Preface

If we split a set into two parts, will at least one of the parts behave like the whole? Certainly not in every aspect. But if we are interested only in the persistence of *certain small regular substructures*, the answer turns out to be "yes".

A famous example is the persistence of *arithmetic progressions.* The numbers $1, 2, \ldots, N$ form the most simple arithmetic progression imaginable: The next number differs from the previous one by exactly 1. But the numbers $4, 7, 10, 13, \ldots$ also form an arithmetic progression, where each number differs from its predecessor by 3.

So, if we split the set $\{1, \ldots, N\}$ into two parts, will one of them contain an arithmetic progression, say of length 7? *Van der Waerden's theorem*, one of the central results of Ramsey theory, tells us precisely that: *For every k there exists a number N such that if we split the set $\{1, \ldots, N\}$ into two parts, one of the parts contains an arithmetic progression of length k.*

Van der Waerden's theorem exhibits the two phenomena, the interplay of which is at the heart of Ramsey theory:

- **Principle 1:** If we split a large enough object with a certain regularity property (such as a set containing a long arithmetic progression) into two parts, one of the parts will also exhibit this property (to a certain degree).

- **Principle 2:** When proving Principle 1, "large enough" often means *very, very, very large.*

The largeness of the numbers encountered seems intrinsic to Ramsey theory and is one of its most peculiar and challenging features. Many great results in Ramsey theory are actually new proofs of known results, but the new proofs yield much better bounds on *how large* an object has to be in order for a Ramsey-type persistence under partitions to take place. Sometimes, "large enough" is even so large that the numbers become difficult to describe using axiomatic arithmetic—so large that they venture into the realm of *metamathematics.*

One of the central issues of metamathematics is *provability.* Suppose we have a set of *axioms*, such as the group axioms or the axioms for a vector space. When you open a textbook on group theory or linear algebra, you will find results (theorems) that follow from these axioms by means of logical deduction. But how does one know whether a certain statement about groups is provable (or refutable) from the axioms at all? A famous instance of this problem is Euclid's fifth postulate (axiom), also known as the *parallel postulate.* For more than two thousand years, mathematicians tried to derive the parallel postulate from the first four postulates. In the 19th century it was finally discovered that the parallel postulate is *independent* of the first four axioms, that is, neither the postulate nor its negation is entailed by the first four postulates.

Toward the end of the 19th century, mathematicians became increasingly disturbed as more and more strange and paradoxical results appeared. There were different sizes of infinity, one-dimensional curves that completely fill two-dimensional regions, and subsets of the real number line that have no reasonable measure of length, or there was the paradox of a set containing all sets not containing themselves. It seemed increasingly important to lay a solid foundation for mathematics. David Hilbert was one of the foremost leaders of this movement. He suggested finding axiom systems from which all of mathematics could be formally derived and in which it would be impossible to derive any logical inconsistencies.

An important part of any such foundation would be axioms which describe the natural numbers and the basic operations we perform on them, addition and multiplication. In 1931, Kurt Gödel published his famous *incompleteness theorems*, which dealt a severe blow to Hilbert's program: For any reasonable, consistent axiomatization of arithmetic, there are independent statements—statements which can be neither proved nor refuted from the axioms.

The independent statements that Gödel's proof produces, however, are of a rather artificial nature. In 1977, Paris and Harrington found a result in Ramsey theory that is independent of arithmetic. In fact, their theorem is a seemingly small variation of the original Ramsey theorem. It is precisely the *very rapid growth of the Ramsey numbers* (recall Principle 2 above) associated with this variation of Ramsey's theorem that makes the theorem unprovable in Peano arithmetic.

But if the Paris-Harrington principle is unprovable in arithmetic, how do we convince ourselves that it is true? We have to pass from the finite to the *infinite*. Van der Waerden's theorem above is of a finitary nature: All sets, objects, and numbers involved are finite. However, basic Ramsey phenomena also manifest themselves when we look at infinite sets, graphs, and so on. Infinite Ramsey theorems in turn can be used (and, as the result by Paris and Harrington shows, sometimes have to be used) to deduce finite versions using the *compactness principle*, a special instance of topological compactness. If we are considering only the infinite as opposed to the finite, Principle 2 in many cases no longer applies.

