
Preface

This book serves as both an introduction to discrete Morse theory and a general introduction to concepts in topology. I have tried to present the material in a way accessible to undergraduates with no more than a course in mathematical proof writing. Although some books such as [102, 132] include a single chapter on discrete Morse theory, and one [99] treats both smooth and discrete Morse theory together, no book-length treatment is dedicated solely to discrete Morse theory. Discrete Morse theory deserves better: It serves as a tool in applications as varied as combinatorics [16, 41, 106, 108], probability [57], and biology [136]. More than that, it is fascinating and beautiful in its own right. Discrete Morse theory is a discrete analogue of the “smooth” Morse theory developed in Marston Morse’s 1925 paper [124], but it is most popularly known via John Milnor [116]. Fields medalist Stephen Smale went so far as to call smooth Morse theory “the single greatest contribution of American mathematics” [144]. This beauty and utility carries over to the discrete setting, as many of the results, such as the Morse inequalities, have discrete analogues. Discrete Morse theory not only is topological but also involves ideas from combinatorics and linear algebra. Yet it is easy to understand, requiring no more than familiarity with basic set theory and mathematical proof techniques. Thus we find several online introductions to discrete Morse theory written by undergraduates. For

example, see the notes of Alex Zorn for his REU project at the University of Chicago [158], Dominic Weiller's bachelor's thesis [150], and Rachel Zax's bachelor's thesis [156].

From a certain point of view, discrete Morse theory has its foundations in the work of J. H. C. Whitehead [151, 152] from the early to mid-20th century, who made the deep connection between simple homotopy and homotopy type. Building upon this work, Robin Forman published the original paper introducing and naming discrete Morse theory in 1998 [65]. His extremely readable *A user's guide to discrete Morse theory* is still the gold standard in the field [70]. Forman published several subsequent papers [66, 68–71] further developing discrete Morse theory. The field has burgeoned and matured since Forman's seminal work; it is certainly established enough to warrant a book-length treatment.

This book further serves as an introduction, or more precisely a first exposure, to topology, one with a different feel and flavor from other introductory topology books, as it avoids both the point-set approach and the surfaces approach. In this text, discrete Morse theory is applied to simplicial complexes. While restriction to only simplicial complexes does not expose the full generality of discrete Morse theory (it can be defined on regular CW complexes), simplicial complexes are easy enough for any mathematically mature student to understand. A restriction to simplicial complexes is indeed necessary for this book to act as an exposure to topology, as knowledge of point-set topology is required to understand CW complexes. The required background is only a course in mathematical proofs or an equivalent course teaching proof techniques such as mathematical induction and equivalence relations. This is not a book about smooth Morse theory either. For smooth Morse theory, one can consult Milnor's classic work [116] or a more modern exposition in [129]. A discussion of the relations between the smooth and discrete versions may be found in [27, 29, 99].

One of the main lenses through which the text views topology is homology. A foundational result in discrete Morse theory consists of the (weak) discrete Morse inequalities; it says that if K is a simplicial complex and $f : K \rightarrow \mathbb{R}$ a discrete Morse function with m_i critical simplices of dimension i , then $b_i \leq m_i$ where b_i is the i th Betti number. To prove this theorem and do calculations, we use \mathbb{F}_2 -simplicial homology and

build a brief working understanding of the necessary linear algebra in Chapter 3. Chapter 1 introduces simplicial complexes, collapses, and simple homotopy type, all of which are standard topics in topology.

Any book reflects the interests and point of view of the author. Combining this with space considerations, I have regrettably had to leave much more out than I included. Several exclusions are worth mentioning here. Discrete Morse theory features many interesting computational aspects, only a few of which are touched upon in this book. These include homology and persistent homology computations [53, 80, 82], matrix factorization [86], and cellular sheaf cohomology computations [48]. Mathematicians have generalized and adapted discrete Morse theory to various settings. Heeding a call from Forman at the end of *A user's guide to discrete Morse theory* [70], several authors have extended discrete Morse theory to certain kinds of infinite objects [8, 10, 12, 15, 105]. Discrete Morse theory has been shown to be a special case [155] of Bestvina-Brady discrete Morse theory [34, 35] which has extensive applications in geometric group theory. There is an algebraic version of discrete Morse theory [87, 102, 142] involving chain complexes, as well as a version for random complexes [130]. E. Minian extended discrete Morse theory to include certain collections of posets [118], and B. Benedetti developed discrete Morse theory for manifolds with boundary [28]. There is also a version of discrete Morse theory suitable for reconstructing homotopy type via a certain classifying space [128]. K. Knudson and B. Wang have recently developed a stratified version of discrete Morse theory [100]. The use of discrete Morse theory as a tool to study other kinds of mathematics has proved invaluable. It has been applied to study certain problems in combinatorics and graph theory [16, 41, 49, 88, 106, 108] as well as configuration spaces and subspace arrangements [60, 122, 123, 139]. It is also worth noting that before Forman, T. Banchoff also developed a discretized version of Morse theory [17–19]. This, however, seemed to have limited utility. E. Bloch found a relationship between Forman's discrete Morse theory and Banchoff's [36].

