Chapter 1

Symmetric Polynomials, the Monomial Symmetric Polynomials, and Symmetric Functions

Symmetric polynomials and symmetric functions arise naturally in a variety of settings. Certain symmetric functions appear as characters of polynomial irreducible representations of the general linear groups. The algebra of symmetric polynomials in n variables is isomorphic to the algebra generated by the characters of the irreducible representations of the symmetric group S_n. Certain graph invariants related to proper colorings of graphs are symmetric functions, and recently knot invariants like the Jones polynomial have been found to be related to symmetric functions. A variety of symmetric functions arise as generating functions for various families of combinatorial objects, and certain generalizations of symmetric polynomials represent cohomology classes of Schubert cycles in flag varieties.

In short, symmetric polynomials and symmetric functions are worth studying for the many places they turn up in algebra, geometry, topology, graph theory, combinatorics, and elsewhere. But symmetric functions and symmetric polynomials are also worth studying for their own structure, and for the rich way the algebraic questions they generate have natural and elegant combinatorial answers. Our goal will be to reveal as much of this structure as we can, without worrying too much about the rich connections symmetric functions have elsewhere.

In this chapter we start our study of symmetric functions and symmetric polynomials with the symmetric polynomials. We explain what it means for a polynomial to be symmetric, we describe how to construct symmetric functions, and we explain how symmetric functions are related to symmetric polynomials. We will see that the set of all symmetric functions with coefficients in $\mathbb{Q}$ is a finite-dimensional vector space over $\mathbb{Q}$, and we will use a natural construction to give the first of several bases we will see for this space.

1.1. Symmetric Polynomials

We often think of each permutation $\pi \in S_n$ as a bijection from $[n] := \{1, 2, \ldots, n\}$ to $[n]$, meaning π permutes the elements of $[n]$. (See Appendix C for background on permutations.) If instead of $[n]$ we have a set of variables $x_1, \ldots, x_n$, then we can also view π as a permutation of these variables, so that $\pi(x_j) = x_{\pi(j)}$ for $1 \le j \le n$. Pushing this a bit further, we might eventually want to combine our variables into polynomials, and then try to view π as a function on these polynomials. This turns out to be reasonably easy to do: if $f(x_1, x_2, \ldots, x_n)$ is a polynomial in $x_1, \ldots, x_n$, then we define $\pi(f)$ by setting $\pi(f) := f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$. That is, π acts on f by permuting (the subscripts of) its variables.

Example 1.1. Suppose that in one-line notation we have $\pi = 2413$ and $\sigma = 4312$. If we also have $f(x_1, x_2, x_3, x_4) = x_1^2 + 3x_2x_4 - 2x_3^2x_4$ and $g(x_1, x_2, x_3, x_4) = 3x_3x_4^2$, then compute $\pi(f)$, $\pi(g)$, $\pi(f + g)$, $\pi(f) + \pi(g)$, $\pi(fg)$, $\pi(f)\pi(g)$, $\sigma(f)$, $\pi(\sigma(f))$, and $(\pi\sigma)(f)$.

Solution. By definition we have

$$\begin{aligned}\pi(f) &= f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, x_{\pi(4)}) \\ &= x_{\pi(1)}^2 + 3x_{\pi(2)}x_{\pi(4)} - 2x_{\pi(3)}^2 x_{\pi(4)}.\end{aligned}$$

Since $\pi(1) = 2$, $\pi(2) = 4$, $\pi(3) = 1$, and $\pi(4) = 3$, we find

$$\pi(f) = x_2^2 + 3x_4x_3 - 2x_1^2x_3.$$

Similarly, $\pi(g) = 3x_1x_3^2$.

