Chapter 1

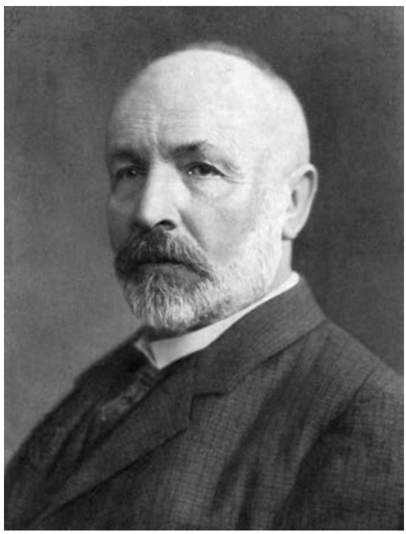
Cantor and Infinity

1.1. Countable Sets

There are some philosophers that deny the existence of infinity. These are the "finitists". They argue that since we have never seen an infinite collection of things, infinity does not exist. When presented with notions of mathematics that lead one to the idea of infinity, they argue that it may be that the universe is working modulo p for some large prime p!

Georg Cantor (1845–1918) can be said to be the founder of set theory and, more generally, the mathematical theory of infinite numbers. He was born in St. Petersburg into a merchant family that settled in Germany in 1856. He studied in Zürich and then Berlin where he obtained his degree in 1867. In 1869 he became a lecturer at the University of Halle in Germany and served there as a professor from 1879 to 1913. He put forth the *continuum hypothesis* (which will be described at the end of this chapter) and attempted to solve it. Perhaps under the strain of these efforts as well as initial opposition to his new ideas concerning infinity, he suffered from depression which may have eventually contributed to his death. The celebrated physicist, Stephen Hawking [**Haw**] wrote, "Georg Cantor scaled the peaks of infinity and then plunged into the deepest abysses of the mind: mental depression."[1]

Cantor's doctoral thesis was in number theory. Later, he introduced the concepts of *ordinal numbers* and *cardinal numbers*, which we discuss in Chapter 2. Using this theory, he proved a number of results that compare the sizes of infinite sets, many of which are given here.



Georg Cantor (Photo source: Wikipedia)

A set S is said to be *countably infinite* if it can be put in one-to-one correspondence with the natural numbers (that is, if there is a bijection between $\mathbb{N} = \{0, 1, 2, \ldots\}$ and S). A *countable* set is either finite or countably infinite. If a set is not countable, it is called *uncountable*. Since the function that sends n to $2n$ is a

[1] Hawking edited the book *God Created the Integers* in which he penned short essays on about two dozen mathematical giants, with Cantor being one of them. Unfortunately, these essays were not copy edited properly and there is a serious error on page 1132. Responding to a question of Dedekind as to whether an infinite set can be defined without referring to the natural numbers, Hawking tries to give Cantor's reply. Thus, the first sentence of the last paragraph on page 1132 should be: "Cantor answered his first question by defining a set as being infinite if it could be put into a one to one correspondence with a proper subset of itself."

bijection between $\mathbb{N}$ and the set of even natural numbers, the set of even natural numbers is countably infinite. The set of integers, denoted by $\mathbb{Z}$, is also countably infinite since we may define a map $f : \mathbb{N} \to \mathbb{Z}$ by setting $f(0) = 0$, $f(1) = 1$, $f(2) = -1$, $f(3) = 2$, $f(4) = -2$, $f(5) = 3$, $f(6) = -3$, and so on.

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$f(n)$	0	1	-1	2	-2	3	-3	4	-4	5	-5	$\cdots$

More generally, we set

$$f(n) = (-1)^{n+1} \left\lfloor \frac{n+1}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ is the floor function, which returns the greatest integer less than or equal to x. This function is easily verified to be injective and surjective.

