Chapter 5

Katona's circle

Trying to solve extremal problems by looking straight at the whole of $2^{[n]}$ or even $\binom{[n]}{k}$ might be too difficult or very complicated. Katona's ingenious idea was to look at such problems on a much smaller family of subsets, that is, the circle. Let us define this circle.

For an arbitrary ordering of the n elements of $[n]$ on a circle, we consider only subsets that form an arc on this circle, that is, the elements of this set are consecutive. More formally, let $(a_1, \ldots, a_n)$ be a circular permutation of $1, 2, \ldots, n$. By circular we mean that we consider the element a_1 to be the consecutive element of a_n *and* we do not distinguish the n possible permutations $(a_1, \ldots, a_n)$, $(a_2, \ldots, a_n, a_1)$, $(a_3, \ldots, a_n, a_1, a_2)$, $\ldots$, $(a_n, a_1, a_2, \ldots, a_{n-1})$ that yield the same circular arrangement. There are $(n-1)!$ circular permutations and each of them contains n subsets of size i, $1 \le i < n$, forming an arc of the circle. For a circular permutation π of $[n]$, let $\mathcal{C}(\pi)$ denote the family of all arcs of π. So $|\mathcal{C}(\pi)| = n^2 - n$ because $\mathcal{C}(\pi)$ consists of n arcs of size i for each $1 \le i < n$.

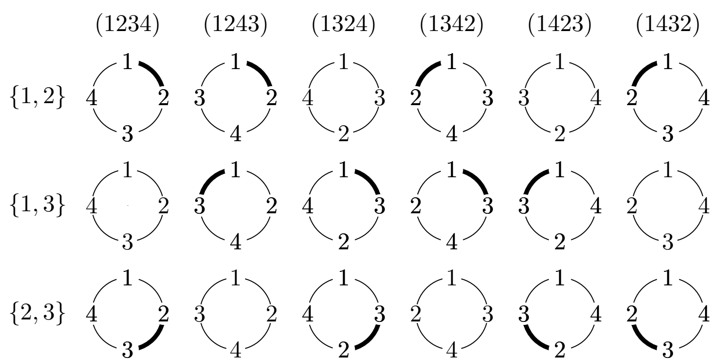
We will always imagine that the n elements are arranged on the circle in clockwise order. This will allow us to speak of the first and last elements of an arc.

Let us illustrate the use of the circle by giving an alternate proof of the Erdős–Ko–Rado Theorem.

Theorem 5.1 (Erdős–Ko–Rado Theorem on the circle)**.** *Let* $n \geq 2k$ *be positive integers and fix a circular permutation of* $[n]$*. Let* $\mathcal{E} = \{E_1, \ldots, E_m\}$ *be a collection of circular arcs of* k *elements such that* $E_i \cap E_j \neq \emptyset$ *for any* i, j*. Then* $m \leq k$*.*

Proof. Without loss of generality let $A = (a_1, \ldots, a_k)$ be a member of $\mathcal{E}$. Since $\mathcal{E}$ consists of subsets of size k, no circular arc of size k (except for A itself) contains A completely. By the intersection property all the remaining arcs E_i have either their first or their last element in A. The candidates for the first element except for A are $a_2, \ldots, a_k$. For $1 \leq i \leq k-1$, we denote the arc starting at a_{i+1} by S_i and the arc terminating at a_i by T_i. Note that $2k \leq n$ implies that $S_i \cap T_i = \emptyset$. Thus for each $1 \leq i \leq k-1$ at most one of the two subsets S_i and T_i is in $\mathcal{E}$. Together with A this gives us $m \leq (k-1)+1 = k$. □

Katona deduced the Erdős–Ko–Rado Theorem from the circle version by simple double counting. Let $n \geq 2k$. Note that a fixed k-element subset is an arc in exactly $k!(n-k)!$ circular permutations. Let $\mathcal{F} \subset \binom{[n]}{k}$ be an intersecting family. Let us count, in two different ways, the number N of pairs (F, π), where $F \in \mathcal{F}$ appears as an arc of a circular permutation π, that is, $F \in \mathcal{C}(\pi)$. The following picture shows an example for the case $n = 4, k = 2$, and $\mathcal{F} = 1, 2, 1, 3, 2, 3$. In this example the members of $\mathcal{F}$ are indicated by thick arcs, and we see that $N = 12$.



On the one hand, each $F \in \mathcal{F}$ appears in $k!(n-k)!$ circular permutations, implying $N = |\mathcal{F}|k!(n-k)!$. On the other hand, there are $(n-1)!$ circular permutations and each of them contains at most k arcs of $\mathcal{F}$ by Theorem 5.1. Thus $N \le (n-1)!k$. Therefore we get

$$|\mathcal{F}|k!(n-k)! \le (n-1)!k,$$

or equivalently

$$|\mathcal{F}| \le \binom{n-1}{k-1}.$$

Probably, a simpler way is by comparing $\mathcal{F}$ and the family

$$\mathcal{A}_1 = \left\{A \in \binom{[n]}{k} : 1 \in A\right\}$$

for each circular permutation π. Since $|\mathcal{C}(\pi) \cap \mathcal{A}_1| = k$ for all π, we always have

$$|\mathcal{C}(\pi) \cap \mathcal{F}| \le |\mathcal{C}(\pi) \cap \mathcal{A}_1|.$$

This implies

$$|\mathcal{F}| \le |\mathcal{A}_1| = \binom{n-1}{k-1}$$

because each k-element subset of $[n]$ occurs with the same multiplicity $k!(n-k)!$. This completes an alternative proof of Theorem 4.1. □

Let us next prove the following important theorem concerning antichains. Recall that a family $\mathcal{S} \subset 2^{[n]}$ is an antichain if $\mathcal{S}$ contains no two members S and S' such that $S \subsetneq S'$.

Theorem 5.2 (Yamamoto's inequality[1] [**112**]). *Suppose that $\mathcal{S} \subset 2^{[n]}$ is an antichain. Then*

$$\sum_{S \in \mathcal{S}} \frac{1}{\binom{n}{|S|}} \le 1. \tag{5.1}$$

Noting that $\emptyset \in \mathcal{S}$ (resp. $[n] \in \mathcal{S}$) implies $\mathcal{S} = \{\emptyset\}$ (resp. $\mathcal{S} = \{[n]\}$), in proving (5.1) we may assume that $\emptyset, [n] \notin \mathcal{S}$. Then we shall consider arcs on a circular permutation π again, that is, we look at $\mathcal{S} \cap \mathcal{C}(\pi)$.

Proposition 5.3. *Suppose that $\mathcal{S}_0 \subset \mathcal{C}(\pi)$ is an antichain. Then $|\mathcal{S}_0| \le n$, and moreover, equality holds only if all members of $\mathcal{S}_0$ have the same size.*

[1]This inequality has been rediscovered several times by several authors, including Lubell [**90**] and Meshalkin [**93**], and it is also called the LYM inequality.

Proof. Note that if the last element of two arcs is the same, then the longer one contains the shorter one. Consequently, for the antichain $\mathcal{S}_0$ there can be at most one member of $\mathcal{S}_0$ for every last element. This proves $|\mathcal{S}_0| \leq n$.

Should we have equality, there must be *exactly* one member of $\mathcal{S}_0$ for every last element. If not all are of the same size, then we can find two consecutive last elements, say a_i and a_{i+1}, such that the arc in $\mathcal{S}_0$ corresponding to a_{i+1} is longer. However, this means that this arc contains the shorter arc ending in a_i, a contradiction. $\square$

Proof of Theorem 5.2. We count the number N of pairs (S, π), where $S \in \mathcal{S}$ appears as an arc on a circular permutation π. Since each $S \in \mathcal{S}$ appears in $|S|!(n-|S|)!$ circular permutations, it follows that $N = \sum_{S \in \mathcal{S}} |S|!(n-|S|)!$. On the other hand, there are $(n-1)!$ circular permutations and each of them contains at most n arcs of $\mathcal{S}$ by Proposition 5.3. This yields $N \leq (n-1)!n = n!$. Thus

$$\sum_{S \in \mathcal{S}} |S|!(n-|S|)! \leq n!,$$

and dividing both sides by $n!$ gives (5.1). $\square$

Let us next deduce Sperner's Theorem (Theorem 1.1) from Yamamoto's inequality. For convenience we restate it here.

Theorem. *Let $\mathcal{S} \subset 2^{[n]}$ be an antichain. Then $|\mathcal{S}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Moreover, equality holds if and only if $\mathcal{S} = \binom{[n]}{\frac{n}{2}}$ for n even and $\mathcal{S} = \binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ or $\binom{[n]}{\lceil \frac{n}{2} \rceil}$ for n odd.*

Proof. Noting that $\binom{n}{k}$ attains its maximum (as a function of k) for $k = \frac{n}{2}$ in the even case, and for $k = \frac{n-1}{2}, \frac{n+1}{2}$ in the odd case, (5.1) implies

$$\frac{|\mathcal{S}|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \sum_{S \in \mathcal{S}} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq \sum_{S \in \mathcal{S}} \frac{1}{\binom{n}{\lfloor \frac{n}{|S|} \rfloor}} \leq 1,$$

that is,

$$|\mathcal{S}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Moreover, in the case of equality one must have $|S| = \frac{n}{2}$ for all $S \in \mathcal{S}$ if n is even. If n is odd, we can only deduce that $|S| = \lfloor \frac{n}{2} \rfloor$ or

$|S| = \lceil \frac{n}{2} \rceil$ for all $S \in \mathcal{S}$. However, by Proposition 5.3, $|\mathcal{S}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ is only possible if all $S \in \mathcal{S}$ have the same size. That is, $\mathcal{S} = \binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ or $\mathcal{S} = \binom{[n]}{\lceil \frac{n}{2} \rceil}$. $\square$

Our next target is an old theorem of Erdős. A family $\mathcal{S} \subset 2^{[n]}$ is called a *t-antichain* if $\mathcal{S}$ contains no $t+1$ members $S_0, S_1, \ldots, S_t$ forming a chain, that is, $S_0 \subsetneqq S_1 \subsetneqq \cdots \subsetneqq S_t$.

Let $k_1, k_2, \ldots, k_t$ be such that $\binom{n}{k_1}, \ldots, \binom{n}{k_t}$ are the largest t binomial coefficients. Note that for $n-t$ odd, $\{k_1 \ldots, k_t\} = [\frac{n-t+1}{2}, \frac{n+t-1}{2}]$, and for $n-t$ even, $\{k_1 \ldots, k_t\}$ is either $[\frac{n-t}{2}, \frac{n+t}{2} - 1]$ or $[\frac{n-t}{2}+1, \frac{n+t}{2}]$.

Proposition 5.4. *Suppose that $\mathcal{S} \subset 2^{[n]}$ is a t-antichain and π is a circular permutation of $[n]$. Then we have*

(i) $|\mathcal{S} \cap \mathcal{C}(\pi)| \le tn$,

(ii) $$\sum_{S \in \mathcal{S} \cap \mathcal{C}(\pi)} \frac{1}{|S|!(n-|S|)!} \le \sum_{i=1}^{t} \frac{n}{k_i!(n-k_i)!}.$$

Proof. The arcs with identical last element form a chain. This proves that in $\mathcal{S} \cap \mathcal{C}(\pi)$ there can be at most t members with the same last element, proving (i).