To compute $\pi(f+g)$, first note that $f + g = x_1^2 + 3x_2x_4 - 2x_3^2x_4 + 3x_3x_4^2$, so $\pi(f+g) = x_2^2 + 3x_4x_3 - 2x_1^2x_3 + 3x_1x_3^2$. On the other hand, we have $\pi(f) = x_2^2 + 3x_4x_3 - 2x_1^2x_3$ and $\pi(g) = 3x_1x_3^2$, so $\pi(f) + \pi(g) = x_2^2 + 3x_4x_3 - 2x_1^2x_3 + 3x_1x_3^2$.

To compute $\pi(fg)$, first note that $fg = 3x_1^2x_3x_4^2 + 9x_2x_3x_4^3 - 6x_3^3x_4^3$, so $\pi(fg) = 3x_1x_2^2x_3^2 + 9x_1x_3^3x_4 - 6x_1^3x_3^3$. On the other hand, from our previous computations we find $\pi(f)\pi(g) = 3x_1x_2^2x_3^2 + 9x_1x_3^3x_4 - 6x_1^3x_3^3$.

Computing as we did for π, we find $\sigma(f) = x_4^2 + 3x_2x_3 - 2x_1^2x_2$. If we now apply π to $\sigma(f)$, we find $\pi(\sigma(f)) = x_3^2 + 3x_1x_4 - 2x_2^2x_4$. On the other hand, $\pi\sigma = 3124$, so $(\pi\sigma)(f) = x_3^2 + 3x_1x_4 - 2x_2^2x_4$. □

As we can already see in Example 1.1, we will be considering polynomials in many variables. To make our notation less cumbersome, for all $n \geq 1$ we will write X_n to denote the set of variables $x_1, \ldots, x_n$.

The computations in Example 1.1 suggest that the action of the permutations on polynomials is compatible with our usual arithmetic operations on polynomials, as well as with composition of permutations. In our next result we make this observation precise.

Proposition 1.2. *Suppose $f(X_n)$ and $g(X_n)$ are polynomials, c is a constant, and $\pi, \sigma \in S_n$ are permutations. Then*

(i) $\pi(cf) = c\pi(f)$;

(ii) $\pi(f+g) = \pi(f) + \pi(g)$;

(iii) $\pi(fg) = \pi(f)\pi(g)$;

(iv) $(\pi\sigma)(f) = \pi(\sigma(f))$.

Proof. (i) This follows from the fact that π permutes the subscripts of the variables without changing any coefficients.

(ii) This follows from the fact that if f is a sum of monomials t_j, then $\pi(f)$ is the sum of the monomials $\pi(t_j)$.

(iii) Suppose first that f and g are each a single term, so that $f(X_n) = ax_1^{a_1}\cdots x_n^{a_n}$ and $g(X_n) = bx_1^{b_1}\cdots x_n^{b_n}$ for constants a,b,$a_1,\dots,$ a_n, $b_1,\dots,b_n$. Then we have

$$\begin{aligned}\pi(fg) &= \pi(abx_1^{a_1+b_1}\cdots x_n^{a_n+b_n})\\ &= abx_{\pi(1)}^{a_1+b_1}\cdots x_{\pi(n)}^{a_n+b_n}\\ &= ax_{\pi(1)}^{a_1}\cdots x_{\pi(n)}^{a_n}bx_{\pi(1)}^{b_1}\cdots x_{\pi(n)}^{b_n}\\ &= \pi(f)\pi(g),\end{aligned}$$

and the result holds in this case. To show the result holds in general, suppose $f(X_n) = Y_1 + \cdots + Y_k$ and $g(X_n) = Z_1 + \cdots + Z_m$, where each Y_j and each Z_j is a monomial in $x_1,\dots,x_n$. Using (ii) and the fact that (iii) holds for monomials, we find

$$\begin{aligned}\pi(fg) &= \pi\left(\left(\sum_{j=1}^{k} Y_j\right)\left(\sum_{l=1}^{m} Z_l\right)\right)\\ &= \pi\left(\sum_{j=1}^{k}\sum_{l=1}^{m} Y_jZ_l\right)\\ &= \sum_{j=1}^{k}\sum_{l=1}^{m}\pi(Y_jZ_l)\\ &= \sum_{j=1}^{k}\sum_{l=1}^{m}\pi(Y_j)\pi(Z_l)\\ &= \left(\sum_{j=1}^{k}\pi(Y_j)\right)\left(\sum_{l=1}^{m}\pi(Z_l)\right)\\ &= \pi\left(\sum_{j=1}^{k} Y_j\right)\pi\left(\sum_{l=1}^{m} Z_l\right)\\ &= \pi(f)\pi(g),\end{aligned}$$

which is what we wanted to prove.

