions lead us to consider *lower* bounds for the quantity $|\alpha - p/q|$. The first major result in this direction was given in 1844 by Liouville. As a consequence of Liouville's work, humankind, for the first time, was treated to a proof of the fact that transcendental numbers exist. This spectacular episode is another instance of how diophantine inequalities give rise to incredible, seemingly unrelated, discoveries. Liouville's beautiful theorem has also inspired an enormous amount of new mathematics and led to many important results within number theory and beyond. As we will see for ourselves, Liouville's work is one of the jewels of diophantine approximation.

Exactly 30 years after Liouville's result, Cantor developed his deep and revolutionary theory of sets. We'll close this module with some diophantine consequences of Cantor's work.

A number α is said to be *algebraic* if it is a root of a polynomial of the form

$$f(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0,$$

where $a_n \in \mathbb{Z}$ and $a_N \neq 0$. The *minimal polynomial* associated with an algebraic number α is the polynomial with (relatively prime) integer coefficients, $a_N > 0$, having the smallest degree for which α is a root. In other words, it is the *irreducible polynomial* associated with α—the polynomial that cannot be factored into two polynomials with integer coefficients each having degrees less than N. For example, $\sqrt{3}$ is algebraic because it is a root of the polynomial $h(z) = z^4 - 9$. However, $h(z)$ is not the minimal polynomial for $\sqrt{3}$ since $h(z)$ can be factored further: $h(z) = \left(z^2 - 3\right)\left(z^2 + 3\right)$. The minimal polynomial for $\sqrt{3}$ is in fact $f(z) = z^2 - 3$. Notice that $f(z)$ is irreducible. We define the *degree* of an algebraic number to be the degree of its minimal polynomial. So the degree of $\sqrt{3}$ is 2.

A number that is not algebraic is called *transcendental*. Certainly, as we've seen in the previous paragraph, algebraic numbers exist. Do transcendental numbers exist? This question was one of the major open problems from 19th-century mathematics. This question was finally answered by Liouville who used techniques from diophantine approximation.

Liouville's Theorem provides a measure for how well algebraic numbers can be approximated by rational numbers that are not too complicated. We edge our way up to Liouville's result slowly.

9.1. Suppose that $f(x)$ is a differentiable function and a and b are real numbers satisfying $|a - b| \leq 1$. Use the Mean Value Theorem to give a bound of the form

$$|f(b) - f(a)| \leq C(f, a)|b - a|,$$

where $C(f, a)$ is a constant that just depends upon the function f and the number a, but *not* on b.

9.2. Suppose that $f(x)$ is a nonconstant polynomial. If $0 < |a - b| \leq 1$, then the constant $C(f, a)$ found above is nonzero.

Lemma 9.3. *Let $f(x)$ be a polynomial of degree d, with integer coefficients. Suppose that the rational number p/q is not a root of f. Then*

$$\frac{1}{q^d} \leq \left| f\left(\frac{p}{q}\right) \right|.$$

Lemma 9.4. *Suppose that α is a real number and p/q is a rational number such that $|\alpha - p/q| > 1$. Then for any positive integer d,*

$$\frac{1}{q^d} \leq \left| \alpha - \frac{p}{q} \right|.$$

9.5. Liouville's Theorem. *Let α be a real algebraic number of degree $d \geq 2$. Then there exists a constant $c = c(\alpha) > 0$ such that for all rational numbers p/q,*

$$\frac{c}{q^d} < \left| \alpha - \frac{p}{q} \right|.$$

[$\star\ \star\ \star$ HINT]

9.6. Show that Liouville's Theorem implies that all rational numbers near algebraic numbers are complicated. Specifically, show that if α is an algebraic number and p/q is a rational number that is very, very close to α, then the height of p/q (*i.e.*, q) must be very large.

9.7. Suppose that α is a transcendental number (note that at this point we do not know if such animals exist). Does Liouville's Theorem imply anything about how well α can be approximated by rationals having small height?

9.8. What is the contrapositive of Liouville's Theorem? What does that statement imply about numbers that have amazing approximations by rationals of small height?

An important consequence of Liouville's celebrated result is the existence of transcendental numbers.

Theorem 9.9. *Let*

$$\alpha = \sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001000\cdots.$$

Then α is a transcendental number.

[$\star\star$ HINT]

Congratulations—you've just demonstrated that a particular number is transcendental! Feel free to enjoy your triumph!