To prove (ii) we note that $\frac{1}{|S|!(n-|S|)!} = \binom{n}{|S|}/n!$ and thus the t largest values are for $\{k_1, \ldots, k_t\}$. By (i), the left-hand side (LHS) of (ii) has at most tn terms. Each value of $|S|$ can occur at most n times. These facts imply (ii). $\square$

Theorem 5.5 (The Erdős t-antichain theorem)**.** *Let $n \ge t$, and let $\mathcal{S} \subset 2^{[n]}$ be a t-antichain. Then we have*

$$|\mathcal{S}| \le \sum_{i=1}^{t} \binom{n}{k_i}.$$

Proof. Let us sum (ii) of Proposition 5.4 over all $(n-1)!$ circular permutations. Since each $S \in \mathcal{S}$ is counted $|S|!(n-|S|)!$ times, it follows that

$$|\mathcal{S}| \le (n-1)! \sum_{i=1}^{t} \frac{n}{k_i!(n-k_i)!} = \sum_{i=1}^{t} \binom{n}{k_i},$$

as needed. $\square$

Next we prove a very simple but useful statement about the shadows. Let $k \geq 2$. For a family $\mathcal{S} \subset \binom{[n]}{k}$ of arcs on a circular permutation π, let $\sigma^\pi(\mathcal{S})$ denote the family of arcs of size $k-1$ that are contained in some member of $\mathcal{S}$. If $\mathcal{S}$ consists of all n arcs of size k, then $|\sigma^\pi(\mathcal{S})| = n$.

Proposition 5.6. *Let $k \geq 2$ and let $\mathcal{S} \subset \binom{[n]}{k}$ be a family of arcs on a circular permutation π. Unless $\mathcal{S}$ consists of all n arcs of size k, it follows that*

$$|\sigma^\pi(\mathcal{S})| \geq |\mathcal{S}| + 1.$$

Proof. To every $S \in \mathcal{S}$ associate the arc of size $k-1$ obtained by deleting the last element of S. This shows that $|\sigma^\pi(\mathcal{S})| \geq |\mathcal{S}|$. Let $|\mathcal{S}| < n$. Then there are two consecutive arcs, say $(a_1, \ldots, a_k)$ and $(a_2, \ldots, a_{k+1})$, such that the first is in $\mathcal{S}$ but the second is not. Then the arc $(a_2, \ldots, a_k)$ of size $k-1$ is in $\sigma^\pi(\mathcal{S})$ but has not been counted yet. This proves $|\sigma^\pi(\mathcal{S})| \geq |\mathcal{S}| + 1$. $\square$

The next statement is about *several* — not necessarily distinct — families. Such statements have proved useful in dealing with problems concerning one family as well as being interesting in their own right.

Let $t \geq 2$ and let $\mathcal{F}_1, \ldots, \mathcal{F}_r \subset 2^{[n]}$. We say that $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are *r-cross union* if $F_1 \cup \cdots \cup F_r \neq [n]$ for all choices of $F_i \in \mathcal{F}_i$, $i = 1, \ldots, r$.

Proposition 5.7. *Let $1 \leq k_i < n$ for $1 \leq i \leq r$. Let $\mathcal{F}_i$ be a family of arcs of size k_i on a circular permutation $\pi = (a_1, \ldots, a_n)$. Suppose that $k_1 + \cdots + k_r \geq n$ and $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are r-cross union. Then we have*

$$\sum_{i=1}^{r} (|\mathcal{F}_i| + k_i) \leq rn. \tag{5.2}$$

Proof. We use induction on $k_1 + \cdots + k_r$. First we prove the base case $k_1 + \cdots + k_r = n$. The idea is very simple. For every $1 \leq j \leq n$ consider the r arcs $F_1^{(j)}, \ldots, F_r^{(j)}$, where the first element of $F_1^{(j)}$ is a_j, the first element of $F_2^{(j)}$ is a_{j+k_1}, ..., and the first element of $F_r^{(j)}$ is $a_{j+k_1+k_2+\cdots+k_{r-1}}$. Then these r arcs partition the whole circle.

Therefore not all r can be members of the corresponding families. That is, there is some i such that $F_i^{(j)} \notin \mathcal{F}_i$. For the n choices of j we get altogether at least n missing sets, yielding

$$|\mathcal{F}_1| + |\mathcal{F}_2| + \cdots + |\mathcal{F}_r| \leq rn - n = rn - (k_1 + \cdots + k_r),$$

which is equivalent to (5.2).

Now suppose that $k_1 + \cdots + k_r > n$ and we are in the induction step. Suppose that we can choose i, $1 \leq i \leq r$, such that $k_i \geq 2$ and $|\mathcal{F}_i| < n$. Then we replace $\mathcal{F}_i$ with $\sigma^\pi(\mathcal{F}_i)$ and leave the other families unchanged. In view of Proposition 5.6 we have that

$$|\sigma^\pi(\mathcal{F}_i)| + k_i - 1 \geq |\mathcal{F}_i| + k_i.$$

Consequently, (5.2) follows from the induction hypothesis.

The only remaining case is where after reordering the families we have $k_1 = \cdots = k_s = 1$ and $|\mathcal{F}_{s+1}| = \cdots = |\mathcal{F}_r| = n$ for some $s \geq 0$. We may further assume that $|F_s| < n$. If $k_{s+1} + \cdots + k_r \geq n$, then $|\mathcal{F}_{s+1}| = \cdots = |\mathcal{F}_r| = n$ implies that $\mathcal{F}_{s+1}, \ldots, \mathcal{F}_r$ are *not* $(r-s)$-cross union already, and $\mathcal{F}_1, \ldots, \mathcal{F}_r$ cannot be r-cross union. Thus we may assume that $k_{s+1} + \cdots + k_r \leq n - 1$ and so $s \geq 1$.

Then without loss of generality $|\mathcal{F}_1| < n$ and therefore $|\mathcal{F}_1| + k_1 \leq n$. Now consider the $(r-1)$-cross union families $\mathcal{F}_2, \ldots, \mathcal{F}_r$. Recall that we have assumed $k_1 + \cdots + k_r \geq n + 1$ and $k_1 = 1$. Thus $k_2 + \cdots + k_r \geq (n+1) - k_1 = n$ and we may apply the induction hypothesis and obtain $\sum_{i=2}^r (|\mathcal{F}_i| + k_i) \leq (r-1)n$. Adding $|\mathcal{F}_1| + 1 \leq n$, the inequality (5.2) follows. □

We say that a family $\mathcal{F} \subset 2^{[n]}$ is r*-wise union* if $F_1 \cup \cdots \cup F_r \neq [n]$ for all $F_1, \ldots, F_r \in \mathcal{F}$.

Theorem 5.8 ([**43**])**.** *Let $0 < k < n$, $r \geq 2$ and $kr \geq n$. If $\mathcal{F} \subset \binom{[n]}{k}$ is r-wise union, then $|\mathcal{F}| \leq \binom{n-1}{k}$.*

Proof. Let π be a circular permutation. Applying Proposition 5.7 to $\mathcal{F}_1 = \mathcal{F}_2 = \cdots = \mathcal{F}_r := \mathcal{F} \cap \mathcal{C}(\pi)$, we get $|\mathcal{F} \cap \mathcal{C}(\pi)| \leq n - k$. Adding up this inequality over all $(n-1)!$ cyclic permutations gives $k!(n-k)!|\mathcal{F}| \leq (n-k)(n-1)!$, or equivalently $|\mathcal{F}| \leq \binom{n-1}{k}$. □

Let us note that the above theorem is best possible. Indeed, more careful analysis shows that $|\mathcal{F}| = \binom{n-1}{k}$ if and only if $\mathcal{F} = \binom{Y}{k}$ for some $Y \in \binom{[n]}{n-1}$, except for the case $r = 2$ and $n = 2k$; cf. [**50**].

Chapter 9

The Erdős matching conjecture

After the publication of the Erdős–Ko–Rado Theorem, Erdős arrived at the following very natural generalization of it. Let n, k, and s be positive integers. The *matching number* of a family $\mathcal{F} \subset \binom{[n]}{k}$ is defined as the maximum number s such that $\mathcal{F}$ contains s pairwise disjoint members, and it is denoted by $\nu(\mathcal{F})$. Suppose that $\nu(\mathcal{F}) \leq s$, that is, there are no $s+1$ pairwise disjoint members in $\mathcal{F}$. How large can $|\mathcal{F}|$ be? If $n < (s+1)k$ then $\nu(\binom{[n]}{k}) \leq s$. Thus we need the assumption $n \geq (s+1)k$ to make the problem non-trivial. Note that the case $s = 1$ corresponds to the Erdős–Ko–Rado Theorem.

Erdős found two natural constructions:

$$\mathcal{A}(k,s) = \binom{[k(s+1)-1]}{k},$$
$$\mathcal{B}(n,k,s) = \left\{ B \in \binom{[n]}{k} : B \cap [s] \neq \emptyset \right\}.$$

Note that $\mathcal{A}(k,s)$ is independent of n. Since $C \in \binom{[n]}{k}$ satisfies $C \notin \mathcal{B}(n,k,s)$ if and only if $C \subset [n] \setminus [s]$, we have $|\mathcal{B}(n,k,s)| = \binom{n}{k} - \binom{n-s}{k}$.

Conjecture 9.1 (Erdős matching conjecture [**32**])**.** *Let $n \geq (s+1)k$ and $\mathcal{F} \subset \binom{[n]}{k}$. If $\nu(\mathcal{F}) \leq s$ then*

$$|\mathcal{F}| \leq \max\{|\mathcal{A}(k,s)|, |\mathcal{B}(n,k,s)|\}. \tag{9.1}$$

Erdős proved that $|\mathcal{F}| \le |\mathcal{B}(n,k,s)|$ for fixed k and s and for n very large. In this chapter we prove the following.

Theorem 9.2. *Let* $n \ge (2s+1)k - s$ *and* $\mathcal{F} \subset \binom{[n]}{k}$. *If* $\nu(\mathcal{F}) \le s$ *then* $|\mathcal{F}| \le |\mathcal{B}(n,k,s)|$.