(iv) By definition we have

$$\begin{aligned}(\pi\sigma)(f) &= f(x_{(\pi\sigma)(1)},\ldots,x_{(\pi\sigma)(n)})\\ &= f(x_{\pi(\sigma(1))},\ldots,x_{\pi(\sigma(n))})\\ &= \pi(f(x_{\sigma(1)},\ldots,x_{\sigma(n)}))\\ &= \pi(\sigma(f)),\end{aligned}$$

which is what we wanted to prove. □

After computing $\pi(f)$ for various permutations π and polynomials f, we eventually notice f and $\pi(f)$ are usually different, but we sometimes get $\pi(f) = f$. When $\pi(f) = f$, we say f is *invariant under* π. If f is invariant under π, then we might also say f is *fixed by* π, or π *fixes* f.

Example 1.3. Find all permutations $\sigma \in S_3$ which fix $x_1x_2^2x_3 + x_1^2x_2x_3$.

Solution. Every polynomial is invariant under the identity permutation. If we check the other five permutations in S_3, then we find $x_1x_2^2x_3 + x_1^2x_2x_3$ is also invariant under (12), but not under any other permutation. □

Every permutation has its own set of invariant polynomials, but some polynomials are invariant under every permutation. These are the polynomials we plan to study.

Definition 1.4. We say a polynomial $f(X_n)$ is a *symmetric polynomial in* $x_1,\ldots,x_n$ whenever $\pi(f) = f$ for all $\pi \in S_n$. We write $\Lambda(X_n)$ to denote the set of all symmetric polynomials in $x_1,\ldots,x_n$ with coefficients in $\mathbb{Q}$.

We leave it as an exercise to use Proposition 1.2 to show $\Lambda(X_n)$ is a vector space over $\mathbb{Q}$ for all n, and that in fact $\Lambda(X_n)$ is infinite dimensional for all $n \geq 1$.

Example 1.5. Which of the following are symmetric polynomials in x_1, x_2, x_3? Which are symmetric polynomials in x_1, x_2, x_3, x_4?

(a) $3x_1x_2x_3 + x_1x_2 + x_2x_3 + x_1x_3 + 5$.

(b) $x_1^2x_2 + x_2^2x_3 + x_1x_3^2$.

Solution. (a) This polynomial is a symmetric polynomial in x_1, x_2, x_3, but it is not invariant under the permutation (14), so it is not a symmetric polynomial in x_1, x_2, x_3, x_4.

(b) The image of this polynomial under the permutation (12) is $x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_2$, which is not the original polynomial, so it is not a symmetric polynomial in either x_1, x_2, x_3 or x_1, x_2, x_3, x_4. □

When we apply a permutation π to a polynomial f, it may alter the terms of f dramatically. But some properties will always be preserved. For example, no matter how much a term is changed by π, its total degree remains the same. As a result, it is useful to break our symmetric polynomials into pieces in which every term has the same total degree.

Definition 1.6. We say a polynomial $f(X_n)$ in which every term has total degree exactly k is *homogeneous of degree* k. We write $\Lambda_k(X_n)$ to denote the set of all symmetric polynomials in $x_1, \ldots, x_n$ which are homogeneous of degree k. Note that for every nonnegative integer k, every term in the polynomial 0 has degree k (since 0 has no terms), so $0 \in \Lambda_k(X_n)$ for all $k \geq 0$ and all $n \geq 1$.

