Preface

This book grew out of the MASS course in algebra in the fall of 2009. As the title suggests, the central theme throughout is that geometry can be used to understand group theory and group theory can be used to understand geometry.

For a little over half of the book, this relationship takes the form of symmetry groups in various geometries. After introducing some of the fundamentals of group theory in the first chapter, we spend Chapter 2 using synthetic methods to explore Euclidean geometry in low dimensions, where we classify isometries, show that all isometries are products of reflections, and explain the algebraic structure of the isometry group as a semidirect product, before moving on to finite and discrete groups of isometries, with their connections to polygons, polyhedra, and tilings. Synthetic methods give way to linear algebra in Chapter 3, where we study not just isometries of Euclidean space, but also of the elliptic and hyperbolic planes, as well as projective and affine geometry.

In the second half of the book, we describe an (apparently) different type of relationship between groups and geometry by introducing the fundamental group in Chapter 4. Then in Chapter 5 we describe how the fundamental group in fact arises as a symmetry group of the universal covering space via deck transformations. At this point we also begin to explore relationships in the opposite direction; instead of taking a geometric object and producing a group (such as the symmetry group or the fundamental group), we take a group and produce a geometric object. This is done in several ways in Chapter 5; we discuss Cayley graphs, planar models, and fundamental domains for group actions, all of which produce a geometric object associated to a particular group. Then we use these objects to learn something about groups: we use Cayley graphs to show that subgroups of free groups are free; we use planar models to show that every finitely presented group appears as a fundamental group; and we use fundamental domains in the hyperbolic plane to study Fuchsian groups.

Finally, in Chapter 6 we outline some of the results that can be obtained by viewing the group itself as a geometric object. In particular, we introduce growth rates in groups and give a gentle introduction to some of the ideas behind two landmark results: Gromov's Theorem that polynomial growth is equivalent to being virtually nilpotent and Grigorchuk's construction of a group with intermediate growth. We conclude with a brief discussion of some other ideas that appear in geometric group theory, including amenability, hyperbolicity, and the boundary of a group.

In keeping with the spirit of the original course, this book is not meant to be a comprehensive reference where all details are filled in, but rather is more of a carefully guided tour, where we point out various features of the landscape but do not explore all of them in depth, or in their most general forms. In particular, throughout the book we relegate many routine computations and "tidying up of details" to the exercises, guided by a belief that serious readers are best served by doing some of the dirty work themselves and also by a desire to focus on the overall picture and conceptual understanding. As befits an undergraduate course, the presentation here is a gateway to many interesting topics but is not the final word on any of them.

With that said, we provide proofs of nearly all formally stated theorems, propositions, etc., which are either complete as they stand or depend only on certain exercises that should be within the reach of a dedicated student. We deviate from this approach in the final chapter, where we state some results whose proofs would require more machinery than could be developed here but which are nevertheless too elegant to omit entirely. Throughout the book we occasionally give informal discussions of results that lie beyond our scope.

We assume throughout the book a certain mathematical maturity and energy on the part of the reader. Beyond this, the main prerequisite is a familiarity with basic real analysis, abstract linear algebra, and Euclidean geometry. In particular, we work with metric spaces and with their associated topologies, including concepts of limits, connectedness, and interiors and boundaries; we discuss quotient spaces but do not assume that the reader has encountered these in the most general topological setting. It would be helpful to have seen linear algebra up to Jordan normal form, but for most of what we do, it will suffice to be familiar with matrices as linear transformations and with eigenvectors and eigenvalues. We use complex numbers frequently, but concepts such as "holomorphic" appear only in passing. The book's focus is on groups; other algebraic objects such as rings and fields will make cameo appearances, and so some familiarity with them would be helpful, but not essential.

The course notes from which this book developed were a straightforward transcription of the contents of fourteen weeks' worth of lectures (less several holidays and exam days), with three 50-minute lectures each week. The book you hold in your hands is faithful to the overall structure and content of those course notes, deviating from them only in some small ways: we have made minor reorganizations to streamline the presentation; various explanations have been added for clarity; more exercises have been added; and the final chapter on geometric group theory has been expanded to include more discussion of Gromov's Theorem and Grigorchuk's example than appeared in the lectures.

In the "Guide for instructors" we discuss possible ways to use this book as a textbook; we also hope that a student working outside of a formal course setting will find in this book a window onto some beautiful mathematical landscapes and a map with which to explore them further.

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