Chapter 1

Introduction

Apart from numbers, sets are the most fundamental notion in mathematics. Arguably the simplest sets are *finite sets*, that is, sets with a finite number of elements. The present book deals with combinatorial, mostly extremal problems concerning systems of subsets of a given finite set. Let us introduce the basic notation first.

For a set X let $|X|$ denote its *size*, that is, the number of its elements. If $|X| = n$, then X is called an n-element set. For a positive integer n let $[n] = \{1, 2, \ldots, n\}$ denote the standard n-element set. The *power set* $2^{[n]}$ consists of the 2^n subsets (including $[n]$ and the *empty set* $\emptyset$) of $[n]$. For $0 \leq k \leq n$ let $\binom{[n]}{k}$ denote the collection of all k-element subsets of $[n]$. A subset $\mathcal{F}$ of $2^{[n]}$ is called a *family* of subsets. We can think of $\mathcal{F} \subset 2^X$ as a hypergraph. In this case the vertex set is X and the edge set is $\mathcal{F}$ itself. In this sense we sometimes call a member of $\mathcal{F}$ an edge of $\mathcal{F}$. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called *k-uniform* (or a k-uniform hypergraph).

To explain the topics in this book, probably it is best to state and prove Sperner's Theorem, the first result in this area. We need some simple notions. If

$$F_0 \subsetneqq F_1 \subsetneqq \cdots \subsetneqq F_l$$

then $(F_0, \ldots, F_l)$ is called a *chain* of length l. If for a family $\mathcal{F}$ no $F, G \in \mathcal{F}$ satisfy $F \subsetneqq G$, $\mathcal{F}$ is called an *antichain*. The reason is that $\mathcal{F}$ contains no chains. Obviously, $\binom{[n]}{k}$ is an antichain for every fixed k, $0 \le k \le n$. Among these very specific antichains, $\binom{[n]}{\frac{n}{2}}$ is the largest for n even. For n odd, $\binom{[n]}{\frac{n-1}{2}}$ and $\binom{[n]}{\frac{n+1}{2}}$ are the largest (note that they have the same size).

In 1928 Sperner [**103**] proved that for all n these are the largest antichains.

Theorem 1.1 (Sperner's Theorem)**.** *Let $\mathcal{F} \subset 2^{[n]}$ be an antichain. Then*

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor},$$

and in the case of equality, $\mathcal{F} = \binom{[n]}{\frac{n}{2}}$ for n even and $\mathcal{F} = \binom{[n]}{\frac{n-1}{2}}$ or $\mathcal{F} = \binom{[n]}{\frac{n+1}{2}}$ for n odd.

Proof. Let us suppose that $\mathcal{F} \subset 2^{[n]}$ is an antichain of maximum possible size, that is, $|\mathcal{F}|$ is as large as possible. Let

$$p = \max\{|F| : F \in \mathcal{F}\} \text{ and } q = \min\{|F| : F \in \mathcal{F}\}.$$

We want to show that $p \le \frac{n+1}{2}$ and $q \ge \frac{n-1}{2}$. If we succeed, the proof for n even is complete. Indeed the above inequalities imply $p = q = \frac{n}{2}$. Thus $\mathcal{F} \subset \binom{[n]}{\frac{n}{2}}$. If n is odd, the two inequalities imply $\mathcal{F} \subset \binom{[n]}{\frac{n-1}{2}} \cup \binom{[n]}{\frac{n+1}{2}}$ and we have some more work to do.

Let us prove $p \le \frac{n+1}{2}$ by an indirect argument. Suppose $p > \frac{n+1}{2}$ and define

$$\mathcal{G} = \{G \in \mathcal{F} : |G| = p\}.$$

Define the *immediate shadow* $\mathcal{H}$ of $\mathcal{G}$ by

$$\mathcal{H} = \{H : |H| = p - 1 \text{ and } H \subset G \text{ for some } G \in \mathcal{G}\}.$$

Claim 1.2. We have the following.

(i) Let $\mathcal{F}' = (\mathcal{F} \setminus \mathcal{G}) \cup \mathcal{H}$. Then $\mathcal{F}'$ is an antichain.

(ii) $|\mathcal{F}'| > |\mathcal{F}|$.

Once we prove the claim, the proof of $p \le \frac{n+1}{2}$ is complete because $|\mathcal{F}'| > |\mathcal{F}|$ contradicts the maximal choice of the antichain $\mathcal{F}$.