Note that if f is any polynomial in $x_1, \ldots, x_n$, then we can group the terms of f by their total degree, so f can be written uniquely as a sum $f_0 + f_1 + \cdots$, where f_k is homogeneous of degree k for each $k \geq 0$. If f happens to be a symmetric polynomial in $x_1, \ldots, x_n$, then for every $\pi \in S_n$, we have

$$\begin{aligned} f_0 + f_1 + \cdots &= f \\ &= \pi(f) \\ &= \pi(f_0 + f_1 + \cdots) \\ &= \pi(f_0) + \pi(f_1) + \cdots. \end{aligned}$$

Since π does not change the total degree of any monomial, we must have $\pi(f_j) = f_j$ for every $j \geq 0$. That is, if f is a symmetric polynomial in $x_1, \ldots, x_n$ and $f = f_0 + f_1 + \cdots$ is the decomposition of f into its homogeneous parts f_j, then each f_j is also a symmetric polynomial in $x_1, \ldots, x_n$. With this in mind, we often restrict our attention to the homogeneous symmetric polynomials of a given total degree.

Example 1.7. We saw in Example 1.5 that $3x_1x_2x_3 + x_1x_2 + x_2x_3 + x_1x_3 + 5 \in \Lambda(X_3)$. Write this symmetric polynomial as a sum $f_0 + f_1 + \cdots$, where $f_k \in \Lambda_k(X_3)$ for all $k \geq 0$.

Solution. In general f_k is the sum of all of the terms of total degree k, so in this case we have $f_0 = 5$, $f_1 = 0$, $f_2 = x_1x_2 + x_2x_3 + x_1x_3$, $f_3 = 3x_1x_2x_3$, and $f_k = 0$ for $k \geq 4$. □

As we did for $\Lambda(X_n)$, we leave it as an exercise to show $\Lambda_k(X_n)$ is a vector space over $\mathbb{Q}$ for all $k \geq 0$ and all $n \geq 1$. In contrast with $\Lambda(X_n)$, we will see the space $\Lambda_k(X_n)$ is finite dimensional.

1.2. The Monomial Symmetric Polynomials

So far we have seen exactly one symmetric polynomial, so it is natural to ask for more examples. One simple way to construct more symmetric polynomials is to be even more demanding: pick a set of variables $x_1, \ldots, x_n$, pick a monomial in those variables, and try to construct a symmetric polynomial that includes your monomial as a term.

Example 1.8. Find a symmetric polynomial $f \in \Lambda(X_3)$ that includes $x_1^3x_2$ as one of its terms. Similarly, find a symmetric polynomial $g \in \Lambda(X_3)$ that includes $3x_1^2x_2x_3^2$ as one of its terms. Use as few other monomials as possible in both f and g.

Solution. If f is a symmetric polynomial, then it must be invariant under (12). So if one of its terms is $x_1^3x_2$, then another of its terms must be $(12)(x_1^3x_2) = x_1x_2^3$. Similarly, f must also include the terms $x_2x_3^3$, $x_1^3x_3$, $x_1x_3^3$, and $x_2^3x_3$. On the other hand, $x_1^3x_2 + x_1x_2^3 + x_2x_3^3 + x_1^3x_3 + x_1x_3^3 + x_2^3x_3$ is a symmetric polynomial, so it must be the one we are looking for.

When we reason in the same way for g as we did for f, we find we get some duplicate terms. Since we do not need these duplicates, we find $g = 3x_1^2x_2x_3^2 + 3x_1x_2^2x_3^2 + 3x_1^2x_2^2x_3$. □

When we look more closely at our work in Example 1.8, we see we can construct a symmetric polynomial in $x_1, \ldots, x_n$ by starting with a monomial in these variables, and then adding the distinct images of this monomial under the permutations in S_n. If one monomial is the image of another under some permutation, then both monomials will give us the same symmetric polynomial under this construction. This means we can rearrange the factors in our monomial to ensure that when we list the variables in the order $x_1, \ldots, x_n$, their exponents form a partition. (See Appendix B for background on partitions.) With this in mind, we set some notation and terminology for the symmetric polynomials we've found.