Proof. It is easy to see that the (i,j)-shift does not increase $\nu(\mathcal{F})$. So we may assume that $\mathcal{F}$ is shifted. For a subset $A \subset [s+1]$ and a family $\mathcal{G} \subset 2^{[n]}$, let

$$\mathcal{G}(A) = \{G \setminus [s+1] : G \in \mathcal{G},\ G \cap [s+1] = A\}.$$

Note that if $\mathcal{G}$ is k-uniform and $|A| \le k$, then $\mathcal{G}(A) \subset \binom{[n]\setminus[s+1]}{k-|A|}$. Note also that $A \cap [s] \ne \emptyset$ for all $A \subset [s+1]$ with $|A| \ge 2$. We simply use $\mathcal{B}$ for our target family $\mathcal{B}(n,k,s)$. Then we have $\mathcal{B}(A) = \binom{[n]\setminus[s+1]}{k-|A|}$ for all $A \subset [s+1]$ with $|A| \ge 2$. In particular,

$$|\mathcal{F}(A)| \le |\mathcal{B}(A)| \text{ for } A \subset [s+1] \text{ with } |A| \ge 2. \tag{9.2}$$

The equality

$$|\mathcal{G}| = \sum_{A \subset [s+1]} |\mathcal{G}(A)| \tag{9.3}$$

should be obvious. For simplicity, let $f_0 = |\mathcal{F}(\emptyset)|$ and let $f_i = |\mathcal{F}(\{i\})|$ for $1 \le i \le s+1$. Then (9.1) will follow once we show the following:

$$f_0 + \sum_{i=1}^{s+1} f_i \le s\binom{n-s-1}{k-1}. \tag{9.4}$$

Indeed, $\mathcal{B}(\emptyset) = \mathcal{B}(\{s+1\}) = \emptyset$, showing that the RHS of (9.4) is equal to $|\mathcal{B}(\emptyset)| + \sum_{i=1}^{s+1} |\mathcal{B}(\{i\})|$. This, together with (9.2) and (9.3), yields $|\mathcal{F}| \le |\mathcal{B}(n,k,s)|$.

We deduce (9.4) from the following two inequalities:

$$f_0 \le s f_{s+1}, \tag{9.5}$$

$$\sum_{i=1}^{s} f_i + (s+1) f_{s+1} \le s\binom{n-s-1}{k-1}. \tag{9.6}$$

Let us see how these inequalities imply (9.4). In view of (9.5), the LHS of (9.4) does not exceed the LHS of (9.6). Thus (9.4) follows from (9.6).

For the proof of (9.5) and (9.6) we need to use the fact that $\mathcal{F}$ is shifted. Recall that the immediate shadow is defined by

$$\sigma(\mathcal{F}) = \left\{ G \in \binom{[n]}{k-1} : G \subset F \text{ for some } F \in \mathcal{F} \right\}.$$

We use the following general result to prove (9.5).

Theorem 9.3 ([**52**]). *Let* $\mathcal{H} \subset \binom{[n]}{k}$. *If* $\nu(\mathcal{H}) \leq s$ *then* $|\mathcal{H}| \leq s|\sigma(\mathcal{H})|$.

We will prove the above theorem at the end of this chapter. Since $\nu(\mathcal{F}(\emptyset)) \leq \nu(\mathcal{F}) \leq s$, the above theorem yields

$$f_0 = |\mathcal{F}(\emptyset)| \leq s|\sigma(\mathcal{F}(\emptyset))|. \tag{9.7}$$

Then (9.5) follows from (9.7) via the following easy observation.

Claim 9.4. $\sigma(\mathcal{F}(\emptyset)) \subset \mathcal{F}(\{s+1\})$.

Proof. Suppose, to the contrary, that there is an $H \in \sigma(\mathcal{F}(\emptyset))$ such that $H \notin \mathcal{F}(\{s+1\})$. Since $H \in \sigma(\mathcal{F}(\emptyset))$, we have $H \in \binom{[n]\setminus[s+1]}{k-1}$ and $H \subset F$ for some $F \in \mathcal{F}$. Then we can write $F = H \sqcup \{y\}$ for some $y \geq s+2$. Since $\mathcal{F}$ is shifted, it follows that $(F \setminus \{y\}) \sqcup \{s+1\} = \{s+1\} \sqcup H \in \mathcal{F}$. This means that $H \in \mathcal{F}(\{s+1\})$, a contradiction. □

Next we show (9.6). For the $(k-1)$-uniform families $\mathcal{F}(\{i\})$, $1 \leq i \leq s+1$, shiftedness implies

$$\mathcal{F}(\{s+1\}) \subset \mathcal{F}(\{s\}) \subset \cdots \subset \mathcal{F}(\{1\}). \tag{9.8}$$

The assumption $n \geq (2s+1)k - s$ is equivalent to $n - (s+1) \geq (2s+1)(k-1)$. Therefore one can choose $2s+1$ pairwise disjoint $(k-1)$-element subsets $H_0, H_1, \ldots, H_{2s} \subset [n] \setminus [s+1]$.

Let us construct an auxiliary bipartite graph G with vertex partition $V(G) = V_1 \sqcup V_2$, where $V_1 = [s+1], V_2 = \{H_0, \ldots, H_{2s}\}$, and there is an edge joining $i \in V_1$ and $H_j \in V_2$ if and only if $H_j \in \mathcal{F}(\{i\})$. Note that $s+1$ pairwise independent edges in G would immediately yield $s+1$ pairwise disjoint edges in $\mathcal{F}$, a contradiction. Now, by invoking the Köning Theorem (see, e.g., Theorem 2.1.1 in [**25**]), we can choose an s-element subset $C \subset V(G)$ covering all edges of G. That is, at least one endpoint of every edge of G is contained in C. We denote by d_i the degree of $i \in V_1$. Note that $d_1 + \cdots + d_{s+1}$ is the number of edges $|E(G)|$. Note also that $d_{s+1} \le d_s \le \cdots \le d_1$ by (9.8).

Claim 9.5. $|E(G)| + sd_{s+1} \le s(2s+1)$.

Proof. Let us define x by $|C \cap V_1| = s - x$. Since $|C| < s+1$, one can choose $i \in V_1 \setminus C$. Then $|C \cap V_2| = x$ implies $d_i \le x$ and, in particular, $d_{s+1} \le x$. Noting that $|V_1 \setminus C| = x + 1 \le s + 1$, we have

$$|E(G)| \le |C \cap V_1||V_2| + |V_1 \setminus C|x \le (s-x)(2s+1) + (s+1)x.$$

Together with $sd_{s+1} \le sx$, we get $|E(G)| + sd_{s+1} \le s(2s+1)$. □

We want to deduce (9.6) from Claim 9.5. Choose $H_0, \ldots, H_{2s}$ uniformly at random. Then the probability of $H_j \in \mathcal{F}(\{i\})$, or equivalently the probability that $i \in V_1$ and $H_j \in V_2$ are adjacent, is

$$\frac{f_i}{\binom{n-s-1}{k-1}}.$$

Thus the expected value of d_i is

$$\frac{(2s+1)f_i}{\binom{n-s-1}{k-1}}.$$

Therefore the expected value of

$$\sum_{i=1}^{s+1} d_i + sd_{s+1} = |E(G)| + sd_{s+1}$$

is exactly

$$\frac{2s+1}{\binom{n-s-1}{k-1}} \left(\sum_{i=1}^{s+1} f_i + sf_{s+1} \right).$$

By Claim 9.5, this never exceeds $s(2s+1)$, which proves (9.6). □

Finally we prove Theorem 9.3.

Proof. Let $\mathcal{H} \subset \binom{[n]}{k}$. We may assume that $\mathcal{H}$ is shifted and $\nu(\mathcal{H}) = s$. For fixed $s \ge 2$ we prove that

$$|\mathcal{H}| \le s|\sigma(\mathcal{H})| \tag{9.9}$$

for all (n, k) with $n \ge k \ge 1$ by double induction on k and n.

First consider the case $k = 1$. In this case we may assume that $\mathcal{H} = \{1, 2, \ldots, s\}$. Then $\sigma(\mathcal{H}) = \{\emptyset\}$ and $|\sigma(\mathcal{H})| = 1$. So we always have equality in (9.9).

Next we deal with the case $n \leq (s+1)k-1$, that is, $sk \geq n-k+1$. Let G be a bipartite graph with vertex partition $V(G) = V_1 \sqcup V_2$, where $V_1 = \mathcal{H}$ and $V_2 = \sigma(\mathcal{H})$, and $H \in V_1$ and $K \in V_2$ are adjacent if and only if $H \supset K$. For $H \in V_1$ the degree of H is k, while for $K \in V_2$ the degree of K is at most $|[n] \setminus K| = n - (k-1)$. Thus by counting the number of edges from both sides we get $|V_1|k \leq |V_2|(n-k+1)$, that is,

$$|\mathcal{H}| \leq \frac{n-k+1}{k}|\sigma(\mathcal{H})| \leq s|\sigma(\mathcal{H})|.$$

Finally let $n \geq (s+1)k$. We prove (9.9) for (n, k) by assuming that (9.9) holds for $(*, k-1)$ and $(n-1, k)$. We want to apply the induction hypothesis to the following two families:

$$\mathcal{C} = \{H : n \notin H \in \mathcal{H}\} \subset \binom{[n-1]}{k},$$
$$\mathcal{D} = \{H \setminus \{n\} : n \in H \in \mathcal{H}\}\} \subset \binom{[n-1]}{k-1}.$$

Since $\mathcal{C} \subset \mathcal{H}$ we have $\nu(\mathcal{C}) \leq \nu(\mathcal{H}) = s$, and it follows from the hypothesis that $|\mathcal{C}| \leq s|\sigma(\mathcal{C})|$.

Claim 9.6. $\nu(\mathcal{D}) \leq s$.

Proof. Assume, on the contrary, that we have $s+1$ pairwise disjoint subsets $D_1, \ldots, D_{s+1} \in \mathcal{D}$. Then $D_i \sqcup \{n\} \in \mathcal{H}$ for $1 \leq i \leq s+1$, and

$$|[n-1] \setminus (D_1 \sqcup \cdots \sqcup D_{s+1})| = (n-1) - (s+1)(k-1) \geq s.$$

Thus we can find s vertices $x_1, \ldots, x_s$ in $[n-1] \setminus (D_1 \sqcup \cdots \sqcup D_{s+1})$. Since $\mathcal{H}$ is shifted, we have $D_i \sqcup \{x_i\} \in \mathcal{H}$ for all $1 \leq i \leq s$, and moreover $D_{s+1} \sqcup \{n\} \in \mathcal{H}$. This contradicts $\nu(\mathcal{H}) = s$. □

By the hypothesis we get $|\mathcal{D}| \leq s|\sigma(\mathcal{D})|$ and

$$|\mathcal{H}| = |\mathcal{C}| + |\mathcal{D}| \leq s(|\sigma(\mathcal{C})| + |\sigma(\mathcal{D})|).$$

Thus it remains to show the following.

Claim 9.7. $|\sigma(\mathcal{C})| + |\sigma(\mathcal{D})| = |\sigma(\mathcal{H})|$.

Proof. Let $\sigma(\mathcal{H}) = \tilde{\mathcal{C}} \sqcup \tilde{\mathcal{D}}$, where

$$\tilde{\mathcal{C}} = \{K \in \sigma(\mathcal{H}) : n \notin K\},$$
$$\tilde{\mathcal{D}} = \{K \in \sigma(\mathcal{H}) : n \in K\}.$$

Then clearly $\sigma(\mathcal{C}) \subset \tilde{\mathcal{C}}$. Moreover, if $K \sqcup \{n\} \in \mathcal{H}$, then since $\mathcal{H}$ is shifted we have $K \sqcup \{i\} \in \mathcal{H}$ for all $i < n$ and $K \in \sigma(\mathcal{C})$. This means that $\sigma(\mathcal{C}) = \tilde{\mathcal{C}}$. On the other hand, it follows (without using shiftedness) that $\sigma(\mathcal{D}) = \{K \setminus \{n\} : K \in \tilde{\mathcal{D}}\}$ and $|\sigma(\mathcal{D})| = |\tilde{\mathcal{D}}|$. Thus we have $|\sigma(\mathcal{C})| + |\sigma(\mathcal{D})| = |\tilde{\mathcal{C}}| + |\tilde{\mathcal{D}}| = |\sigma(\mathcal{H})|$. □

This completes the proof of Theorem 9.3. □

Exercise 9.8. In Theorem 9.3 suppose further that $\mathcal{H}$ is shifted. Show that if $|\mathcal{H}| = s|\sigma(\mathcal{H})|$ then $\mathcal{H} = \binom{[sk+k-1]}{k}$.

