
Introduction

The aim of this book, which originates from a course that we taught successively at École Normale Supérieure (FL in 2007-2010 and MH in 2010-2013), is to present a broad panorama of Mathematical Logic to students who feel curious about this field but have no intent to specialize in it. As a consequence we have deliberately chosen not to write another comprehensive textbook, of which there already exist quite a few excellent ones, but instead to deliver a slim text which provides direct routes to some significant results of general interest.

Our point of view is to treat Logic on an equal footing to any other topic in the mathematical curriculum. Since one does not have to define natural numbers when teaching Number Theory, or sets when teaching Analysis, why should we in a Logic course? For this reason we start the book with a presentation of naive Set Theory, that is, the theory of sets that mathematicians use on a daily basis. It is only in the last chapter that we discuss the Zermelo-Fraenkel axioms, which in fact most mathematicians who are not Set Theorists or teaching a logic course are not so familiar with.

In each chapter we have tried to present at least a few juicy highlights, outside Logic whenever possible, either in the main text, or as exercises or appendices. We consider exercises as an essential component of the book, and we encourage the reader to work them out thoroughly;

they should be seen not only as a tool to check that the course is correctly assimilated, but also as a way to provide an opening to additional topics of interest.

The book is organized as follows. In the first chapter, in addition to the basic theory of ordinal and cardinal numbers, we cover more exotic topics like Goodstein sequences, infinite combinatorics (clubs and Solovay's Theorem) and Hindman's Theorem (a striking result in additive combinatorics). In Chapter 2 we introduce First-order Logic and formal proofs. We prove Gödel Completeness via Henkin witnesses. Craig Interpolation and Beth Definability are treated in exercises. The next chapter delves deeper inside Model Theory, with detailed coverage of Quantifier Elimination. In particular we prove Quantifier Elimination for algebraically closed fields, which allows one to state and prove the Lefschetz Principle in Algebraic Geometry and Ax's Theorem on surjectivity of injective polynomial mappings. Chapter 4 is devoted to basic Recursion Theory and culminates with the existence of universal recursive functions, undecidability of the Halting Problem and Rice's Theorem. In Chapter 5 we prove the classical undecidability and incompleteness results of Tarski and Church and provide a complete proof of Gödel's Second Incompleteness Theorem which we found in Martin Ziegler's book [13]. We also present, as an exercise, a theorem of Tennenbaum about the inexistence of non-standard countable recursive models of Peano. Finally in Chapter 6 we develop Axiomatic Set Theory, including the Reflection Principle and some proofs of independence and relative consistency.

This book is intended towards advanced undergraduate students, graduate students at any stage, or working mathematicians, who seek a first exposure to core material of mathematical logic and some of its applications. Prerequisites are minimal: besides familiarity with abstract reasoning and basic mathematical concepts, some acquaintance with General Topology and Algebra, especially Field Theory, is required at various places.

For the interested reader, here are a few suggestions for further reading, providing more comprehensive and advanced material, ordered by increasing difficulty:

Model Theory: The books by Marker [8], Poizat [9] and Tent-Ziegler [12].

Set Theory: The books by Krivine [6], Kunen [7] and Jech [5].

Recursion Theory: The books by Cooper [1], Rogers [10] and Soare [11].

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