Definition 1.9. Suppose $n \geq 1$ and λ is a partition. Then the *monomial symmetric polynomial* $m_\lambda(X_n)$ indexed by λ is the sum of the monomial $\prod_{j=1}^{l(\lambda)} x_j^{\lambda_j}$ and all of its distinct images under the elements of S_n. Here we take $x_j = 0$ for all $j > n$, so if $l(\lambda) > n$, then $m_\lambda(X_n) = 0$.

We will often have partitions as subscripts, as we do for the monomial symmetric polynomials. When all of the parts of these partitions are less than 10, we will save some space by omitting the commas and parentheses. So, for example, we will write $m_{4431}(X_n)$ instead of $m_{(4,4,3,1)}(X_n)$.

Example 1.10. Compute the four monomial symmetric polynomials $m_{21}(X_2)$, $m_{21}(X_3)$, $m_{3311}(X_3)$, and $m_{3311}(X_4)$.

Solution. The monomial $x_1^2 x_2$ has only one image (other than itself) under the elements of S_2, namely $x_1 x_2^2$. Therefore,

$$m_{21}(X_2) = x_1^2 x_2 + x_1 x_2^2.$$

Notice that if we hold λ constant and increase the number of variables, then we get more images of $x_1^2 x_2$, and therefore a new monomial symmetric polynomial, namely

$$m_{21}(X_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$

The length of the partition $(3^2, 1^2)$ is greater than three, so $m_{3311}(X_3)$ is 0. The requirement that we take only distinct images of our monomial comes into play when we compute $m_{3311}(X_4)$, which turns out to be

$$x_1^3 x_2^3 x_3 x_4 + x_1^3 x_2 x_3^3 x_4 + x_1^3 x_2 x_3 x_4^3 + x_1 x_2^3 x_3^3 x_4 + x_1 x_2^3 x_3 x_4^3 + x_1 x_2 x_3^3 x_4^3. \qquad \square$$

We saw in our solution to Example 1.10 that the polynomials $m_{21}(X_2)$, $m_{21}(X_3)$, $m_{3311}(X_3)$, and $m_{3311}(X_4)$ are all symmetric polynomials in their respective variables, and it may already be clear that $m_\lambda(X_n)$ is a symmetric polynomial in $x_1, \ldots, x_n$ for all n and all λ. Nevertheless, the following proof of this fact uses ideas and techniques which will be useful later on, so we include it here.

Proposition 1.11. *For all positive integers n and all partitions λ, the polynomial $m_\lambda(X_n)$ is a symmetric polynomial in $x_1, \ldots, x_n$.*

Proof. When $n < l(\lambda)$, we have $m_\lambda(X_n) = 0$, in which case the result is clear, so we assume $n \geq l(\lambda)$.

In view of Proposition C.5, it is sufficient to show $\sigma_j(m_\lambda(X_n)) = m_\lambda(X_n)$ for all j with $1 \leq j \leq n-1$, where σ_j is the adjacent transposition $(j, j+1)$. To do this, it is sufficient to show that every term $x_1^{\mu_1} \cdots x_n^{\mu_n}$ has the same coefficient in $\sigma_j(m_\lambda(X_n))$ as it has in $m_\lambda(X_n)$. Note that the coefficient of $x_1^{\mu_1} \cdots x_n^{\mu_n}$ in $m_\lambda(X_n)$ is 1 if $\mu_1, \ldots, \mu_n$ is a reordering of $\lambda_1, \ldots, \lambda_n$ and 0 otherwise. On the other hand, the coefficient of $x_1^{\mu_1} \cdots x_n^{\mu_n}$ in $\sigma_j(m_\lambda(X_n))$ is the coefficient of $x_1^{\mu_1} \cdots x_j^{\mu_{j+1}} x_{j+1}^{\mu_j} \cdots x_n^{\mu_n}$ in $m_\lambda(X_n)$. Furthermore, this coefficient is 1 if $\mu_1, \ldots, \mu_{j+1}, \mu_j, \ldots, \mu_n$ is a reordering of $\lambda_1, \ldots, \lambda_n$ and 0 otherwise. But $\mu_1, \ldots, \mu_n$ is a reordering of $\lambda_1, \ldots, \lambda_n$ if and only if $\mu_1, \ldots, \mu_{j+1}, \mu_j, \ldots, \mu_n$ is a reordering of $\lambda_1, \ldots, \lambda_n$, and the result follows. □