In Theorem 9.2 we only proved the inequalities, but one can further prove the uniqueness of the optimal families, that is, if $|\mathcal{F}| = |\mathcal{B}(n,k,s)|$ then $\mathcal{F} \cong \mathcal{B} := \mathcal{B}(n,k,s)$. The idea of the proof is as follows. First, it is not so difficult to show this for shifted families. Second, one can also show that if $\nu(\mathcal{F}) = s$ and $S_{i,j}(\mathcal{F}) = \mathcal{B}$ then $\mathcal{F} \cong \mathcal{B}$. The proof of this part is similar to the proof of Lemma 10.5.

Chapter 14

r-Cross union families

Let n, k, and r be positive integers. We say that families $\mathcal{F}_1, \dots, \mathcal{F}_r \subset \binom{[n]}{k}$ are r-cross intersecting if $F_1 \cap \cdots \cap F_r \neq \emptyset$ for all $F_i \in \mathcal{F}_i$, $1 \le i \le r$. In this chapter we prove the following extension of the Erdős–Ko–Rado Theorem.

Theorem 14.1 ([**62**]). *Let $(r-1)n \ge rk$ and let $\mathcal{F}_1, \dots, \mathcal{F}_r \subset \binom{[n]}{k}$ be r-cross intersecting families. Then $\prod_{i=1}^{r} |\mathcal{F}_i| \le \binom{n-1}{k-1}^r$.*

Recall that families $\mathcal{G}_1, \dots, \mathcal{G}_r \subset \binom{[n]}{l}$ are r-cross union if $G_1 \cup \cdots \cup G_r \neq [n]$ for all $G_i \in \mathcal{G}_i$, $1 \le i \le r$. Note that $\mathcal{F}_1, \dots, \mathcal{F}_r \subset \binom{[n]}{k}$ are r-cross intersecting if and only if the complement families $\mathcal{F}_1^{\mathrm{c}}, \dots, \mathcal{F}_r^{\mathrm{c}} \subset \binom{[n]}{l}$ are r-cross union, where $n = k + l$. So considering r-cross intersecting families is equivalent to considering the corresponding r-union families. The former is more popular, but sometimes the latter is easier to handle in notation. So we will consider the r-union version of Theorem 14.1 in a stronger form. To state the result let us introduce an important invariant, which is a key to the proof. For $\mathcal{G} \subset \binom{[n]}{l}$ choose a unique real $x \ge l$ so that $|\mathcal{G}| = \binom{x}{l}$, and let $\|\mathcal{G}\|_l := x$.

Theorem 14.2. *Let $n \le rl$ and let $\mathcal{G}_1, \dots, \mathcal{G}_r \subset \binom{[n]}{l}$ be r-cross union families. Then we have the following.*

(i) $\sum_{i=1}^{r} \|\mathcal{G}_i\|_l \le r(n-1)$.

(ii) $\prod_{i=1}^{r} |\mathcal{G}_i| \le \binom{n-1}{l}^r$.

Theorem 14.1 follows from (ii) of Theorem 14.2 by considering the complement families. Note that the condition $n \leq rl$ in the theorem cannot be weakened. (Consider, say, r copies of $\binom{[n]}{l}$ for $n > rl$.)

The proof of Theorem 14.2 is simple and is taken from [**62**]. In fact a referee of the paper said "The study of cross-intersecting families has a long history ... It is therefore somewhat shocking that a statement as simple as Theorem 14.1 is being proven for the first time only in 2010. One's surprise only increases after reading the proof: it uses little more than the 'circle method' of Katona, plus the AM-GM inequality and a simple induction on n." This is a nice summary of the proof except that the induction is actually on $rl - n$, which is a tricky point of the proof. We also use the Lovász version of the Kruskal–Katona Theorem, that is, Theorem 2.14 with Remark 2.17. For convenience we restate it here.

Theorem 14.3. *Let $0 \leq p < k$ be integers, and let x be a positive real number with $k \leq x$. If $\mathcal{F} \subset \binom{[n]}{k}$ and $|\mathcal{F}| \geq \binom{x}{k}$, then $|\sigma_p(\mathcal{F})| \geq \binom{x}{p}$.*

We mention that the case $r = 2$ of Theorem 14.1 was obtained by Pyber [**97**], and Theorem 5.8 is exactly for the case $\mathcal{F}_1 = \cdots = \mathcal{F}_r$.

Proof of Theorem 14.2. For $1 \leq i \leq r$ let $x_i = \|\mathcal{G}_i\|_l$, that is, $|\mathcal{G}_i| = \binom{x_i}{l}$. First we show that (i) implies (ii). Indeed we have

$$\prod_{i=1}^{r} |\mathcal{G}_i| = \binom{x_1}{l} \cdots \binom{x_r}{l}$$

$$= \frac{1}{(l!)^r} \prod_{i=0}^{l-1} (x_1 - i) \cdots (x_r - i)$$

$$\leq \frac{1}{(l!)^r} \prod_{i=0}^{l-1} \left(\frac{x_1 + \cdots + x_r}{r} - i \right)^r \tag{14.1}$$

$$= \binom{\frac{x_1 + \cdots + x_r}{r}}{l}^r$$

$$\leq \binom{n-1}{l}^r, \tag{14.2}$$

where we used the AM-GM (arithmetic mean and geometric mean) inequality in (14.1) and (i) in (14.2).

Now we prove (i) by induction on $s := rl - n$.

First we consider the initial step $s = 0$, that is, $n = rl$. Recall Katona's circle from Chapter 5. Fix a circular permutation $\tau = (a_1, a_2, \ldots, a_n)$, and let $\mathcal{C}(\tau) = \{A_1, A_2, \ldots, A_n\}$ be the set of arcs of size l in τ, where $A_i = \{a_i, a_{i+1}, \ldots, a_{i+l-1}\}$ (the indices are read mod n). For $1 \le i \le r$, let $\mathcal{G}_i^\tau = \mathcal{G}_i \cap \mathcal{C}(\tau)$ denote the subfamily of $\mathcal{G}_i$ consisting of arcs in τ.

Claim 14.4. $\sum_{i=1}^r |\mathcal{G}_i^\tau| \le r(n-l)$.

Proof. This follows from Proposition 5.7, but we repeat the simple proof for convenience. For every $1 \le j \le n$ consider the r arcs $F_1^{(j)}, \ldots, F_r^{(j)}$, where the first element of $F_1^{(j)}$ is a_j, the first element of $F_2^{(j)}$ is a_{j+l}, $\ldots$, and the first element of $F_r^{(j)}$ is $a_{j+(r-1)l}$. Then these r arcs partition $[n]$. Therefore not all r can be members of the corresponding families. That is, there is some i such that $F_i^{(j)} \notin \mathcal{G}_i^\tau$. For the n choices of j we get altogether at least n missing sets, yielding $\sum_{i=1}^r |\mathcal{G}_i^\tau| \le rn - n = rn - rl = r(n-l)$. □

Claim 14.5. If $n = rl$, then we have $\sum_{i=1}^r |\mathcal{G}_i| \le r\binom{n-1}{l}$.

Proof. Let $\mathcal{C}_n$ denote the set of all circular permutations. Then $|\mathcal{C}_n| = (n-1)!$, and by Claim 14.4 we get

$$\sum_{\tau \in \mathcal{C}_n} \sum_{i=1}^r |\mathcal{G}_i^\tau| \le (n-1)!\, r(n-l).$$

Each $G \in \mathcal{G}_i$ is counted as an arc $l!(n-l)!$ times in $\sum_{\tau \in \mathcal{C}_n} |\mathcal{G}_i^\tau|$, and

$$\sum_{\tau \in \mathcal{C}_n} |\mathcal{G}_i^\tau| = l!(n-l)!|\mathcal{G}_i|.$$

Combining these two statements, we have

$$\begin{aligned}\sum_{i=1}^r |\mathcal{G}_i| &= \sum_{i=1}^r \frac{1}{l!(n-l)!} \sum_{\tau \in C_n} |\mathcal{G}_i^\tau| = \frac{1}{l!(n-l)!} \sum_{\tau \in C_n} \sum_{i=1}^r |\mathcal{G}_i^\tau| \\ &\le \frac{(n-1)!\, r(n-l)}{l!(n-l)!} = r\binom{n-1}{l},\end{aligned}$$

as required. □

Since $\binom{x}{l}$ is convex for $x \geq l$, we have

$$\binom{\frac{x_1+\cdots+x_r}{r}}{l} \leq \frac{1}{r}\sum_{i=1}^{r}\binom{x_i}{l} = \frac{1}{r}\sum_{i=1}^{r}|\mathcal{G}_i| \leq \binom{n-1}{l},$$

where the last inequality follows from Claim 14.5. By comparing the first and last terms we get $\frac{x_1+\cdots+x_r}{r} \leq n-1$, that is, $\sum_{i=1}^{r}\|\mathcal{G}_i\|_l \leq r(n-1)$. This proves (i) for the case $s = 0$.

Next, for the induction step let $s > 0$. Suppose that (i) is true for the case $rl - n = s$, and now we consider the case $rl - n = s + 1$. Let $\mathcal{G}_1, \ldots, \mathcal{G}_r \subset \binom{[n]}{l}$ be r-cross union families, and let $x_i = \|\mathcal{G}_i\|_l$ for $1 \leq i \leq r$.

Here is the tricky part. Define $\mathcal{H}_i = \mathcal{G}_i \sqcup \mathcal{D}_i \subset \binom{[n+1]}{l}$ by

$$\mathcal{D}_i = \{D \sqcup \{n+1\} : D \in \sigma_{l-1}(\mathcal{G}_i)\}.$$

Then, by Theorem 14.3, we have $|\mathcal{D}_i| = |\sigma_{l-1}(\mathcal{G}_i)| \geq \binom{x_i}{l-1}$, and

$$|\mathcal{H}_i| = |\mathcal{G}_i| + |\mathcal{D}_i| \geq \binom{x_i}{l} + \binom{x_i}{l-1} = \binom{x_i+1}{l},$$

that is,

$$z_i := \|\mathcal{H}_i\|_l \geq x_i + 1. \tag{14.3}$$

Since $\mathcal{H}_1, \ldots, \mathcal{H}_r \subset \binom{[n+1]}{l}$ are r-cross union families and $rl - (n+1) = s$, we can apply the induction hypothesis to these families. Therefore we obtain

$$\sum_{i=1}^{r} z_i \leq r((n+1)-1) = rn. \tag{14.4}$$

Finally it follows from (14.3) and (14.4) that $\sum_{i=1}^{r} x_i \leq r(n-1)$. This completes the proof of the theorem. □

Chapter 15

Random walk method

Suppose that we are given a family $\mathcal{F}$ of subsets which satisfies some conditions, say, t-intersecting, and we want to bound the size of $\mathcal{F}$. To this end we relate subsets in $\mathcal{F}$ with walks in the plane. These walks should satisfy some corresponding conditions, say, hitting a certain line. Then, by counting the number of such walks, we get some information about $|\mathcal{F}|$. This is the core idea of the random walk method. The idea looks simple, but combined with the shifting operation it becomes a strong, far-reaching tool for studying intersecting families. In this chapter we present some simple applications of the method.

