
Introduction

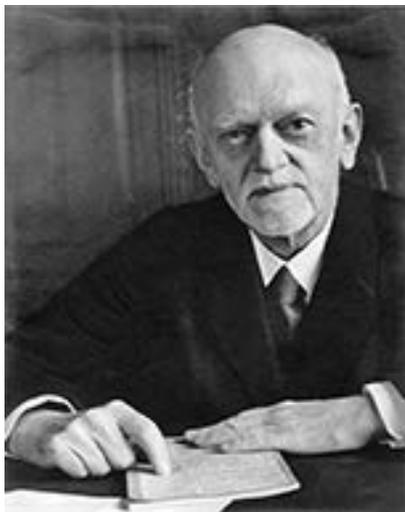
In 1900, at the second International Congress of Mathematicians (ICM), taking place in Paris, David Hilbert (1862–1943) presented a list of twenty-three problems that he felt were fundamentally important, and would influence the direction of mathematics in the 20th century.¹ In his own words (in translation):²

The supply of problems in mathematics is inexhaustible, and as soon as one problem is solved numerous others come forth in its place. Permit me in the following, tentatively as it were, to mention particular definite problems, drawn from various branches of mathematics, from the discussion of which an advancement of science may be expected.

In his tenth problem, Hilbert asked for an algorithm that, when given an arbitrary Diophantine equation, will determine whether the equation has integer solutions or not. The solution to this problem, that there is no such algorithm, is one of the remarkable achievements of 20th-century mathematics. When such dramatic advances in a

¹The address was given in German, and he presented ten of the problems. In a paper written in French appearing in the proceedings of the congress, he included the full list of twenty-three problems.

²Hilbert's address was translated and published in the *Bulletin of the American Mathematical Society*; see [Hil02].



DAVID HILBERT (Photo courtesy the Archives of the Mathematisches Forschungsinstitut Oberwolfach)

discipline are made, it is rare that one can explain the solution in simple terms to the nonexpert. Fortunately, this is not the case with Hilbert's tenth problem. All that is needed to understand how the solution has been put together is an elementary knowledge of the rudiments of logic and elementary number theory, both topics being at a level accessible to an undergraduate student of mathematics. It is the purpose of this monograph to explain this work in as simple a language as possible. In fact, we feel it is a splendid way to introduce the student to both logic and number theory through such a motivated introduction with Hilbert's tenth problem as the focus.

That Hilbert's tenth problem has a negative solution, in that the algorithm that Hilbert sought does not exist, might have surprised Hilbert. Still, he expected that such things may occur. In his 1900 address, he commented:

Occasionally it happens that we seek the solution under insufficient hypotheses or in an incorrect sense, and for this reason do not succeed. The problem then arises: to show the impossibility of

the solution under the given hypotheses, or in the sense contemplated. Such proofs of impossibility were effected by the ancients, for instance when they showed that the ratio of the hypotenuse to the side of an isosceles right triangle is irrational. In later mathematics, the question as to the impossibility of certain solutions plays a pre-eminent part, and we perceive in this way that old and difficult problems, such as the proof of the axiom of parallels, the squaring of the circle, or the solution of equations of the fifth degree by radicals, have finally found fully satisfactory and rigorous solutions, although in another sense than that originally intended. It is probably this important fact along with other philosophical reasons that gives rise to the conviction (which every mathematician shares, but which no one has as yet supported by a proof) that every definite mathematical problem must necessarily be susceptible of an exact settlement, either in the form of an actual answer to the question asked or by the proof of the impossibility of its solution and therewith the necessary failure of all attempts.

In this book, we touch on several of Hilbert's problems. His first problem, the continuum hypothesis, is discussed in Sections 2.3 and 4.3. In his second problem, Hilbert asked for a proof of the consistency of the axioms of arithmetic. We discuss Kurt Gödel's momentous result on this problem in Section 4.4. We briefly mention Hilbert's seventh problem, on the transcendence of a^b when $a \neq 0, 1$ is algebraic and b is irrational and algebraic, in Section 1.2. Hilbert's eighth problem includes the Riemann hypothesis, Goldbach's conjecture, and the twin prime conjecture, which are discussed in Section 6.3.

The scope of this book is broad. A self-contained rigorous treatment of all the topics covered by this book would more than triple its length. Our hope is to introduce the reader to numerous topics in

logic, number theory, and computability. The interested reader can then undertake further study in these areas. To that end, a list of references for further reading is presented at the end of most chapters.

The first four chapters develop the rudimentary notions of set theory, elementary number theory, and logic needed for a complete self-contained proof of Hilbert's tenth problem in Chapter 5. Some applications of the solution to Hilbert's tenth problem are covered in Chapter 6. This material is accessible to the undergraduate student and can be covered in a semester course.³ The final chapter aims to introduce the aspiring student to current research on this topic, namely Hilbert's tenth problem over number fields. This chapter requires more mathematical maturity and is intended for the advanced student. Undoubtedly, there is more research to be done and it is our hope that the reader is thus taken to the frontier of the existing knowledge on this topic so that he or she may survey what is known and what is unknown.

If one is just looking to understand the solution to Hilbert's tenth problem, which is given in Chapter 5, then Sections 3.4 and 4.2 are necessary, provided one is comfortable using an informal definition of an algorithm. A more careful discussion of algorithms and computability is given in Section 4.1. In Chapter 6, some applications of the solution to Hilbert's tenth problem are given. These use the material developed in Chapter 2 and Sections 4.3 and 4.4. However, we feel that by reading through the entire book, one will get a better sense of the interplay between the topics covered by this book and an understanding of some of the most important problems in logic and number theory of the past 150 years.

³An appendix containing preliminary material is included.