Note that if $\lambda \vdash k$, then the monomial symmetric polynomial $m_\lambda(X_n)$ is homogeneous of degree k, so $m_\lambda(X_n) \in \Lambda_k(X_n)$. In fact, immediately after Example 1.7 we promised we would see that $\Lambda_k(X_n)$ is finite dimensional. We conclude this section by keeping that promise, showing that if $n \geq k$, then the monomial symmetric polynomials form a basis for this space.

Proposition 1.12. *If $n \geq 1$, $k \geq 0$, and $n \geq k$, then the set*

$$\{m_\lambda(X_n) \mid \lambda \vdash k\}$$

of monomial symmetric polynomials is a basis for $\Lambda_k(X_n)$. In particular, $\dim \Lambda_k(X_n) = p(k)$, *the number of partitions of k.*

Proof. First observe that for any monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, there is only one partition λ whose parts are a rearrangement of $\alpha_1, \ldots, \alpha_n$, and $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a term in $m_\lambda(X_n)$. In particular, if λ and μ are partitions with $\lambda \neq \mu$, then $m_\lambda(X_n)$ and $m_\mu(X_n)$ have no terms in common. Therefore, since each $m_\lambda(X_n)$ is nonzero, $\sum_{\lambda \vdash k} a_\lambda m_\lambda(X_n) = 0$ can only occur if $a_\lambda = 0$ for all $\lambda \vdash k$. In other words, $\{m_\lambda(X_n) \mid \lambda \vdash k\}$ is linearly independent.

To see that this set also spans $\Lambda_k(X_n)$, suppose $f \in \Lambda_k(X_n)$. If $f = 0$, then $f = \sum_{\lambda \vdash k} 0 \cdot m_\lambda(X_n)$, so suppose $f \neq 0$. We argue by induction on the number of terms in f. Since f is symmetric, f has a term of the form $\alpha \prod_{j=1}^n x_j^{\mu_j}$ for some $\mu \vdash k$ and some constant α. Moreover, by the symmetry of f all of the distinct images of this monomial under the action of S_n also appear in f. Therefore $f - \alpha m_\mu(X_n) \in \Lambda_k(X_n)$ and $f - \alpha m_\mu(X_n)$ has fewer terms than f. By induction $f - \alpha m_\mu(X_n)$ is a linear combination of the elements of $\{m_\lambda(X_n) \mid \lambda \vdash k\}$, and thus f is as well.

The fact that $\dim \Lambda_k(X_n) = p(k)$, the number of partitions of k, now follows from the fact that there is exactly one monomial symmetric polynomial in $\Lambda_k(X_n)$ for each partition of k. □

We note that Proposition 1.12 also tells us the monomial symmetric polynomials form a basis for $\Lambda(X_n)$.

1.3. Symmetric Functions

In Proposition 1.12 we saw that if $n \geq k$, then the dimension of $\Lambda_k(X_n)$ is independent of n. This fact is an example of a more general phenomenon: if we have enough variables, then the algebraic properties of $\Lambda_k(X_n)$ do not depend on exactly how many variables we have.