Notation: For integers a and b we use $[a,b]_2$ to denote the set

$$\{a + 2i \in \mathbb{Z} : 0 \le i \le \lfloor \tfrac{b-a}{2} \rfloor\}.$$

15.1. Walks and some basic facts

For a subset $F \subset [n]$ we define its *walk* in $\mathbb{R}^2$. This is an n-step walk starting from the origin, and the ith step is *up* if $i \in F$ or *right* if $i \notin \mathcal{F}$. More precisely, suppose that after $(i-1)$ steps we are at (x,y), then at the ith step we go from (x,y) to $(x,y+1)$ if $i \in F$, or we go from (x,y) to $(x+1,y)$ if $i \notin F$. For example, if $F = \{2,3\} \subset [4]$, then its walk consists of four line segments (each segment has length one) connecting $(0,0),(1,0),(1,1),(1,2)$, and $(2,2)$ in this order. We identify a subset with its walk. For example, we say that $F = \{2, 3\}$ hits the line $y = x + 1$ only at $(1, 2)$.

Exercise 15.1. Let x_0, y_0, and t be nonnegative integers with $t \le y_0 \le x_0 + t$. Show that the number of walks from $(0, 0)$ to (x_0, y_0) which hit the line $y = x + t$ is $\binom{x_0+y_0}{y_0-t}$. (Hint: Use the reflection principle. Find a bijection between the set of those walks and the set of walks from $(-t, t)$ to (x_0, y_0).)

Exercise 15.2. Let $F \subset [n]$. Suppose that F hits the line $y = x + t$. Show that $|F \cap [i]| \ge \frac{i+t}{2}$ for some i. (Hint: Suppose that F hits the line at (x_0, y_0) and put $i = x_0 + y_0$.)

We introduce some definitions and notation. For $A \subset [n]$, let $(A)_i$ be the ith smallest element of A. For $A, B \subset [n]$, we say that A *shifts to* B, denoted by $A \rightsquigarrow B$, if $|A| \le |B|$ and $(A)_i \ge (B)_i$ for each $i \le |A|$. In other words, the walk of A is in the lower right area with respect to the walk of B, that is, in the area bounded by the walk of B and the lines $y = 0$ and $x + y = n$. For example, $\{2, 4, 6\} \rightsquigarrow \{1, 4, 5, 7\}$. We say that $G \in \mathcal{F}$ is the *shift-end* in $\mathcal{F}$ if $F \rightsquigarrow G$ for all $F \in \mathcal{F}$.

Exercise 15.3. Let $\mathcal{F}$ be a shifted family in $2^{[n]}$. Show that $F \in \mathcal{F}$ and $F \rightsquigarrow F'$ imply $F' \in \mathcal{F}$ provided $|F| = |F'|$ or $\mathcal{F}$ is an upset.

Claim 15.4. Let $\mathcal{F} \subset 2^{[n]}$ be a shifted t-intersecting family and let $F \in \mathcal{F}$. Then F hits the line $y = x + t$.

Proof. Suppose to the contrary that $F \in \mathcal{F}$ does not hit the line. Let $|F| = k$. Define $F' \subset [n]$ with $|F'| = k$ by

$$F' = [t-1] \sqcup \{t+1, t+3, \ldots\} = [t-1] \sqcup [t+1, 2k-t+1]_2.$$

Then F' is the shift-end in the k-uniform family whose walks do not hit the line. Since $\mathcal{F}$ is shifted, we have $F \rightsquigarrow F'$ and $F' \in \mathcal{F}$. Let

$$F'' = [t-1] \sqcup \{t, t+2, \ldots\} = [t-1] \sqcup [t, 2k-t]_2.$$

Then $F' \rightsquigarrow F''$ and $F'' \in \mathcal{F}$. But $|F' \cap F''| = t-1$, a contradiction. $\square$

Walks in a shifted t-intersecting family hit the line $y = x + t$, and we can bound the size by counting such walks as follows.

Theorem 15.5. *Let $n \geq 2k - t$. If $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting, then $|\mathcal{F}| \leq \binom{n}{k-t}$.*

Proof. We may assume that $\mathcal{F}$ is shifted. Then by Claim 15.4 all $F \in \mathcal{F}$ hit the line $y = x + t$. Apply Exercise 15.1 with $x_0 = n - k$ and $y_0 = k$, to get that the number of such walks is at most $\binom{n}{k-t}$. □

The bound $\binom{n}{k-t}$ is not sharp, but it is not too bad and we get it easily. This already shows the potential of the random walk method, and actually we can go further. If $n \geq (t+1)(k-t+1)$ then the exact upper bound for the size of t-intersecting families is $\binom{n-t}{k-t}$, which we call the Full Erdős–Ko–Rado Theorem (Full EKR). The first serious application of the random walk method, due to Frankl [**45**], was to obtain this sharp bound for the case $t \geq 15$. In the next section we refine the original proof to get the corresponding result in the measure version. Our proof works for $t \geq 6$, and one can get the corresponding Full EKR in a similar way; c.f. [**58**].

We recall some basic facts to deal with the measure version. Let p be a real number with $0 < p < 1$, and let $q = 1 - p$. The product measure of a family $\mathcal{F} \subset 2^{[n]}$ is defined by

$$\mu_p(\mathcal{F}) = \sum_{F \in \mathcal{F}} p^{|F|} q^{n-|F|}.$$

Exercise 15.6. Let $p < 1/2$. Consider an infinite random walk in the plane starting at the origin, where the ith step is *up* with probability p and *right* with probability q. For a positive integer t let $P(t)$ denote the probability that the walk hits the line $y = x + t$. Show that $P(t) = (p/q)^t$. (Hint: After the first step we are at $(0, 1)$ with probability p, or at $(1, 0)$ with probability q. This gives $P(t) = pP(t-1) + qP(t+1)$. Assume that $P(t)$ is given in the form x^t and solve the equation.)

Theorem 15.5 has the following corresponding measure version.

Theorem 15.7. *Let $p < 1/2$. If $\mathcal{F} \subset 2^{[n]}$ is t-intersecting, then $\mu_p(\mathcal{F}) < (p/q)^t$.*

Proof. We may assume that $\mathcal{F}$ is shifted. Then the walks in $\mathcal{F}$ hit the line $y = x + t$. Thus $\mu_p(\mathcal{F})$ is at most the product measure of the set of n-step walks hitting the line. The latter is precisely the probability that the random walk defined in Exercise 15.6 hits the line in the first n steps, and it is less than $(p/q)^t$. □

This bound $(p/q)^t$ is not sharp. In fact, if $p \leq \frac{1}{t+1}$ then p^t is the exact upper bound for $\mu_p(\mathcal{F})$, and this is a special case of Theorem 12.4. There are several ways to prove the result; for example, Friedgut [**68**] gave a spectral proof; see also Chapter 30. In the next section we present a proof using the random walk method. For the proof we gather some useful facts about intersecting families.

Lemma 15.8. *Let $\mathcal{F} \subset 2^{[n]}$ be a shifted t-intersecting family. For every $F, G \in \mathcal{F}$ there is some i such that $|F \cap [i]| + |G \cap [i]| \geq i + t$.*

Proof. Suppose the contrary, and choose a pair of counterexamples $F, G \in \mathcal{F}$ so that $|F \cap G|$ is minimal. Let j be the tth element of $F \cap G$. Then

$$|F \cap [j]| + |G \cap [j]| < j + t = j + |F \cap G \cap [j]|,$$

and $|(F \cup G) \cap [j]| < j$. Thus we can find $i \in [j] \setminus (F \cup G)$. By the shiftedness, $G' = (G \setminus \{j\}) \sqcup \{i\} \in \mathcal{F}$, and F and G' are also counterexamples because $|G \cap [j]| = |G' \cap [j]|$. But $|F \cap G'| < |F \cap G|$, contradicting the minimality of the choice. □

Exercise 15.9. Let $f_n = \max\{\mu_p(\mathcal{F}) : \mathcal{F} \subset 2^{[n]}$ is t-intersecting$\}$. Show that $f_n \leq f_{n+1}$. (Hint: If $\mathcal{F} \subset 2^{[n]}$ is t-intersecting, then $\mathcal{F}' = \mathcal{F} \sqcup \{F \sqcup \{n+1\} : F \in \mathcal{F}\} \subset 2^{[n+1]}$ is t-intersecting as well.)

Exercise 15.10. Let $\mathcal{G}, \mathcal{H} \subset 2^{[n]}$. Show that if $\mathcal{G}$ is a shifted upset and $H \in \mathcal{H} \setminus \mathcal{G}$ is the shift-end in $\mathcal{H}$, then $\mathcal{G} \cap \mathcal{H} = \emptyset$.

In the next section we will bound the measure of a t-intersecting shifted upset $\mathcal{G} \subset 2^{[n]}$. For this we use the following simple fact. Let $A, W, W' \subset [n]$, and suppose that $W \in \mathcal{G}$ and $W' \notin \mathcal{G}$. Then $A \notin \mathcal{G}$ if $|A \cap W| < t$ or $A \rightsquigarrow W'$. If $\mathcal{G}$ satisfies some additional conditions, then by choosing W and W' appropriately we get a good estimation for the measure of $\mathcal{G}$, as we will see.

In Chapter 12 we mentioned the relation between the uniform measure μ_k of t-intersecting families in $\binom{[n]}{k}$ and the product measure μ_p of those in $2^{[n]}$, where $p = k/n$. For walks hitting the line $y = x + t$ we also have the correspondence. That is, the family of walks $\mathcal{W} \subset$

$\binom{[n]}{k}$ hitting the line has the uniform measure $\mu_k(\mathcal{W}) = \binom{n}{k-t}/\binom{n}{k}$, while the corresponding family of walks $\mathcal{W}' \subset 2^{[n]}$ has the product measure $\mu_p(\mathcal{W}') < (p/q)^t$. Then for fixed $p = k/n$ it follows that $\binom{n}{k-t}/\binom{n}{k} \to (p/q)^t$ as $n \to \infty$. This suggests that if we prove a result concerning the measure of t-intersecting families either in $\binom{[n]}{k}$ or in $2^{[n]}$ using walks, for example, if we deduce something about μ_k by counting the number of walks satisfying some conditions, then we may expect to get the corresponding result about μ_p in a similar way, and vice versa. In fact this is true for some important cases; see, e.g., [**58**, **108**].