- **Principle 1 (infinite version):** If we split an infinite object with a certain regularity property (such as a set containing arbitrarily long arithmetic progressions) into two parts, one infinite part will exhibit this property, too.

If we take into account, on the other hand, that there are different sizes of infinity, as reflected by Cantor's theory of ordinals and cardinals, Principle 2 reappears in a very interesting way. Moreover, as with the Paris-Harrington theorem, it leads to metamathematical issues, this time in *set theory*.

It is the main goal of this book to introduce the reader to the interplay between Principles 1 and 2, from finite combinatorics to set theory to metamathematics. The book is structured as follows.

In Chapter 1, we prove Ramsey's theorem and study Ramsey numbers and how large they can be. We will make use of the probabilistic methods of Paul Erdős to give lower bounds for the Ramsey numbers and a result in extremal graph theory.

In Chapter 2, we prove an infinite version of Ramsey's theorem and describe how theorems about infinite sets can be used to prove theorems about finite sets via compactness arguments. We will use such a strategy to give a new proof of Ramsey's theorem. We also connect these arguments to topological compactness. We introduce ordinal and cardinal numbers and consider generalizations of Ramsey's theorem to uncountable cardinals.

Chapter 3 investigates other classical Ramsey-type problems and the large numbers involved. We will encounter fast-growing functions and make an analysis of these in the context of primitive recursive functions and the Grzegorczyk hierarchy. Shelah's elegant proof of the Hales-Jewett theorem, and a Ramsey-type theorem with truly explosive bounds due to Paris and Harrington, close out the chapter.

Chapter 4 deals with metamathematical aspects. We introduce basic concepts of mathematical logic such as proof and truth, and we discuss Gödel's completeness and incompleteness theorems. A large part of the chapter is dedicated to formulating and proving the Paris-Harrington theorem.

The results covered in this book are all cornerstones of Ramsey theory, but they represent only a small fraction of this fast-growing field. Many important results are only briefly mentioned or not addressed at all. The same applies to important developments such as ultrafilters, structural Ramsey theory, and the connection with dynamical systems. This is done in favor of providing a more complete narrative explaining and connecting the results.

The unsurpassed classic on Ramsey theory by Graham, Rothschild, and Spencer [**24**] covers a tremendous variety of results. For those especially interested in Ramsey theory on the integers, the book by Landman and Robertson [**43**] is a rich source. Other reading suggestions are given throughout the text.

The text should be accessible to anyone who has completed a first set of proof-based math courses, such as abstract algebra and analysis. In particular, no prior knowledge of mathematical logic is required. The material is therefore presented rather informally at times, especially in Chapters 2 and 4. The reader may wish to consult a textbook on logic, such as the books by Enderton [**13**] and Rautenberg [**54**], from time to time for more details.

This book grew out of a series of lecture notes for a course on Ramsey theory taught in the MASS program of the Pennsylvania State University. It was an intense and rewarding experience, and the authors hope this book conveys some of the spirit of that semester back in the fall of 2011.

It seems appropriate to close this introduction with a few words on the namesake of Ramsey theory. Frank Plumpton Ramsey (1903–1930) was a British mathematician, economist, and philosopher. A prodigy in many fields, Ramsey went to study at Trinity College Cambridge when he was 17 as a student of economist John Maynard Keynes. There, philosopher Ludwig Wittgenstein also served as a mentor. Ramsey was largely responsible for Wittgenstein's *Tractatus Logico-Philosophicus* being translated into English, and the two became friends.

Ramsey was drawn to mathematical logic. In 1928, at the age of 25, Ramsey wrote a paper regarding consistency and decidability. His paper, *On a problem in formal logic*, primarily focused on solving certain problems of axiomatic systems, but in it can be found a theorem that would become one of the crown jewels of combinatorics.

> *Given any r, n, and μ we can find an m_0 such that, if $m \geq m_0$ and the r-combinations of any Γ_m are divided in any manner into μ mutually exclusive classes C_i $(i = 1, 2, \ldots, \mu)$, then Γ_m must contain a sub-class Δ_n such that all the r-combinations of members of Δ_n belong to the same C_i.* [**53**, Theorem B, p. 267]

Ramsey died young, at the age of 26, of complications from surgery and sadly did not get to see the impact and legacy of his work.

Acknowledgment. The authors would like to thank Jennifer Chubb for help with the manuscript and for many suggestions on how to improve the book.

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April 2018

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