To see another illustration of this central general principle, suppose $n \geq 2$ and consider the product $m_{11}(X_n)m_{21}(X_n)$. When $n = 2$, we have

$$m_{11}(X_2)m_{21}(X_2) = x_1^3x_2^2 + x_1^2x_2^3.$$

Table 1.1. The product $m_{11}(X_n)m_{21}(X_n)$ for $2 \le n \le 7$, as a linear combination of monomial symmetric polynomials

n	$m_{11}(X_n)m_{21}(X_n)$
2	m_{32}
3	$m_{32} + 2m_{311} + 2m_{221}$
4	$m_{32} + 2m_{311} + 2m_{221} + 3m_{2111}$
5	$m_{32} + 2m_{311} + 2m_{221} + 3m_{2111}$
6	$m_{32} + 2m_{311} + 2m_{221} + 3m_{2111}$
7	$m_{32} + 2m_{311} + 2m_{221} + 3m_{2111}$

When $n = 3$ we have

$$\begin{aligned} m_{11}(X_3)m_{21}(X_3) = {} & x_1^3x_2^2 + x_1^2x_2^3 + x_1^3x_3^2 + x_1^2x_3^3 + x_2^3x_3^2 + x_2^2x_3^3 \\ & + 2x^3x_2x_3 + 2x_1x_2^3x_3 + 2x_1x_2x_3^3 \\ & + 2x_1^2x_2^2x_3 + 2x_1^2x_2x_3^2 + 2x_1x_2^2x_3^2. \end{aligned}$$

Our products are all homogeneous symmetric polynomials of degree 5, so we can write them as linear combinations of the monomial symmetric polynomials. When we do this for $n = 2$ and $n = 3$, we find

$$m_{11}(X_2)m_{21}(X_2) = m_{32}(X_2)$$

and

$$m_{11}(X_3)m_{21}(X_3) = m_{32}(X_3) + 2m_{311}(X_3) + 2m_{221}(X_3).$$

In Table 1.1 we write $m_{11}(X_n)m_{21}(X_n)$ as a linear combination of the monomial symmetric polynomials, suppressing the arguments X_n in the answers. It appears from these data that if $n \ge 4$, then $m_{11}(X_n)m_{21}(X_n) = m_{32}(X_n) + 2m_{311}(X_n) + 2m_{221} + 3m_{2111}(X_n)$. In addition, it seems that if $n < 4$, then we have the same sum, except we remove those monomial symmetric polynomials whose indexing partition has more than n parts. We can prove this happens for any product $m_\lambda(X_n)m_\mu(X_n)$ by using a simple relationship between $m_\lambda(x_1, \ldots, x_n)$ and $m_\lambda(x_1, \ldots, x_n, 0)$.

Proposition 1.13. *For any partition λ if $n \ge l(\lambda)$, then*

$$m_\lambda(x_1, \ldots, x_n, 0) = m_\lambda(x_1, \ldots, x_n).$$

More generally, if $n \geq l(\lambda)$, then

$$m_\lambda(x_1, \dots, x_n, \underbrace{0, \dots, 0}_{j}) = m_\lambda(x_1, \dots, x_n)$$

for any $j \geq 1$.

Proof. Since $n \geq l(\lambda)$, both of the polynomials $m_\lambda(x_1, \dots, x_n, 0)$ and $m_\lambda(x_1, \dots, x_n)$ include the term $x_1^{\lambda_1} \cdots x_l^{\lambda_l}$, where $l = l(\lambda)$. In particular, neither $m_\lambda(x_1, \dots, x_n, 0)$ nor $m_\lambda(x_1, \dots, x_n)$ is 0. In fact, the terms in $m_\lambda(x_1, \dots, x_n, 0)$ are exactly those for which the exponents of $x_1, \dots, x_n$ are some permutation of $\lambda_1, \dots, \lambda_l$ and $n - l$ zeros. These are exactly the terms in $m_\lambda(x_1, \dots, x_n)$ as well, so $m_\lambda(x_1, \dots, x_n, 0) = m_\lambda(x_1, \dots, x_n)$.