15.2. The measure of t-intersecting families

In this section we fix a real number p with $0 < p < 1$. Let $q = 1 - p$ and $\alpha = p/q$. Let t be a positive integer and define t-intersecting families $\mathcal{F}_s^t$ for $0 \leq s \leq (n-t)/2$ by

$$\mathcal{F}_s^t = \{F \subset [n] : |F \cap [t+2s]| \geq t+s\}.$$

In the language of walks, after $t + 2s$ steps, $F \in \mathcal{F}_s^t$ is at one of the following points: $(0, t+2s)$, $(1, t+2s-1), \ldots, (t, t+s)$. Among these $t+1$ points, only $(t, t+s)$ is on the line $y = x + t$.

Exercise 15.11. Let $0 < p \leq \frac{1}{t+1}$.

(i) Show that $\mu_p(\mathcal{F}_0^t) = p^t$.

(ii) Let $s \geq 0$ be an integer. Show that $\mu_p(\mathcal{F}_s^t) \geq \mu_p(\mathcal{F}_{s+1}^t)$, with equality holding if and only if $s = 0$ and $p = \frac{1}{t+1}$. (Hint: Compare $\mu_p(\mathcal{F}_s^t \setminus \mathcal{F}_{s+1}^t)$ and $\mu_p(\mathcal{F}_{s+1}^t \setminus \mathcal{F}_s^t)$.)

Theorem 15.12. *Let $t \geq 6$ and $0 < p \leq \frac{1}{t+1}$. If $\mathcal{G} \subset 2^{[n]}$ is t-intersecting, then $\mu_p(\mathcal{G}) \leq p^t$. Moreover, equality holds if and only if either* (i) *$\mathcal{G} \cong \mathcal{F}_0^t$ or* (ii) *$p = \frac{1}{t+1}$ and $\mathcal{G} \cong \mathcal{F}_1^t$.*

Proof. Here we prove the result for $t \geq 8$; see Exercise 15.15 for the improvement. Let $\mathcal{G} \subset 2^{[n]}$ be t-intersecting. First we prove the inequality $\mu_p(\mathcal{G}) \leq p^t$. For this we may assume that $\mathcal{G}$ is a shifted upset. Thus $G \in \mathcal{G}$ and $G \rightsquigarrow G'$ imply $G' \in \mathcal{G}$. Let λ be the maximal integer such that all walks in $\mathcal{G}$ hit the line $y = x + \lambda$. By Claim 15.4 we have $\lambda \geq t$. Also $\mu_p(\mathcal{G}) \leq \alpha^\lambda$ by Exercise 15.6. If $\lambda \geq t+1$, then simple computation shows that $\alpha^\lambda \le \alpha^{t+1} < p^t$ for $t \ge 3$, and we are done. (Here and hereafter we omit the details of computations, which are just standard calculus.)

Thus we may assume that $\lambda = t$. In other words, all walks in $\mathcal{G}$ hit the line $L : y = x + t$ and some walks do not hit $L' : y = x + t + 1$. So we can divide $\mathcal{G}$ into three subfamilies:

$$\mathcal{G} = \dot{\mathcal{G}} \sqcup \ddot{\mathcal{G}} \sqcup \tilde{\mathcal{G}},$$

where walks in $\tilde{\mathcal{G}}$ hit the line L', walks in $\dot{\mathcal{G}}$ hit L only once, and walks in $\ddot{\mathcal{G}}$ hit L at least twice. If $G \in \dot{\mathcal{G}} \sqcup \ddot{\mathcal{G}}$ then G does not hit L', and $|G \cap [j]| \le \frac{j+t}{2}$ for all j; see Exercise 15.2.

Let $\mathcal{W}$ be the set of all n-step walks starting from the origin, that is, $\mathcal{W} = 2^{[n]}$, and we also divide $\mathcal{W}$ into $\dot{\mathcal{W}} \sqcup \ddot{\mathcal{W}} \sqcup \tilde{\mathcal{W}}$ in the same manner. Then it follows that $\mu_p(\ddot{\mathcal{W}}) = \mu_p(\tilde{\mathcal{W}})$. In fact there is a bijection between $\ddot{\mathcal{W}}$ and $\tilde{\mathcal{W}}$. That is, for $W \in \ddot{\mathcal{W}}$, by reflecting the walk between the first and second hitting points on the line L with respect to this line, we get the corresponding walk $W' \in \tilde{\mathcal{W}}$.

Suppose that $\dot{\mathcal{G}} = \emptyset$. Then we have

$$\mu_p(\mathcal{G}) = \mu_p(\ddot{\mathcal{G}}) + \mu_p(\tilde{\mathcal{G}}) \le \mu_p(\ddot{\mathcal{W}}) + \mu_p(\tilde{\mathcal{W}}) = 2\mu_p(\tilde{\mathcal{W}}) \le 2\alpha^{t+1}.$$

Since $2\alpha^{t+1} < p^t$ for $t \ge 5$, we are done.

So we may assume that $\dot{\mathcal{G}} \ne \emptyset$.

Claim 15.13. There is a unique s such that $\dot{\mathcal{G}} \sqcup \ddot{\mathcal{G}} \subset \mathcal{F}_s^t$.

Proof. Let $G \in \dot{\mathcal{G}}$. Suppose that G hits the line $y = x + t$ only at $(s, t+s)$. Then $|G \cap [t+2s]| = t + s$, $G \in \mathcal{F}_s^t$, and $G \notin \mathcal{F}_{s'}^t$ for $s' \ne s$. Choose $G' \in \dot{\mathcal{G}} \sqcup \ddot{\mathcal{G}}$ arbitrarily. Then $|G' \cap [j]| \le \frac{j+t}{2}$ for all j. On the other hand, by Lemma 15.8 $|G \cap [i]| + |G' \cap [i]| \ge i + t$ for some i. Thus $|G \cap [i]| = |G' \cap [i]| = \frac{i+t}{2}$ must hold. This is possible only when $i = t + 2s$ because G hits the line only once. Then $G' \in \mathcal{F}_s^t$. □

Let s be guaranteed by the above claim. If $s \ge 2$ then

$$\mu_p(\mathcal{G}) = \mu_p(\dot{\mathcal{G}} \sqcup \ddot{\mathcal{G}}) + \mu_p(\tilde{\mathcal{G}}) \le \mu_p(\mathcal{F}_s^t) + \mu_p(\tilde{\mathcal{G}}) \le \mu_p(\mathcal{F}_2^t) + \mu_p(\tilde{\mathcal{W}})$$

$$\le \binom{t+4}{t+2} p^{t+2} q^2 + \binom{t+4}{t+3} p^{t+3} q + p^{t+4} + \alpha^{t+1}, \tag{15.1}$$

where we use the fact that walks in $\mathcal{F}_2^t$ must hit one of $(2, t+2), (1, t+3)$, and $(0, t+4)$. The RHS of (15.1) is less than p^t for $t \geq 8$. (This is the only point where we need $t \geq 8$; see Exercise 15.15.)

So we may assume that $s = 0$ or 1. First let $s = 1$. We will compare $\mathcal{G}$ and $\mathcal{F}_1^t$ using a walk $W_i \in \dot{\mathcal{W}}$ defined by

$$W_i = ([t+2] \setminus \{t\}) \sqcup [t+i+4, n]_2.$$

Let $W_{i_{\max}} = [t+2] \setminus \{t\}$. If $W_{i_{\max}} \in \mathcal{G}$, then by the shiftedness both $[t+2] \setminus \{t+1\}$ and $[t+1]$ are also members of $\mathcal{G}$. In this case $\mathcal{G} \subset \mathcal{F}_1^t$. So we may assume that $W_{i_{\max}} \notin \mathcal{G}$. Note that W_1 is the shift-end in $\dot{\mathcal{W}} \cap \mathcal{F}_1^t$ and $\dot{\mathcal{W}} \cap \mathcal{F}_1^t \supset \dot{\mathcal{G}} \neq \emptyset$. Thus $G \rightsquigarrow W_1$ holds for any $G \in \dot{\mathcal{G}}$, and $W_1 \in \mathcal{G}$ because $\mathcal{G}$ is a shifted upset. Therefore we can define $I = \max\{i : W_i \in \mathcal{G}\} \geq 1$. Then $W_I \in \mathcal{G}$ and $W_{I+1} \notin \mathcal{G}$.

Claim 15.14. $\mu_p(\mathcal{F}_1^t \setminus \mathcal{G}) \geq tp^{t+1}q^{I+2}(1-\alpha) > \alpha^{t+I} \geq \mu_p(\mathcal{G} \setminus \mathcal{F}_1^t)$.

Proof. To show the first inequality, let $\mathcal{H}$ be the set of walks H satisfying the following conditions:

(i) H hits all of $(1, t-1), (1, t+1)$, and $(I+2, t+1)$;

(ii) after $t+I+3$ steps it does not hit the line $L : y = x+t-I$.

We note that $\mathcal{H} \subset \mathcal{F}_1^t \setminus \mathcal{G}$, which follows from (iii) $\mathcal{H} \subset \mathcal{F}_1^t$ and (iv) $\mathcal{H} \cap \mathcal{G} = \emptyset$. Indeed (iii) follows from (i), and (iv) follows from the fact that $W_{I+1} \in \mathcal{H} \setminus \mathcal{G}$ is the shift-end in $\mathcal{H}$; see Exercise 15.10. Now we estimate $\mu_p(\mathcal{H})$. For (i) there are t ways from $(0, 0)$ to $(1, t-1)$, and part (i) (the first $I+t+3$ steps) consists of $t+1$ up steps and $I+2$ right steps. So (i) contributes $tp^{t+1}q^{I+2}$ to $\mu_p(\mathcal{H})$. The probability that the random walk starting from $(I+2, t+1)$ hits the line L is at most α, and (ii) contributes at least $1-\alpha$ to $\mu_p(\mathcal{H})$. Thus we get $\mu_p(\mathcal{F}_1^t \setminus \mathcal{G}) \geq \mu_p(\mathcal{H}) \geq tp^{t+1}q^{I+2}(1-\alpha)$.

To show the last inequality, let $E = [t] \sqcup [t+3, t+I+3] \sqcup [t+I+5, n]_2$. Since $W_I \in \mathcal{G}$ and $W_I \cap E = [t-1]$, we have $E \notin \mathcal{G}$. If $G \in \mathcal{G} \setminus \mathcal{F}_1^t$ then G hits neither $(0, t+1)$ nor $(1, t)$. Thus G must hit the line $y = x+t+I$. (Otherwise $G \rightsquigarrow E$ and $E \in \mathcal{G}$, a contradiction.) This means that all walks $\mathcal{G} \setminus \mathcal{F}_1^t$ hit $y = x+t+I$, and $\mu_p(\mathcal{G} \setminus \mathcal{F}_1^t) \leq \alpha^{t+I}$.

Finally we show the middle inequality, that is,

$$tpq^{t+2}(1-\alpha)(q^2/p)^I > 1.$$

Since $q^2/p > 1$ for $t \geq 2$, it suffices to check the inequality at $I = 1$, that is, $tq^{t+1}(1-\alpha) > 1$, and this is true for $t \geq 6$. □

By the claim we have $\mu_p(\mathcal{F}_1^t \setminus \mathcal{G}) > \mu_p(\mathcal{G} \setminus \mathcal{F}_1^t)$, that is, $\mu_p(\mathcal{G}) < \mu_p(\mathcal{F}_1^t)$.