What about $\mathbb{Q}$, the set of rational numbers? Could it be that $\mathbb{Q}$ is countably infinite? Since any positive rational number can be written as a/b for some natural numbers a and b, we are led to consider the problem of determining if the set $\mathbb{N} \times \mathbb{N}$ of ordered pairs of natural numbers is countably infinite. That is, is there a one-to-one correspondence between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$? Cantor discovered an explicit function, given by

$$P(x, y) = \frac{(x+y)(x+y+1)}{2} + x,$$

that sets up a one-to-one correspondence between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$. This function is called Cantor's pairing function. A table for the first few values of $P(x, y)$ is given below.

		y					
		0	1	2	3	4	5
	0	0	1	3	6	10	15
	1	2	4	7	11	16	22
x	2	5	8	12	17	23	30
	3	9	13	18	24	31	39
	4	14	19	25	32	40	49
	5	20	26	33	41	50	60

Cantor found the pairing function via a diagonal method of enumeration. That is, he began his list of pairs as

$$(0,0),(0,1),(1,0),(0,2),(1,1),(2,0),(0,3),(1,2),(2,1),(3,0),\ldots.$$

Let us note that we can group the pairs (a, b) according to the sum $a + b$. There are only finitely many such pairs in any group. Corresponding to the sum k, we see that there are $k + 1$ such pairs. Now given an ordered pair (x, y), the group it lies in is determined by $k = x + y$. Before we reach this group, the number of ordered pairs we encounter is

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Having reached the group with sum k, to reach (x, y) we have to proceed through

$$(0, k), (1, k-1), \ldots, (x, y),$$

which encompass $x+1$ additional pairs. Thus the pair (x, y) is in the

$$\frac{k(k+1)}{2} + x + 1 = \frac{(x+y)(x+y+1)}{2} + x + 1$$

position in the listing. Since we want the first listed pair to be mapped to 0, the second to be mapped to 1, and so on, subtracting 1 yields the pairing function $P(x, y)$.[2]

Using the above functions f and P, it follows that $\mathbb{Z} \times \mathbb{Z}$ is also countably infinite, for one may show that $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ given by $h(x, y) = P(f^{-1}(x), f^{-1}(y))$ is bijective. By a similar argument, $A \times B$ is also countably infinite for countably infinite A and B. By iteration, it follows that $\mathbb{Z}^n$, the set of all ordered n-tuples of elements of $\mathbb{Z}$, is countably infinite for any positive natural number n, as is $\mathbb{N}^n$.

Since Cantor's pairing function $P(x, y)$ is bijective, $F = P^{-1}$ is a bijective map from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. To the ordered pair (a, b) we may associate the positive rational number $(a+1)/(b+1)$, and thus use F to list the positive rational numbers, agreeing to skip any previously

[2] In this context, we mention a result of Rudolf Fueter (1880–1950) and George Pólya (1887–1985). It is an open question to determine all bijective polynomial maps between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$. Fueter and Pólya showed that if we restrict our attention to quadratic polynomials, then essentially Cantor's pairing function (up to permutation) is the only one. Their proof uses the transcendence of π, as proved by Ferdinand von Lindemann (1852–1939), as well as nontrivial analytic number theory regarding error terms in lattice point enumerations.

listed numbers. This listing yields a bijection between $\mathbb{N}$ and the positive rational numbers. In particular, since

$$F(0) = (0,0),\ F(1) = (0,1),\ F(2) = (1,0),\ F(3) = (0,2),$$
$$F(4) = (1,1),\ F(5) = (2,0),\ \ldots,$$

our bijection g between $\mathbb{N}$ and the positive rational numbers begins as

$$g(0) = 1/1 = 1, \quad g(1) = 1/2, \quad g(2) = 2/1 = 2,$$
$$g(3) = 1/3, \quad g(4) = 3/1 = 3,\ \ldots$$

($2/2 = 1$ was skipped since it was previously listed).

Now that we have a bijection g from $\mathbb{N}$ to the positive rational numbers, we can define a bijection $q : \mathbb{N} \to \mathbb{Q}$ as follows. We define $q(0) = 0$, $q(2n+1) = g(n)$, and $q(2n+2) = -g(n)$. Thus $\mathbb{Q}$ is a countably infinite set.

To summarize, we have shown the following theorem.

Theorem 1.1. *The following sets are countably infinite:* $\mathbb{N}$, $\mathbb{Z}$, *and* $\mathbb{Q}$.