I originally developed these ideas for a course in discrete Morse theory taught at Ursinus College for students whose only prerequisite was a proof-writing course. An introductory course might cover Chapters 1–5 and Chapter 8. For additional material, Chapters 6 and 9 are good

choices for a course with students who have an interest in computer science, while Chapters 7 and 10 are better for students interested in pure math. Some of the more technical proofs in these chapters may be skipped. A more advanced course could begin at Chapter 2 and cover the rest of the book, referring back to Chapter 1 when needed. This book could also be used as a supplemental text for a course in algebraic topology or topological combinatorics, an independent study, or a directed study, or as the basis for an undergraduate research project. It is also intended for research mathematicians who need a primer in or reference for discrete Morse theory. This includes researchers in not only topology but also combinatorics who would like to utilize the tools that discrete Morse theory provides.

Exercises and Problems

The structure of the text reflects my philosophy that “mathematics is not a spectator sport” and that the best way to learn mathematics is to actively do mathematics. Scattered throughout each chapter are tasks for the reader to work on, labeled “Exercise” or “Problem.” The distinction between the two is somewhat artificial. The intent is that an Exercise is a straightforward application of a definition or a computation of a simple example. A Problem is either integral to understanding, necessary for other parts of the book, or more challenging. The level of difficulty of the Problems can vary substantially.

A note on the words “easy,” “obvious,” etc.

In today’s culture, we often avoid using words such as “easily,” “clearly,” “obviously,” and the like. It is thought that these words can be stumbling blocks for readers who do not find it clear, causing them to become discouraged. For that reason, I have attempted to avoid using these words in the text. However, the text is not completely purged of such words, and I would like to convey what I mean when I use them. I often tell my students that a particular mathematical fact is “easy but it is very difficult to see that it is easy.” By this I mean that one may need to spend a significant amount of time struggling to understand the meaning of the claim before it “clicks.” So when the reader sees words like “obviously,” she should not despair if it is not immediately obvious

to her. Rather, the word is an indication that should alert the reader to write down an example, rewrite the argument in her own words, or stare at the definition until she gets it.

Erratum

A list of typos, errors, and corrections for the book will be kept at <http://webpages.ursinus.edu/nscoville/DMTerratum.html>.

Acknowledgements

Many people helped and supported my writing of this book. Ranita Biswas, Sebastiano Cultrera di Montesano, and Morteza Saghafian worked through the entire book and offered detailed comments and suggestions for improvement. Steven Ellis, Mark Ellison, Dominic Klyve, Max Lin, Simon Rubinstein-Salzedo, and Jonathan Webster gave helpful mathematical feedback and caught several typos and errors. I am grateful for the help of Kathryn Hess, Chris Sadowski, Paul Pollack, Lisbeth Fajstrup, Andy Putman, and Matthew Zaremsky for answering specific questions, as well as Dana Williams and Ezra Miller for their help developing the index. A very special thanks to Pawel Dłotko, Mimi Tsuruga, Chris Tralie, and David Millman for their help with Chapter 9. Of course, any errors in the book are solely *mea culpa*, *mea culpa*, *mea maxima culpa*. I would like to thank Kevin Knudson for his encouragement and support of this book, as well as Ursinus College, the Mathematics and Computer Science department, and all my students for supporting this project. I would like to express my appreciation to Ina Mette at AMS for her help and patience throughout the submission and publication process, as well as several anonymous reviewers. My family has provided amazing support for this project, through both encouragement and feedback. I am grateful for edits and suggestions from my maternal cousin Theresa Goenner and my paternal cousin Brooke Chantler who resisted the urge to change all the spellings to reflect the Queen's English. A special thanks to both my father Joseph Scoville and my aunt Nancy Mansfield, neither of whom has taken a course in calculus, but who read every single word of this book and gave detailed feedback which greatly improved its readability. Finally, this project could not have been completed without the love and support of my family: my children Gianna,

Aniela, Beatrix, Felicity, and Louisa-Marie, and most especially my wife Jennifer, *mio dolce tesoro*.

Nick Scoville
Feast of Louis de Montfort