Next let $s = 0$. We can argue similarly to the previous case, and the present case is even easier. This time let $W_i = [t] \sqcup [t+i+2, n]_2$. If $W_{i_{\max}} = [t] \in \mathcal{G}$ then clearly $\mathcal{G} \subset \mathcal{F}_0^t$. Otherwise we can define $I := \max\{i : W_i \in \mathcal{G}\}$. Then one can show that

$$\mu_p(\mathcal{F}_0^t \setminus \mathcal{G}) \geq p^t q^{I+1}(1-\alpha) > \alpha^{t+I} \geq \mu_p(\mathcal{G} \setminus \mathcal{F}_0^t)$$

for $t \geq 5$. This yields $\mu_p(\mathcal{G}) < \mu_p(\mathcal{F}_0^t)$.

Consequently we have shown that if $\mathcal{G}$ is shifted and $t \geq 8$, then one of the following holds: (a) $\mu_p(\mathcal{G}) < p^t$, (b) $\mathcal{G} \subset \mathcal{F}_0^t$, or (c) $\mathcal{G} \subset \mathcal{F}_1^t$. Moreover, in view of Exercise 15.11, even in the cases (b) and (c) we have $\mu_p(\mathcal{G}) \leq p^t$, and equality holds only if $\mathcal{G} = \mathcal{F}_0^t$ or $p = 1/(t+1)$ and $\mathcal{G} = \mathcal{F}_1^t$. Finally we use Exercise 10.10, which tells us that if we start from $\mathcal{G}$ and obtain $\mathcal{F}_s^t$ by shifting, then $\mathcal{G}$ is isomorphic to $\mathcal{F}_s^t$. This completes the proof. □

Exercise 15.15. Improve (15.1) so that $\mu_p(\mathcal{G}) < p^t$ for $t \geq 5$ and $s \geq 2$ in the following way.

(i) Let $s \geq 3$. Prove that $\mu_p(\mathcal{G}) < p^t$ for $t \geq 5$ by showing that

$$\mu_p(\mathcal{F}_3^t) + \mu_p(\tilde{\mathcal{W}}) \leq \sum_{j=t+3}^{t+6} \binom{t+6}{j} p^j q^{t+6-j} + \alpha^{t+1} < p^t.$$

(ii) Let $s = 2$. First show that

$$\begin{aligned}\mu_p(\mathcal{G}) &\leq \mu_p(\mathcal{F}_2^t \setminus \mathcal{G}) + \mu_p(\mathcal{G}) \leq \mu_p(\mathcal{F}_2^t \setminus \mathcal{W}) + \mu_p(\mathcal{W}) \\ &\leq \frac{(t+4)(t+1)}{2} p^{t+2} q^2 (1-\alpha+\epsilon) + \alpha^{t+1},\end{aligned}$$

where $\epsilon \to 0$ as $n \to \infty$. Next show that the RHS is $< p^t$ for $t \geq 4$ (assuming $\epsilon > 0$ is small enough). Finally use Exercise 15.9 to conclude that $\mu_p(\mathcal{G}) < p^t$ for all $n \geq t \geq 4$.

The random walk method is applicable to cross t-intersecting families, r-wise t-intersecting families, and so on, see [**58**, **87**] and references therein.

Chapter 31

Capsets and sunflowers

In this chapter we present a recent development of a polynomial method (the slice rank method). We give two applications. One is that a set containing no 3-term arithmetic progressions in $\mathbb{F}_3^n$ has size less than $(2.8)^n$ for n sufficiently large, and the other is that a family of subsets of $[n]$ containing no sunflowers has size less than $(1.9)^n$ for n sufficiently large.

31.1. Things happened in May 2016

Finding large subsets of an abelian group G with no 3-term arithmetic progressions is one of the central problems in additive number theory. In May of 2016 Croot, Lev, and Pach [**20**] proved a remarkable result stating that the size of such subsets for the case $G = (\mathbb{N}/4\mathbb{N})^n$ is at most c^n for some $c < 4$. This was a starting point of successive breakthrough results all obtained in the same month. Soon after they put the preprint on arXiv, Ellenberg and Gijswijt independently found how to use the idea of [**20**] for the case $G = \mathbb{F}_3^n$ and obtained the following result.

Theorem 31.1 (Ellenberg–Gijswijt [**27**])**.** *Let $X \subset \mathbb{F}_3^n$. Suppose that for $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X$,*

$$\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} = \boldsymbol{0} \text{ if and only if } \boldsymbol{x} = \boldsymbol{y} = \boldsymbol{z}. \tag{31.1}$$

Then $|X| < c^n$ for some $c < 3$.

Exercise 31.2. We say that three elements $\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{a}, \boldsymbol{x}+2\boldsymbol{a} \in \mathbb{F}_3^n$ form a 3-term arithmetic progression if $\boldsymbol{a} \neq \boldsymbol{0}$. Show that $X \subset \mathbb{F}_3^n$ contains no 3-term arithmetic progressions if and only if (31.1) holds.

A subset $X \subset \mathbb{F}_3^n$ satisfying (31.1) is called a *capset*. Bounding the size of capsets is important in itself, and it is also related to the following two problems. One is a problem of determining the exponent ω of matrix multiplication; see [**12**]. The other is a conjecture concerning the size of families without sunflowers. Recall that three distinct subsets $A, B, C \subset [n]$ form a *sunflower* if $A \cap B = B \cap C = C \cap A$. The following result was conjectured by Erdős and Szemerédi [**37**].

Theorem 31.3. *Let $\mathcal{F} \subset 2^{[n]}$. If $\mathcal{F}$ contains no sunflower, then there exist $c < 2$ and n_0 such that $|\mathcal{F}| < c^n$ for all $n > n_0$.*

Alon, Shpilka, and Umans [**2**] had proved that if Theorem 31.1 were true (and we now know that it is indeed true), then it would imply Theorem 31.3.

Tao reformulated the proof of Theorem 31.1 in his blog [**106**] by introducing a slice rank. Then using this slice rank method Naslund and Sawin [**94**] gave a direct proof of Theorem 31.3 with a better constant c than the constant obtained from [**2**] with [**27**]. In the following sections we explain the slice rank and its basic property, and then we apply the property to prove Theorems 31.1 and 31.3. Our presentation is partially based on a set of lecture notes by Zeeuw [**17**], which is also recommended for other topics including the density Hales–Jewett Theorem.

31.2. Slice rank

A diagonal matrix without zero diagonal entries is of full rank. We will extend this fact to a "hypermatrix" in some sense. For this purpose let us introduce the slice rank of a function.

Let X be a finite set, and let $\mathbb{F}$ be a field. Let $f : X^k \to \mathbb{F}$ be a function in k variables. We say that f is *sliced* if f can be written as a product of two functions a and b, where a is in one variable and b is in the other $k-1$ variables, for example,

$$f(x_1, x_2, \ldots, x_k) = a(x_3)\, b(x_1, x_2, x_4, \ldots, x_k).$$

We can always write f as a sum of at most $|X|$ sliced functions; indeed,

$$f(x_1, \dots, x_k) = \sum_{z \in X} a_z(x_1) b_z(x_2, \dots, x_k),$$

where $a_z(x) = \delta_{z,x}$ and $b_z(x_2, \dots, x_k) = f(z, x_2, \dots, x_k)$. Now the *slice rank* of f, denoted by $\mathrm{sr}(f)$, is the least number of sliced functions such that f is represented as the sum of them, that is,

$$\mathrm{sr}(f) = \min \Big\{ r : f = \sum_{i=1}^{r} g_i, \text{ where } g_1, \dots, g_r \text{ are sliced functions} \Big\}.$$

As we just noted, $\mathrm{sr}(f) \le |X|$.

Lemma 31.4 (Tao [**106**])**.** *Let X be a finite set, $\mathbb{F}$ be a field, and $k \ge 2$ be an integer. Let $f : X^k \to \mathbb{F}$ be a function such that $f(x_1, \dots, x_k) \ne 0$ if and only if $x_1 = \cdots = x_k$. Then* $\mathrm{sr}(f) = |X|$.

Proof. We prove the statement by induction on k. Let $k = 2$. It suffices to show that $\mathrm{sr}(f) \ge |X|$. Suppose, to the contrary, that $r := \mathrm{sr}(f) < |X|$. Then we can write

$$f(x, y) = \sum_{i=1}^{r} a_i(x) b_i(y)$$

for some nonzero functions a_i and b_i. Let F (resp. F_i) be an $|X| \times |X|$ matrix whose (x, y)-entry is $f(x, y)$ (resp. $a_i(x) b_i(y)$). Then

$$F = \sum_{i=1}^{r} F_i.$$

Since the usual matrix rank of F_i is one, it follows that

$$\operatorname{rank} F \le \sum_{i=1}^{r} \operatorname{rank}(F_i) = r.$$

On the other hand, the assumption that $f(x, y) \ne 0$ if and only if $x = y$ implies that F is a diagonal matrix without zero diagonal entries. Thus $\operatorname{rank} F = |X| > r$, a contradiction.

Next we move on to the induction step, but to make things notationally easier we consider the case $k = 3$. Let $f : X^3 \to \mathbb{F}$ and suppose that $r := \mathrm{sr}(f) < |X|$. Then we can write

(31.2)
$$f(x,y,z) = \sum_{i\in I} a_i(x)b_i(y,z) + \sum_{i\in J} a_i(y)b_i(x,z) + \sum_{i\in K} a_i(z)b_i(x,y)$$

for some nonzero functions a_i and b_i, where $I \sqcup J \sqcup K = [r]$. Without loss of generality we may assume that $I \neq \emptyset$.

Let V be a vector space consisting of all functions $v : X \to \mathbb{F}$ such that $\sum_{x\in X} v(x)a_i(x) = 0$ for all $i \in I$. Thus V is defined by $|I|$ linear equations in $|X|$ variables, and

$$\dim V \geq |X| - |I| > r - |I|. \tag{31.3}$$

Choose $v \in V$ so that the support $S = \{x \in X : v(x) \neq 0\}$ is maximal. We claim that

$$|S| \geq \dim V. \tag{31.4}$$

In fact, if $|S| < \dim V$ then we can find a nonzero $w \in V$ such that $w(x) = 0$ for all $x \in S$. But then $v + w$ has a strictly larger support than S, and this contradicts the maximality of the choice of S.