Note that any infinite subset of a countably infinite set is also countably infinite. The elements of a countably infinite set may be listed as $a_1, a_2, a_3, \ldots$. Then the elements of an infinite subset may be listed as $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ for $n_1, n_2, n_3, \ldots$, an infinite subsequence of $1, 2, 3, \ldots$. Thus, for example, any infinite set of rational numbers is countably infinite.

If A and B are countably infinite, then $A \cup B$ is also countably infinite. This is seen as follows. For simplicity, we assume the sets are disjoint. Let $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ be bijective maps. Define $h : \mathbb{N} \to A \cup B$ by $h(2n) = f(n)$ and $h(2n+1) = g(n)$. It is easy to show that h is a bijective map.

There are numbers that are not rational numbers. These are called *irrational numbers*. For instance, $\sqrt{2}$ is irrational. To see this, suppose we have a rational number a/b, with the property that $(a/b)^2 = 2$. We may suppose that a/b is in lowest terms; that is, there is no common factor between a and b except 1. Then we get

$$a^2 = 2b^2,$$

showing us that the left-hand side is even. Thus a is even, and we can write $a = 2c$ for some integer c. We now get $4c^2 = 2b^2$, and cancelling the common factor of 2 on both sides of the equation yields

$$2c^2 = b^2.$$

This implies that the right-hand side is even, so that b is even. Thus we have both a and b are even, a contradiction.

$\sqrt{2}$ is an example of an *algebraic number*. A number α is said to be *algebraic* if α satisfies an equation of the form

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0,$$

with a_i rational numbers. That is, the algebraic numbers are roots of polynomials with rational coefficients. Since $\sqrt{2}$ is a root of $x^2 - 2$, it is an algebraic number. One may show that the set of algebraic numbers is a countably infinite set, as is done in the exercises at the end of the chapter.

1.2. Uncountable Sets

From his musings on countable sets, Cantor went on to ask if $\mathbb{R}$, the set of real numbers, is countable. His first proof that the reals are uncountable, published in 1874, used nested intervals. His more famous proof, involving the *diagonal argument*, was published in 1891 and is given below.

Every real number x in the interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ can be written as an infinite decimal:

$$x = 0.x_1x_2x_3 \cdots .$$

Note that a decimal expansion ending in an infinite sequence of 0's $0.x_1x_2x_3 \cdots x_{m-1}x_m000 \cdots$ with $x_m \neq 0$ (called a *terminating expansion*) also has the expansion $0.x_1x_2x_3 \cdots x_{m-1}y_m999 \cdots$, where $y_m = x_m - 1$. If we agree never to allow an infinite sequence of 9's as the *tail* of the expansion, then the decimal expansion is unique. We can establish these assertions as follows. Take a real number $0 < x < 1$. Then $0 < 10x < 10$, so we may write

$$10x = x_1 + y_1,$$

where $0 \leq x_1 \leq 9$, $0 \leq y_1 < 1$, and x_1 is an integer. Then

$$\left| x - \frac{x_1}{10} \right| = \frac{y_1}{10} < \frac{1}{10}.$$

Iterate this procedure with y_1. Thus

$$10y_1 = x_2 + y_2,$$

where $0 \leq x_2 \leq 9$, $0 \leq y_2 < 1$, and x_2 is an integer. Thus

$$x = \frac{x_1}{10} + \frac{x_2}{10^2} + \frac{y_2}{10^2}.$$

Proceeding in this manner, we get

$$x - \frac{x_1}{10} - \frac{x_2}{10^2} - \cdots - \frac{x_n}{10^n} = \frac{y_n}{10^n},$$

where $0 \leq y_n < 1$. We see immediately that the decimal expansion converges to x. To establish uniqueness, let us suppose that

$$\sum_{n=1}^{\infty} \frac{x_n}{10^n} = \sum_{n=1}^{\infty} \frac{y_n}{10^n}$$

with $0 \leq x_n, y_n \leq 9$. Let m be the smallest number for which $x_m \neq y_m$. Without loss of generality, suppose that $x_m > y_m$. Then we have

$$\frac{x_m}{10^m} + \sum_{j=m+1}^{\infty} \frac{x_j}{10^j} = \frac{y_m}{10^m} + \sum_{j=m+1}^{\infty} \frac{y_j}{10^j}.$$

