Preface

The story of this book began in 2002 when Matt, then a postdoc at Princeton University, was given the opportunity to teach an undergraduate class in game theory. Thanks largely to the 2001 release of a Hollywood movie on the life of the famous Princeton mathematician and (classical) game theorist John Nash, this course attracted a large and highly diverse audience. Princeton's mathematics department featured not only Nash, but also John Conway, the father of modern combinatorial game theory. So it seemed only natural to blend the two sides of game theory, combinatorial and classical, into one (rather ambitious) class. The varied backgrounds of the students and the lack of a suitable textbook made for an extremely challenging teaching assignment (that sometimes went awry). However, the simple fun of playing games, the rich mathematical beauty of game theory, and its significant real-world connections still made for an amazing class.

Deborah adopted a variant of this material a few years later and further developed it for a general undergraduate audience. Over the ensuing years, Deborah and Matt have both taught numerous incarnations of this course at various universities. Through exchange and collaboration, the material has undergone a thorough evolution, and this textbook represents the culmination of our process. We hope it will provide an introductory course in mathematical game theory that you will find inviting, entertaining, mathematically interesting, and meaningful.

Combinatorial game theory is the study of games like Chess and Checkers in which two opponents alternate moves, each trying to win the game. This part of game theory focuses on deterministic games with full information and is thus highly amenable to recursive analysis. Combinatorial game theory traces its roots to Charles Bouton's theory of the game Nim and a classification theorem attributed independently to Roland Sprague and Patrick Grundy. The 1982 publication of the classic *Winning Ways for Your Mathematical Plays* by Elwyn Berlekamp, John Conway, and Richard Guy laid a modern foundation for the subject—now a thriving branch of combinatorics. In contrast, classical game theory is an aspect of applied mathematics frequently taught in departments of economics. Classical game theory is the study of strategic decision-making in situations with two or more players, each of whom may affect the outcome. John von Neumann and Oskar Morgenstern are commonly credited with the foundation of classical game theory in their groundbreaking work *Theory of Games and Economic Behavior* published in 1944. This treatise established a broad mathematical framework for reasoning about rational decision-making in a wide variety of contexts and it launched a new branch of academic study. Although there have been many significant developments in this theory, John Nash merits mention for his mathematical contributions, most notably the Nash Equilibrium Theorem.

Traditionally, the classical and combinatorial sides of game theory are separated in the classroom. A strong theme of strategic thinking nonetheless connects them and we have found the combination to result in a rich and engaging class. The great fun we have had teaching this broad mathematical tour through game theory undergirded our decision to write this book. From the very beginning of this project, our goal has been to give an honest introduction to the mathematics of game theory (both combinatorial and classical) that is accessible to an early undergraduate student. Over the years, we have developed an approach to teaching combinatorial game theory that avoids some of the set-theoretic complexities found in advanced treatments yet still holds true to the subject. As a result, we achieve the two cornerstones of the Sprague-Grundy Theorem and the Simplicity Principle in an efficient and student-friendly manner. The classical game theory portion of the book contains numerous carefully sculpted and easy-to-follow proofs to establish the theoretical core of the subject (including the Minimax Theorem, Nash arbitration, Shapley Value, and Arrow's Paradox). Most significantly, Chapter 9 is entirely devoted to an extremely gentle proof of Nash's Equilibrium Theorem. For the sake of concreteness, the chapter focuses on 2×2 matrices, but each argument generalizes and Appendix C contains full details. Sperner's Lemma appears in this chapter as the first step of our proof and we offer an intuitive exposition of this lemma by treating it as a game of solitaire. More broadly, Sperner's Lemma provides a touchstone through other chapters. In addition to using it to prove Nash's Equilibrium Theorem, we also call on it to show that the combinatorial game Hex cannot end in a draw. Later still, Sperner's Lemma allows us to construct an envy-free division of cake.

Beyond including both combinatorial and classical theory, we have sought to provide a broad overview of (both sides of) the subject. Within the world of combinatorial game theory, we begin at a very high level of generality with game trees and Zermelo's Theorem—concepts that apply to Chess, Checkers, and many other 2-player games. We also introduce some widely applicable ideas such as symmetry and strategy stealing before specializing in normal-play games to develop the heart of the theory. On the classical side, in addition to the essential mathematical concepts, we tour a variety of exciting supplementary topics including the Folk Theorem, cake cutting, and stable marriages. Furthermore, we have devoted considerable effort to connecting the theory with applications. Chapter 7 focuses on the modeling capability of a game-theoretic framework in the context of sports, biology, business, politics, and more!

One of our primary goals in this book is to enhance the mathematical development of our student readers. Indeed, we aim to take advantage of the naturally stimulating subject of game theory to teach mathematics. We have found that blending combinatorial and classical game theory has great pedagogical advantages. Beginning with combinatorial games means that student pairs are playing and recursively analyzing games right from the start. These games are not only fun to play, but they provide a perfect environment for working with game trees, proving theorems by induction, and starting to think strategically. This part of the book features numerous rich examples of proofs by induction and also a number of interesting proofs by contradiction. Turning to classical game theory, we encounter basic probability, linear algebra, and convexity in our study of zero-sum matrix games. Our later chapters on general games continue to emphasize probability and geometric methods but also introduce questions of modeling as well as plentiful applications. The proof of Nash's Equilibrium Theorem involves a nice blend of combinatorial and continuous mathematics in addition to a taste of topology. Whenever a significant new mathematical concept is required, we pause to introduce it; accordingly, this book contains elementary introductions to proofs by induction, proofs by contradiction, probability, and convexity.