Thirty years after Liouville proved his result, Cantor produced his work on the cardinality of sets which gave a different, albeit non-effective, proof of the existence of transcendental numbers. Cantor's work demonstrated that *most* numbers are transcendental. But even today it remains extremely difficult to prove that a *particular* number is transcendental. Liouville's Theorem, on the other hand, allows us to exhibit explicit examples. Before closing, we consider the work of Cantor and explore its consequences within the context of diophantine analysis.

An infinite set S is said to be *countable* if it has the same cardinality as the set of the natural numbers, that is, if there exists a one-to-one correspondence between the natural numbers and the elements of S. An infinite set T is called *uncountable* if there does not exist a one-to-one correspondence between the elements of T and the natural numbers.

9.10. Is the collection of algebraic numbers a countable or uncountable set? Justify your answer.

9.11. Are there uncountably many badly approximable numbers? Are there uncountably many numbers that are *not* badly approximable? As always, justify your answers with proofs. Suppose we randomly picked a real number from the number line. Do you think it is more likely that we would have selected a badly approximable number or a non–badly approximable number? Just make a guess (*conjecture*) without any justification.

9.12. Suppose that α and β are both badly approximable. Suppose also that $\alpha + \beta \notin \mathbb{Q}$. Then must $\alpha + \beta$ be badly approximable? Is it possible that $\alpha + \beta$ is not badly approximable? What about $\alpha\beta$? What about $1/\alpha$? What about $b\alpha d$, where b and d are nonzero rational numbers? Give justifications for those questions that you can answer (provide a general proof when you can or an illustative example otherwise). If you cannot justify your answer to a question, then just state your guess.

[$\star\star\star$ REMARK]

9.13. Do badly approximable transcendental numbers exist? If so, explain why. If not, prove why not.

9.14. Do badly approximable algebraic numbers exist? Do badly approximable algebraic irrational numbers of any degree exist?

[$\star\star\star$ REMARK]

We close this module with three amusing remarks regarding e and π. First, if we try to compute the beginning of the continued fraction expansion for e, we would see

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, \ldots].$$

It has been shown that the apparent pattern, in fact, continues and thus the continued fraction expansion for e is completely understood. In particular, we see that e is not a badly approximable number. If we calculate the first few partial quotients of π, we would have

$$\begin{aligned}\pi = [&3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15,\\ &3, 13, 1, 4, 2, 6, 6, 99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1, 7, 1, 2, 3, 7, 1, 2, 1,\\ &1, 12, 1, 1, 1, 3, 1, 1, 8, 1, 1, 2, 1, 6, 1, 1, 5, 2, 2, 3, 1, 2, 4, 4, 16, 1, 161,\\ &45, 1, 22, 1, 2, 2, 1, 4, 1, 2, 24, 1, 2, 1, 3, 1, 2, 1, 1, 10, 2, 5, 4, 1, 2, 2, 8,\\ &1, 5, 2, 2, 26, 1, 4, 1, 1, 8, 2, 42, 2, 1, 7, 3, 3, 1, 1, 7, 2, 4, 9, 7, 2, 3, 1,\\ &57, 1, 18, 1, 9, 19, 1, 2, 18, 1, 3, 7, 30, 1, 1, 1, 3, 3, 3, 1, 2, 8, 1, 1, 2, 1,\\ &15, 1, 2, 13, 1, 2, 1, 4, 1, 12, 1, 1, 3, 3, 28, 1, 10, 3, 2, 20, 1, 1, 1, 1, 4, 1,\\ &1, 1, 5, 3, 2, 1, 6, 1, 4, 1, 120, 2, \ldots].\end{aligned}$$

It is an open question as to whether or not the apparent lack of pattern continues. In particular, the question of whether π is a badly approximable number or not remains open. The general consensus is that π is not badly approximable, but for now that is nothing more than a guess. Finally, you are welcome to ask the same types of questions for various other numbers, for example $e+\pi$. Unfortunately, no one has been able to prove that $e+\pi$ is even an *irrational* number. We'll leave this as a bonus problem.

Big Picture Question. What is the basic idea behind Liouville's argument? Numbers having similar diophantine structure as that of the α from Theorem 9.9 are known as *Liouville numbers.* All Liouville numbers are transcendental, and the proof of Theorem 9.9 can be adapted to show that these numbers are transcendental. What should be the definition of a *Liouville number*?

(See Appendix 1 for some commentary on this question.)

Further Challenge. Suppose that $\alpha = [a_0, a_1, a_2, \ldots]$ is an irrational number with the property that there is an infinite *subsequence* of partial quotients $\{a_{n_i}\}$ that grows incredibly fast. In particular, suppose that $a_{n_i} \geq q_{n_i-1}^{n_i}$, where q_n denotes the denominator of the nth convergent of α. Prove that α must be transcendental.