Thus

$$0 < \frac{x_m - y_m}{10^m} = \sum_{j=m+1}^{\infty} \frac{y_j - x_j}{10^j} \leq \frac{9}{10^{m+1}} \left(1 + \frac{1}{10} + \cdots \right) = \frac{1}{10^m}.$$

Hence $0 < x_m - y_m \leq 1$, which implies $x_m = y_m + 1$. Thus we must have $y_n = 9, x_n = 0$ for $n > m$. Thus uniqueness can fail only if one of our decimal expansions eventually ends in an infinite sequence of 9's.

We may now prove Cantor's theorem on the uncountability of $\mathbb{R}$.

Theorem 1.2. *The set $\mathbb{R}$ of real numbers is uncountable.*

Proof. Suppose that the real interval $(0, 1)$ were countable. We may then list them:

$$\begin{aligned} r_1 &= 0.x_{11}x_{12}x_{13}\cdots \\ r_2 &= 0.x_{21}x_{22}x_{23}\cdots \\ &\vdots \end{aligned}$$

Now consider the number

$$r = 0.y_1y_2y_3\cdots,$$

where

$$y_n = \begin{cases} 1 & \text{if } x_{nn} \neq 1, \\ 2 & \text{if } x_{nn} = 1. \end{cases}$$

In this way, we avoid getting a 9 or 0 as a digit, thereby avoiding repeating 9's and ensuring $r \neq 0$. Then r is in $(0, 1)$ but cannot appear in our listing above since it differs from each r_n in the nth digit. This is a contradiction, and hence the real interval $(0, 1)$ is uncountable.

If a set A is uncountable and $A \subseteq B$, then B is also uncountable. To see this, suppose B were countable. Since A is infinite, B too is infinite, and hence countably infinite. Since A is an infinite subset of the countably infinite set B, it must be countably infinite, a contradiction. Thus, since $(0, 1)$ is an uncountable subset of the real numbers, the set of all real numbers is uncountable. □

Suppose the set of irrational numbers were countable. Since $\mathbb{Q}$ is countably infinite, we would then have $\mathbb{R}$ as the union of two countable sets and hence countable, a contradiction. Thus there are uncountably many irrational numbers.

A real number that is not algebraic is called a *transcendental number*. Recall that the set of algebraic numbers is countably infinite. Suppose the set of transcendental numbers were countable. We would then have $\mathbb{R}$ as the union of two countable sets and hence countable, a contradiction. Thus there are uncountably many transcendental numbers.

In this sense, "most" real numbers are irrational, and in fact transcendental. Cantor showed the uncountability of the transcendental numbers in 1874. Before this, the only numbers known to be transcendental were numbers specifically constructed to be so (called Liouville numbers, named after Joseph Liouville (1809–1882)), and e, which was shown by Charles Hermite (1822–1901) to be transcendental just one year earlier. Thus Cantor proved that most real numbers are transcendental at a time when only a few examples were known! The transcendence of π was shown in 1882 by Lindemann. In his address to the ICM in 1900, Hilbert gave his list of twenty-three important unsolved problems in mathematics. In his seventh problem, he asked if a and b are algebraic numbers with $a \neq 0, 1$ and b irrational, does it follow that a^b is transcendental? The answer is yes, as was proved independently in 1934 by Alexander Gelfond (1906–1968) and Theodor Schneider (1911–1988). There are still many open questions regarding transcendental numbers. For example, we do not know if the numbers $\pi + e$ or πe are transcendental, although both are expected to be. It can be proved that at least one of them must be transcendental. This is an exercise in Chapter 7.

Instead of merely classifying sets as finite, countably infinite, and uncountable, we may refine this by saying that two sets A and B have the *same cardinality*, written $|A| = |B|$, if there is a bijective map between them. One may show that this is an equivalence relation. We say that A has *cardinality less than or equal to that of* B, written $|A| \leq |B|$, if there is an injective map from A to B. If there is an injective map from A to B but no bijective map between the sets is possible, we say A has *smaller cardinality* than B and write $|A| < |B|$. With Theorem 1.1 we showed that that $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$. Since the inclusion map from $\mathbb{N}$ to $\mathbb{R}$ is injective, Theorem 1.2 shows that $|\mathbb{N}| < |\mathbb{R}|$.