We have constructed this textbook for a one-semester undergraduate course aimed at students who have already taken courses in differential calculus and linear algebra. However, we have found this material adaptable to a variety of situations and a range of audiences. In particular, most of the book does not directly call upon either calculus or linear algebra and is thus suitable for students who lack these prerequisites but have a similar level of sophistication. Indeed, calculus is used very rarely, and for a capable student without linear algebra, only the proofs of the Minimax and Equilibrium Theorems would be out of reach after a quick introduction to matrix multiplication. The complete book is likely more material than can be comfortably covered in a standard undergraduate semester 3-credit course. To allow the instructor considerable flexibility in content choices, we have limited dependencies between the chapters (see the diagram on page xv). These limited dependencies also allow for portions of this book to be used in other contexts. For instance, the first four chapters on combinatorial games provide an appealing theme for an introductory proofs course, Chapters 5 and 6 on zero-sum matrix games together with Appendix B on linear programming make a nice addition to a linear algebra course, and all three sections in Chapter 12 can be taught independently.

Further to assist the instructor, each chapter ends with a generous supply of exercises. We have sought to include problems at a variety of levels from basic skills all the way up to challenging proofs, with espe-

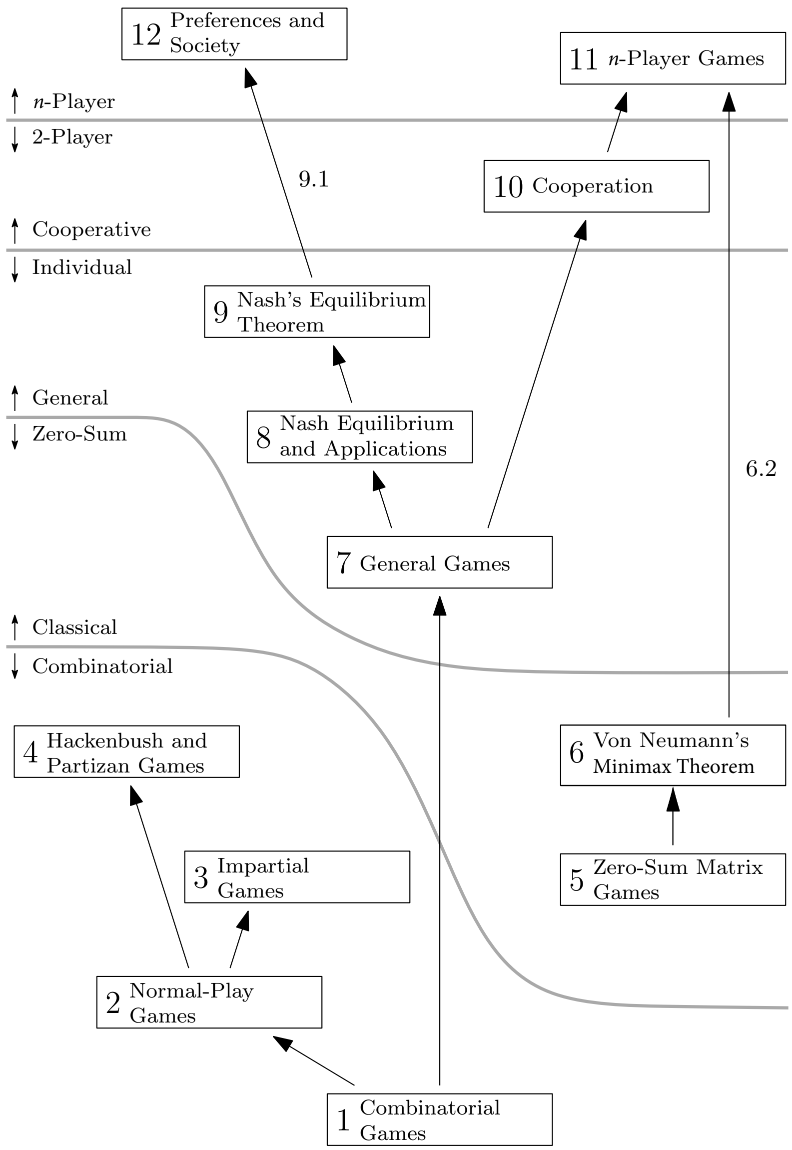

Figure 0.1. Implication Diagram

cially difficult exercises marked with the symbol *. References to exercises in the same chapter are by exercise number, while those to exercises in another chapter also include the chapter number. In addition, game boards and further supplementary material can be found online at

`www.ams.org/bookpages/stml-80`.

This book owes its existence to the many amazing teachers from whom we have been fortunate to learn. Matt's genesis as a combinatorialist is thanks to his incomparable PhD supervisor, Paul Seymour. He also benefited from an inspiring introduction to combinatorial games from John Conway and a detailed initiation to the mathematics of classical game theory under the guidance of Hale Trotter. Deborah deeply appreciates her inimitable dissertation advisor, Karen Parshall, who introduced her to the joys and labors of academic mathematics. She also thanks Tom Archibald for his generous support of this and her other postdoctoral projects. We are so grateful to many of our friends and colleagues who have influenced the development of this book either directly or indirectly: Derek Smith, Drago Bokal, Francis Su, Claude Tardif, and Dave Muraki top this list, but there are countless others. We owe a debt of gratitude to the universities that made it possible for us to teach versions of this class and to the many students who helped to shape this material with their questions, comments, and corrections. We would also like to thank Ina Mette, Arlene O'Sean, Courtney Rose, and the rest of the editorial staff at the AMS whose careful work on our manuscript dramatically improved the final product. Finally, we thank our friends and especially our families for their amazing support throughout the extensive process of creating this book. Although it has taken far more effort and energy than we could ever have foreseen, writing this book has been a labor of love for us. We hope you will enjoy it, too!