Define a function $g : S^2 \to \mathbb{F}$ by $g(y,z) = \sum_{x\in X} v(x)f(x,y,z)$. We estimate the slice rank of g in two ways, which will give rise to a contradiction, showing that the earlier assumption $\mathrm{sr}(f) < |X|$ is false. Compute g from the RHS of (31.2). For the first term we have

$$\sum_{i\in I}\Bigg(\sum_{x\in X} v(x)a_i(x)\Bigg)b_i(y,z) = 0.$$

The second and third terms we can rewrite as

$$\sum_{i\in J} a_i(y)c_i(z) + \sum_{i\in K} a_i(z)c_i(y),$$

where

$$c_i(z) = \sum_{x\in X} v(x)b_i(x,z) \quad \text{for } i \in J,$$
$$c_i(y) = \sum_{x\in X} v(x)b_i(x,y) \quad \text{for } i \in K.$$

Thus we get

$$g(y,z) = \sum_{i\in J} a_i(y)c_i(z) + \sum_{i\in K} a_i(z)c_i(y),$$

that is, g is represented by $|J|+|K|$ sliced functions, and

$$\operatorname{sr}(g) \le |J|+|K| = r-|I|. \tag{31.5}$$

Finally we claim that $g(y,z) \neq 0$ if and only if $y=z$. In fact, if $y \neq z$ then $f(x,y,z)=0$ and $g(y,z)=\sum_x v(x)f(x,y,z)=0$. If $y=z$, then $g(y,y)=\sum_x v(x)f(x,y,y)=v(y)f(y,y,y)\neq 0$ for $y \in S$, as needed. Thus by the induction hypothesis with (31.3) and (31.4) it follows that

$$\operatorname{sr}(g)=|S| \ge \dim V > r-|I|,$$

which contradicts (31.5). □

The next lemma gives us an upper bound for the slice rank of a function. We will use the lemma to solve the capset problem and the sunflower problem.

Lemma 31.5. *Let $\mathbb{F}$ be a field, let $Y \subset \mathbb{F}$ be a finite set, and let $X \subset Y^n$, where $n \in \mathbb{Z}_{>0}$. Define a function $f : X^3 \to \mathbb{F}$ by*

$$f(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = \prod_{i=1}^{n} \big((x_i+y_i+z_i)^s - t\big), \tag{31.6}$$

where $t \in \mathbb{F}$ and $s \in \mathbb{Z}_{>0}$. Let $g(x)=x^{-\frac{s}{3}}(1+x+\cdots+x^s)$ be a real-valued function defined on the interval $(0,1)$, and let α be a unique root of $g'(x)=0$. Then

$$\operatorname{sr}(f) < 3(g(\alpha))^n.$$

In particular the following hold.

(i) *If $s=2, t=1$, and $\mathbb{F}=\mathbb{F}_3$, then* $\operatorname{sr}(f) < 3(2.76)^n$.

(ii) *If $s=1, t=2$, and $\mathbb{F}=\mathbb{R}$, then* $\operatorname{sr}(f) < 3(1.89)^n$.

Proof. By expanding the RHS of (31.6) we can write f as a sum of monomials in the form

$$x_1^{i_1}\cdots x_n^{i_n}\, y_1^{j_1}\cdots y_n^{j_n}\, z_1^{k_1}\cdots z_n^{k_n},$$

with $\|I\|+\|J\|+\|K\| \le sn$, where $\|I\|=\sum_{l=1}^n i_l$, $\|J\|=\sum_{l=1}^n j_l$, and $\|K\|=\sum_{l=1}^n k_l$. For simplicity we write x^I to mean $x_1^{i_1}\cdots x_n^{i_n}$ and define y^J and z^K similarly. Since one of the $\|I\|, \|J\|$, and $\|K\|$

is at most $sn/3$, we can divide f into three parts $f = f_x + f_y + f_z$ as follows. First we collect all monomials in f with $\|I\| \le sn/3$ and let

$$f_x = \sum_{\|I\| \le sn/3} x^I \Big(\sum_{J,K} c_{IJK}\, y^J z^K \Big).$$

Next we collect all monomials in $f - f_{\boldsymbol{x}}$ with $\|J\| \le sn/3$ and let

$$f_y = \sum_{\|J\| \le sn/3} y^J \Big(\sum_{J,K} c_{IJK}\, x^I z^K \Big).$$

Finally let $f_{\boldsymbol{z}} = f - f_{\boldsymbol{x}} - f_{\boldsymbol{y}}$, which is represented as

$$f_z = \sum_{\|K\| \le sn/3} z^K \Big(\sum_{I,J} c_{IJK}\, x^I y^J \Big).$$

We note that f_x is a sum of $\#I$ sliced functions with $\|I\| \le sn/3$. Moreover, $\#I \le N$, where N is the number of integer solutions $(i_1, \dots, i_n)$ satisfying

$$i_1 + \cdots + i_n \le \frac{sn}{3}$$

with $0 \le i_l \le s$ for all $1 \le l \le n$. Suppose that $(i_1, \dots, i_n)$ is one of the solutions, and let $a_u = |\{l : i_l = u, 1 \le l \le n\}|$ for $0 \le u \le s$. Then

(P) $a_0 + \cdots + a_s = n$,

(Q) $a_1 + 2a_2 + \cdots + sa_s \le \dfrac{sn}{3}$.

On the other hand, for a fixed feasible $(a_0, \dots, a_s)$ the number of the corresponding solutions $(i_1, \dots, i_n)$ is

$$\binom{n}{a_0}\binom{n-a_0}{a_1}\cdots\binom{n-(a_0+\cdots+a_{s-1})}{a_s} = \frac{n!}{a_0!a_1!\cdots a_s!}.$$

Thus we have

$$N = \sum \frac{n!}{a_0!a_1!\cdots a_s!},$$

where the sum is taken under the conditions of (P) and (Q). Clearly N is also an upper bound for the number of sliced functions in f_y as well as in f_z. Consequently f is a sum of at most $3N$ sliced functions, and

$$\operatorname{sr}(f) \le 3N.$$

Let $x \in (0,1)$. By the multinomial expansion we have

$$\begin{aligned} x^{-\frac{sn}{3}}(1+x+\cdots+x^s)^n &= \sum_{(\mathrm{P})} \frac{n!}{a_0!\cdots a_s!} x^{a_1+2a_2+\cdots+sa_s-\frac{sn}{3}} \\ &> \sum_{(\mathrm{P}),\,(\mathrm{Q})} \frac{n!}{a_0!\cdots a_s!} x^{a_1+2a_2+\cdots+sa_s-\frac{sn}{3}} \\ &\geq \sum_{(\mathrm{P}),\,(\mathrm{Q})} \frac{n!}{a_0!\cdots a_s!} = N. \end{aligned}$$

Since the LHS is $(g(x))^n$, it follows that $(g(x))^n > N$ for all $x \in (0,1)$. By standard calculus we see that $g'(x) = 0$ has a unique root α in $(0,1)$, and $g(x)$ takes its minimum at $x = \alpha$. Therefore we get

$$\mathrm{sr}(f) \leq 3N < 3(g(\alpha))^n.$$

For (i) let $s = 2$ and $t = 1$. Then $g(x) = x^{-\frac{2}{3}}(1+x+x^2)$ is minimized at $x = \alpha := \frac{1}{8}(\sqrt{33}-1)$, and $g(\alpha) = \frac{3}{8}(33\sqrt{33}+207)^{\frac{1}{3}} < 2.76$.

For (ii) let $s = 1$ and $t = 2$. Then $g(x) = x^{-\frac{1}{3}}(1+x)$ is minimized at $x = \alpha := 1/2$, and $g(\alpha) = 3/2^{2/3} < 1.89$. □

The slice rank method used in this chapter is summarized as follows. Suppose that we want to bound the size of a set X and we know that X^k satisfies some condition. On the one hand, using the condition, we construct a function $f : X^k \to \mathbb{F}$ with the property that $f(x_1,\ldots,x_k) \neq 0$ if and only if $x_1 = \cdots = x_k$. By Lemma 31.4 it follows that $\mathrm{sr}(f) = |X|$. On the other hand, we expand the function f as a sum of sliced functions and count the number N of them using Lemma 31.5. By definition of the slice rank we get $\mathrm{sr}(f) \leq N$. Combining these two things, we obtain an upper bound on $|X|$, that is, $|X| \leq N$. In the final section we apply this method to the capset problem and the sunflower problem.

31.3. Proof of the theorems

Proof of Theorem 31.1. Let $X \subset \mathbb{F}_3^n$ be a capset, that is, X satisfies (31.1). Define a function $f : X^3 \to \mathbb{F}_3$ by

$$f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \prod_{i=1}^{n} \left((x_i + y_i + z_i)^2 - 1 \right).$$

Then it follows from (31.1) that $f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \neq 0$ if and only if $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{z}$. In fact, if $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{z}$ then $f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = f(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}) = (-1)^n \neq 0$. If $\boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{z}$ are not all equal, then $\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} \neq \boldsymbol{0}$, so $x_i + y_i + z_i \neq 0$ for some i, and in this case $(x_i + y_i + z_i)^2 = 1$, yielding $f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = 0$. Thus by Lemma 31.4 it follows that

$$\operatorname{sr}(f) = |X|. \tag{31.7}$$

On the other hand, by (i) of Lemma 31.5, we have

$$\operatorname{sr}(f) < 3(2.76)^n. \tag{31.8}$$

By (31.7) and (31.8) we get $|X| = \operatorname{sr}(f) < 3(2.76)^n$. Then there exists N such that $3(2.76)^n < 2.8^n$ for all $n \geq N$. Since $\mathbb{F}_3^n$ itself is not a capset, we trivially have that $|X| \leq 3^n - 1$, and we can choose $2.8 < c < 3$ so that $3^N - 1 < c^N$. Consequently we have $|X| < c^n$ for all n. □

Proof of Theorem 31.3. Recall that a characteristic vector $\boldsymbol{x} \in \{0,1\}^n$ of a subset $F \subset [n]$ is defined by $x_i = 1$ if $i \in F$ and $x_i = 0$ if $i \notin F$. Let $\mathcal{F} \subset 2^{[n]}$ be sunflower-free and let $X \subset \{0,1\}^n$ be the set of characteristic vectors of $\mathcal{F}$. For $0 \leq k \leq n$ let $\mathcal{F}_k = \mathcal{F} \cap \binom{[n]}{k}$ and let $X_k \subset X$ be the corresponding subset, so if $\boldsymbol{x} \in X_k$ then $\sum_i x_i = k$.

The sunflower-free condition is translated into the following condition[1] for X_k: For any three vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in X_k$ not all the same and $\boldsymbol{w} := \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}$, there is some i such that $w_i = 2$.

Define a function $f : X_k^3 \to \mathbb{R}$ by

$$f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \prod_{i=1}^{n} (x_i + y_i + z_i - 2).$$

[1]We remark that this condition does not hold in X in general. The reason is that a sunflower-free family can have two distinct subsets $A \subsetneq B$ with corresponding characteristic vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, and in this case $\boldsymbol{w} := \boldsymbol{x} + \boldsymbol{x} + \boldsymbol{y}$ satisfies $w_i \neq 2$ for all i.

Then it follows that $f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \neq 0$ if and only if $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{z}$. Thus by Lemma 31.4 we have

$$\mathrm{sr}(f) = |X_k|. \tag{31.9}$$

On the other hand, by (ii) of Lemma 31.5, we have

$$\mathrm{sr}(f) < 3(1.89)^n. \tag{31.10}$$

By (31.9) and (31.10) we get

$$|\mathcal{F}| = |X| = \sum_{k=0}^{n} |X_k| = (n+1)\,\mathrm{sr}(f) < (n+1) \cdot 3(1.89)^n.$$

Thus there exist $c < 2$ and n_0 such that $|\mathcal{F}| < c^n$ for all $n > n_0$. □