Given a set A, consider its *power set* $P(A)$ defined as the set of all subsets of A. It is clear that the function $f : A \to P(A)$ defined by $f(a) = \{a\}$ is injective, and hence $|A| \leq |P(A)|$. Cantor proved the following theorem.

Theorem 1.3. *Let A be a set. There is no bijective map between A and $P(A)$, and hence $|A| < |P(A)|$.*

Proof. The proof is again by contradiction. Suppose there were a bijective map from A to $P(A)$. To each $a \in A$, we can then assign a unique set $T_a \in P(A)$. Consider

$$S = \{a \in A : a \notin T_a\}.$$

Clearly, S is a subset of A. Thus it must correspond to some T_w with $w \in A$. But this leads to a contradiction:

$$w \in S \implies w \notin T_w \implies w \notin S$$

and

$$w \notin S \implies w \in T_w \implies w \in S. \qquad \square$$

In this way, Cantor showed that there is an infinite ladder of infinite sets:

$$|\mathbb{N}| < |P(\mathbb{N})| < |P(P(\mathbb{N}))| < |P(P(P(\mathbb{N})))| < \cdots .$$

1.3. The Schröder–Bernstein Theorem

Instead of seeking a bijective correspondence between two sets A and B, it is sufficient to establish injective maps $f : A \to B$ and $g : B \to A$. In other words, if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. This is known as the Schröder–Bernstein[3] theorem after Ernst Schröder (1841–1902) and Felix Bernstein (1878–1956). Before proving this in general, we first prove a special case. If $B \subseteq A$, then the inclusion map from B to A is an injection. Thus $B \subseteq A$ implies $|B| \leq |A|$. The following lemma is the Schröder–Bernstein theorem in the special case where one set is a subset of the other.

Lemma 1.4. *Let A, B be sets such that $B \subseteq A$, and suppose that we have an injection $f : A \to B$. Then there is a bijection $g : A \to B$.*

[3] There is some controversy on the name of this theorem. It was first stated by Cantor without proof in 1887. It seems Richard Dedekind proved it in 1887 but didn't tell anyone about it. It was discovered in his notes in 1908. In 1895 Cantor published the first proof, but his proof uses the axiom of choice, which is discussed in the next chapter. Dedekind's unpublished proof did not use the axiom of choice. In 1896 Schröder published a proof sketch that was shown to be incorrect a few years later. In 1897, Bernstein proved the theorem—at age 19! Afterwards, Bernstein visited Dedekind, who apparently then independently proved the theorem yet again.

Proof. To prove this, we define sets $D_0, D_1, \ldots$ recursively as follows. $D_0 = A \backslash B$, $D_1 = f(D_0)$, $D_2 = f(D_1)$, and generally $D_{n+1} = f(D_n)$. Now define the map $g : A \to B$ by setting $g(x) = f(x)$ if x is in some D_n and $g(x) = x$ otherwise. If x is not in any D_n, then in particular it is not in D_0, and so x is in B so that $g(x) = x \in B$. We claim that g is a bijection. To see this, we have to show that g is injective and surjective. Suppose $g(x) = g(y)$. If both x and y are in some D_n, then we get $f(x) = f(y)$. Since f is injective, we deduce $x = y$. If both x and y are not in any D_n, then we have $x = g(x) = g(y) = y$, so again g is injective. Now consider the possibility that x is in some D_n and y is not. Then $g(x) = g(y)$ implies $f(x) = y$. Since x is in some D_n, it follows that y is in D_{n+1}, a contradiction. Thus g is injective. To see that g is surjective, let $b \in B$. If b is not in any D_n, then $g(b) = b$. If b is in some D_n, with $n \geq 1$, then $b \in f(D_{n-1})$ and so b is in the range of g. If $b \in D_0$, then $b \notin B$. □

Theorem 1.5 (Schröder–Bernstein theorem)**.** *If $f : A \to B$ and $g : B \to A$ are injective, then there is a bijection between A and B.*