Proof of Claim. The facts that $\mathcal{H} \subset \binom{[n]}{p-1}$ and

$$\max\{|F| : F \in \mathcal{F} \setminus \mathcal{G}\} \le p - 1$$

imply that $\mathcal{H}$ is an antichain, and $H \not\subset F$ for $H \in \mathcal{H}$ and $F \in \mathcal{F} \setminus \mathcal{G}$. We need to show that $H \supset F$ is impossible as well. Here we use that for $H \in \mathcal{H}$ there is some $G \in \mathcal{G}$ with $H \subset G$. Consequently $H \supset F$ would imply $G \supset F$ thereby contradicting the assumption that $\mathcal{F} = (\mathcal{F} \setminus \mathcal{G}) \cup \mathcal{G}$ is an antichain. This proves (i).

To prove (ii) we note that $|\mathcal{F}'| - |\mathcal{F}| = |\mathcal{H}| - |\mathcal{G}|$. Thus we need to prove $|\mathcal{H}| > |\mathcal{G}|$. We do it by a simple double counting argument. Let M be the number of pairs (G, H) with $G \in \mathcal{G}$ and $H \in \mathcal{H}$. Since every p-element subset G has p subsets of size $p - 1$, $M = p|\mathcal{G}|$. On the other hand, for every $(p - 1)$-element subset $H \subset [n]$ there are $n - (p - 1) = n + 1 - p$ choices of $G' \in \binom{[n]}{p}$ with $H \subset G'$. Some of these G' might lie outside of $\mathcal{G}$. Thus

$$M \le (n - p + 1)|\mathcal{H}|.$$

We infer that

$$p|\mathcal{G}| \le (n + 1 - p)|\mathcal{H}|. \tag{1.1}$$

If $p > \frac{n+1}{2}$, then $n+1-p < \frac{n+1}{2}$ and $|\mathcal{G}| < |\mathcal{H}|$ follows. This completes the proof of the claim. □

The argument showing $q \ge \frac{n-1}{2}$ is basically the same. However, one can circumvent it by taking the family of the complements: $\mathcal{F}^{c} = \{[n] \setminus F : F \in \mathcal{F}\}$. It is easy to check that $|\mathcal{F}^{c}| = |\mathcal{F}|$ and that $\mathcal{F}^{c}$ is an antichain, too.

By the above argument, $n - |F| \le \frac{n+1}{2}$ for all $F \in \mathcal{F}$. Equivalently, $|F| \ge n - \frac{n+1}{2} = \frac{n-1}{2}$. In the odd case, $p = \frac{n+1}{2}$ via (1.1) still implies $|\mathcal{G}| \le |\mathcal{H}|$. To avoid contradiction, $|\mathcal{G}| = |\mathcal{H}|$ must hold. This in turn implies $M = (n-p+1)|\mathcal{H}|$, that is, all p-element subsets $G \in \binom{[n]}{p}$ containing $H \in \mathcal{H}$ must be in $\mathcal{F}$. We do not want to go into details here, but by a connectivity argument this implies $\mathcal{G} = \binom{[n]}{p}$ and consequently $\mathcal{F} = \binom{[n]}{p}$. □

We will give an alternative proof of Theorem 1.1 in Chapter 5.

Sperner's Theorem served as the starting point for a lot of research both inside and outside of extremal set theory. There are several books dealing with related results; see, for example [**30**].

In the present book we deal mostly with problems of a slightly different nature. Let us give an example from the famous paper [**36**] written by Erdős, Ko, and Rado. We need one definition: A family $\mathcal{F} \subset 2^{[n]}$ is called *intersecting* if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$.

Theorem 1.3. *Suppose that $\mathcal{F} \subset 2^{[n]}$ is an intersecting family. Then* (i) *and* (ii) *hold.*

(i) $|\mathcal{F}| \leq 2^{n-1}$.

(ii) *If* $\mathcal{F} \subset \binom{[n]}{k}$ *and* $n \geq 2k$, *then* $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

The easiest examples of intersecting families are those for which one fixed element is contained in all members of the family. These show that both (i) and (ii) are best possible.

Let us reproduce the easy proof of (i). For $A \subset [n]$ let $A^{\mathrm{c}} = [n] \backslash A$. If $F \in \mathcal{F}$, then $F \cap F^{\mathrm{c}} = \emptyset$ implies $F^{\mathrm{c}} \notin \mathcal{F}$. Thus $\mathcal{F} \cap \mathcal{F}^{\mathrm{c}} = \emptyset$, where $\mathcal{F}^{\mathrm{c}} = \{F^{\mathrm{c}} : F \in \mathcal{F}\}$. Since $|\mathcal{F}| = |\mathcal{F}^{\mathrm{c}}|$ and $\mathcal{F} \sqcup \mathcal{F}^{\mathrm{c}} \subset 2^{[n]}$, we have

$$2|\mathcal{F}| = |\mathcal{F}| + |\mathcal{F}^{\mathrm{c}}| = |\mathcal{F} \sqcup \mathcal{F}^{\mathrm{c}}| \leq |2^{[n]}| = 2^n,$$

implying (i). The proof of (ii) is much harder, so we postpone it to Chapter 4.