Proof. The composition $g \circ f : A \to g(B)$ is also injective since

$$g(f(x)) = g(f(y)) \implies f(x) = f(y) \implies x = y.$$

$g(B)$ is a subset of A and so, by the previous lemma, there is a bijection $h : A \to g(B)$. Since g is injective, g^{-1} exists, and we have a map $g^{-1} : g(B) \to B$. Define $F : A \to B$ by $F(z) = g^{-1}(h(z))$. We show that F is both injective and surjective. If $F(z_1) = F(z_2)$, then $g^{-1}(h(z_1)) = g^{-1}(h(z_2))$, and applying g to both sides gives $h(z_1) = h(z_2)$. Since h is injective, we deduce $z_1 = z_2$. Let $b \in B$. There is an $a \in A$ such that $h(a) = g(b)$. Then $F(a) = g^{-1}(h(a)) = g^{-1}(g(b)) = b$. □

We have seen that $|\mathbb{N}| < |\mathbb{R}|$ and $|\mathbb{N}| < |P(\mathbb{N})|$. How do the cardinalities of $\mathbb{R}$ and $P(\mathbb{N})$ compare? We will use the Schröder–Bernstein theorem to prove that they are in fact the same.

Theorem 1.6. $|\mathbb{R}| = |P(\mathbb{N})|$.

Proof. Consider the function $f : P(\mathbb{N}) \to \mathbb{R}$ defined by $f(S) = 0.x_1x_2x_3\cdots$, where $x_i = 0$ if $i - 1 \in S$ and $x_i = 1$ if $i - 1 \notin S$.

Let $S, T \in P(\mathbb{N})$ with $f(S) = 0.x_1x_2x_3\cdots$ and $f(T) = 0.y_1y_2y_3\cdots$. Suppose $f(S) = f(T)$. Since we have avoided using the digit 9, the decimal expansions of $f(S)$ and $f(T)$ are unique. Thus $x_i = y_i$ for all $i \geq 1$, and in particular $x_i = 0$ if and only if $y_i = 0$. Thus $i - 1 \in S$ if and only if $i - 1 \in T$ for all $i \geq 1$, and so $S = T$. Thus f is injective, and hence $|P(\mathbb{N})| \leq |\mathbb{R}|$.

We now give an injective function mapping from $\mathbb{R}$ to $P(\mathbb{N})$. We may uniquely represent a real number x as $x = (-1)^n y + z$, where $n \in \{0, 1\}$, $y \in \mathbb{N}$, and $0 \leq z < 1$ (if $y = 0$, we agree to take $n = 0$). Furthermore, we may write $z = 0.d_1d_2d_3\cdots$ and agree to avoid the use of repeating 9's if the decimal expansion terminates. Let p_i be the ith prime number. We define $g : \mathbb{R} \to P(\mathbb{N})$ by $g(x) = \{2^n, 3^y\} \cup \{p_{i+2}^{d_i} : i \in \mathbb{N}\}$. For $x_1, x_2 \in \mathbb{R}$, we write $x_1 = (-1)^{n_1} y_1 + z_1$ and $x_2 = (-1)^{n_2} y_2 + z_2$ as described above. Suppose $g(x_1) = g(x_2)$. By the uniqueness of prime factorization, we have $n_1 = n_2$, $y_1 = y_2$, and all digits of z_1 and z_2 equal, and therefore $x_1 = x_2$. Thus g is injective, and hence $|\mathbb{R}| \leq |P(\mathbb{N})|$. By the Schröder–Bernstein theorem, $|\mathbb{R}| = |P(\mathbb{N})|$. □

Cantor's *continuum hypothesis* is the assertion that for any $\mathbb{N} \subseteq A \subseteq \mathbb{R}$, we either have $|A| = |\mathbb{N}|$ or $|A| = |\mathbb{R}|$. Cantor believed the hypothesis to be true and attempted, unsuccessfully, to prove it for many years. The problem was the first of Hilbert's aforementioned 1900 list of twenty-three unsolved problems. Through the work of Kurt Gödel in 1938 and Paul Cohen in 1963, the continuum hypothesis was shown to be independent of the usual axioms of set theory, which are discussed in the next chapter. The work of Gödel and Cohen is discussed in Section 4.3.

In 1884, the logician Gottlob Frege (1848–1925) defined an equivalence relation on the collection of sets by saying that two sets are equivalent if they have the same cardinality. The equivalence classes were to be thought of as *cardinal numbers.* Thus 0 represents the set of all sets with no elements, 1 represents the set of all sets with one element, and so forth.

However, it was quickly pointed out that the existence of certain large sets, such as the set of all sets with the same cardinality, may lead to some fundamental difficulties. For example, consider the set of all sets that do not contain themselves. Does this set contain itself as an element? Either case yields a contradiction. This is the famous *Russell's paradox* which was brought to Frege's attention by Bertrand Russell (1872–1970). It may be rephrased as the *barber paradox*: a barber (who is male) shaves all men in his town who do not shave themselves. Does the barber shave himself? This self-reference is the fundamental obstacle. These paradoxes led mathematicians to re-examine the definition of a set and what the rules (or axioms) should be for constructing them. Thus it would be hoped that the creation of the set of all sets that do not contain themselves would not be allowed in such an axiomatic system. The most commonly used collection of axioms are discussed in the next chapter.

We end this section with a fun thought experiment, due to Hilbert. We describe an amazing hotel that we name *Hilbert's hotel* in honour of its creator. In this hotel, the number of rooms is countably infinite. There is a room for each positive natural number. Being a popular tourist destination, the hotel is full. On a rainy night, a poor wet soul stumbles into the lobby and pleads for a room. The attendant at the desk, feeling bad for the soaked patron, decides to make room even though the hotel is full. She gets on the hotel intercom and asks each guest to switch to the next room. If a guest is in room n, they are to move to room $n + 1$. The hotel's guests are all very accommodating and happily oblige. Everyone still has a room, but now room 1 is free for the new guest!

Unfortunately, the desk attendant's work is not done for the night. A busload of new guests arrives and all need a separate room. However, this is a rather large bus, as it seats a countably infinite number of people! There is a seat for each positive natural number. Still, the crafty desk attendant is able to make room. On the hotel intercom, she asks all guests to move to the room that is twice their current room number. If a guest is in room n, they are to move to room $2n$. Doing this, all guests still have a room, but now all odd-numbered rooms are empty. The desk attendant then sends the person from bus seat n to room $2n - 1$, which gives everyone a room and leaves the hotel full again.

Things can get even more difficult for our poor desk attendant. A fleet of busses arrives. There are a countably infinite number of busses in the fleet, one for each positive natural number, and each bus has a countably infinite number of seats! Our enterprising desk attendant again sends all current guests to the room twice their number, so that all even-numbered rooms are occupied. The occupant of the nth seat on the first bus is sent to room 3^n. The occupant of the nth seat on the second bus is sent to room 5^n. Similarly, the occupants of the third bus are sent to rooms that are powers of 7. The occupants of the next bus are sent to room that are powers of 11 (room $9 = 3^2$ is already occupied). In general, the nth occupant of bus m is sent to room number p_{m+1}^n, where p_m is the mth prime number (we skip powers of 2 since even-numbered rooms are occupied). The desk attendant is able to give everyone a room. Note that, unlike the previous strategies, this one leaves rooms unoccupied. For example, rooms 1 and 15 are left empty.

The example of Hilbert's hotel shows how counterintuitive infinite sets can be, especially if we are basing our expectations on the behaviour of finite sets.

Exercises

1.1. Show that the function $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = (-1)^{n+1} \left\lfloor \frac{n+1}{2} \right\rfloor$$

is a bijection.

1.2. Give bijections $f : \mathbb{N} \to \mathbb{Z}$ and $P : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, show that the function $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ defined by $h(x, y) = P(f^{-1}(x), f^{-1}(y))$ is bijective.

1.3. If A and B are countably infinite, show that $A \times B$ is countably infinite.

1.4. If A and B are countably infinite, show that $A \cup B$ is countably infinite.

1.5. If $A_1, A_2, \ldots,$ is an infinite sequence of disjoint finite sets, show that the union $\bigcup_{n=1}^{\infty} A_n$ is countably infinite.

1.6. If $A_1, A_2, \ldots$ is an infinite sequence of disjoint countably infinite sets, show that the union $\bigcup_{n=1}^{\infty} A_n$ is countably infinite. (In case you are aware of the *axiom of choice*, you may (and in fact will need to) use it here. The axiom will be discussed in the next chapter).

1.7. Construct an explicit polynomial bijection between $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

1.8. Fix a natural number $b \geq 2$. Show that every positive real number x in $[0, 1]$ has a b-adic expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{x_n}{b^n}$$

with each $0 \leq x_n \leq b - 1$ an integer.

1.9. Suppose

$$\sum_{n=1}^{\infty} \frac{x_n}{b^n} = \sum_{n=1}^{\infty} \frac{y_n}{b^n}$$

with each $0 \leq x_n \leq b - 1$ and $0 \leq y_n \leq b - 1$ integers. Show that either $x_n = y_n$ for all n, or there is an m such that one of the following two cases occurs:

- $x_m = y_m + 1$ and for $n \geq m + 1$, $y_n = b - 1$ and $x_n = 0$;
- $y_m = x_m + 1$ and for $n \geq m + 1$, $x_n = b - 1$ and $y_n = 0$.

1.10. Show that a number $x \in [0, 1]$ is rational if and only if its decimal expansion is eventually periodic. Deduce that irrational numbers have unique decimal expansions.

1.11. Show that the collection of polynomials with rational coefficients is a countably infinite set. Deduce that the set of algebraic numbers is countably infinite.

1.12. Show that the collection of infinite sequences made up of the elements 0 and 1 is uncountable. (*Hint*: Think about the proof that the set of real numbers between 0 and 1 is uncountable, and try something similar.)

1.13. Show that the number of functions mapping from $\mathbb{N}$ to $\mathbb{N}$ is uncountable. (*Hint*: Think about the proof that the number of real numbers between 0 and 1 is uncountable, and try something similar.)

1.14. Define a relation on the collection of sets as follows. A is related to B if there is a bijection f mapping from A to B. Show that this is an equivalence relation. That is, show that the following hold.

- Any set A is related to itself.
- If A is related to B, then B is related to A.
- If A is related to B and B is related to C, then A is related to C.

1.15. Define $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ by $f(a, b) = 2^a 3^b$. Show that f is injective. Use the Schröder–Bernstein theorem to deduce that $\mathbb{N} \times \mathbb{N}$ is countably infinite.

1.16. Let A be the set of all finite subsets of $\mathbb{N}$. Find injective functions from $\mathbb{N}$ to A and from A to $\mathbb{N}$. (*Hint*: For the second function, try to use the uniqueness of prime factorization.) Use the Schröder–Bernstein theorem to deduce that A is countably infinite. Then prove that the number of infinite subsets of $\mathbb{N}$ is uncountable.

1.17. Let $\mathbb{R}^\times = \{x \in \mathbb{R} : x \neq 0\}$. Use the Schröder–Bernstein theorem to deduce that $|\mathbb{R}^\times| = |\mathbb{R}|$. Now try to explicitly define a bijection between the sets.

1.18. Let $A = \{x \in \mathbb{R} \mid 0 < x < 1\}$ and $B = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. Find injective functions $f : A \to B$ and $g : B \to A$. Use the Schröder–Bernstein theorem to deduce that $|A| = |B|$. Now try to explicitly define a bijection between the sets (this is a bit tricky).

1.19. Let $S = \{s_1, \ldots, s_n\}$ be a nonempty set of finitely many symbols. Show that the number of finite strings consisting of elements of S is countably infinite. What happens if S is countably infinite?

1.20. The two questions below refer to Hilbert's hotel, discussed at the end of this chapter.

(a) As in the final scenario, a fleet of a countably infinite number of busses arrives, each with a countably infinite number of seats. Describe a way to assign rooms to everyone, including those currently in the hotel, so that no rooms are left empty.

(b) Multiple fleets of busses arrive at the hotel. There are a countably infinite number of fleets, one for each positive natural number. As before, each fleet contains a countably infinite number of busses, and each bus contains a countably infinite number of people. Find a way for the desk attendant to accommodate all guests.