The fact that $m_\lambda(x_1, \dots, x_n, \underbrace{0, \dots, 0}_{j}) = m_\lambda(x_1, \dots, x_n)$ for any $j \geq 1$ follows by induction on j, since setting $x_{n+j} = 0$ and then setting $x_{n+j-1} = 0$ in the resulting polynomial gives us the same result as setting $x_{n+j} = 0$ and $x_{n+j-1} = 0$ all at once. □

Now suppose we have $n \geq 1$ and partitions λ and μ. In Problem 1.15 you will have a chance to show that for any $n \geq 1$, the product $m_\lambda(X_n) m_\mu(X_n)$ is in $\Lambda(X_n)$. Since the monomial symmetric polynomials are a basis for $\Lambda(X_n)$, we know there are rational numbers $a^\nu_{\lambda,\mu}(n)$ such that

$$m_\lambda(X_n) m_\mu(X_n) = \sum_\nu a^\nu_{\lambda,\mu}(n) m_\nu(X_n). \tag{1.1}$$

In principle ν could be any partition, but all of the terms in $m_\lambda(X_n)$ have total degree $|\lambda|$, and all of the terms in $m_\mu(X_n)$ have total degree $|\mu|$, so all of the terms in their product have degree $|\lambda| + |\mu|$. This means we can assume ν is a partition of $|\lambda| + |\mu|$.

We would like to say $a^\nu_{\lambda,\mu}(n)$ is independent of n for all $n \geq 1$, but this is not quite true. In particular, if $n = l(\lambda) + l(\mu)$, then one term in $m_\lambda(X_n)$ is $x_1^{\lambda_1} \cdots x_{l(\lambda)}^{\lambda_{l(\lambda)}}$ and one term in $m_\mu(X_n)$ is $x_{l(\lambda)+1}^{\mu_1} \cdots x_{l(\lambda)+l(\mu)}^{\mu_{l(\mu)}}$. This means one term in their product is

$$x_1^{\lambda_1} \cdots x_{l(\lambda)}^{\lambda_{l(\lambda)}} x_{l(\lambda)+1}^{\mu_1} \cdots x_{l(\lambda)+l(\mu)}^{\mu_{l(\mu)}}.$$

Therefore, if we write $\lambda \cup \mu$ to denote the partition we obtain by sorting $\lambda_1, \ldots, \lambda_{l(\lambda)}, \mu_1, \ldots, \mu_{l(\mu)}$ into weakly decreasing order, then we see $a^{\lambda\cup\mu}_{\lambda,\mu}(l(\lambda)+l(\mu)) \neq 0$, even though $a^{\lambda\cup\mu}_{\lambda,\mu}(l(\lambda)+l(\mu)-1) = 0$. In other words, the best we can hope for is that $a^{\nu}_{\lambda,\mu}(n)$ is independent of n for all $n \geq l(\lambda)+l(\mu)$.

Corollary 1.14. *For any partitions λ, μ, and $\nu \vdash |\lambda| + |\mu|$ and any $n \geq 1$, let the numbers $a^{\nu}_{\lambda,\mu}(n)$ be defined by*

$$m_\lambda(X_n)m_\mu(X_n) = \sum_\nu a^{\nu}_{\lambda,\mu}(n)m_\nu(X_n).$$

If $n \geq l(\lambda)+l(\mu)$, then $a^{\nu}_{\lambda,\mu}(n)$ does not depend on n.

Proof. Suppose $n = l(\lambda)+l(\mu)$ and fix $j \geq 1$. By definition,

$$m_\lambda(X_{n+j})m_\mu(X_{n+j}) = \sum_\nu a^{\nu}_{\lambda,\mu}(n+j)m_\nu(X_{n+j}).$$

If we set $x_{n+1} = x_{n+2} = \cdots = x_{n+j} = 0$ and use Proposition 1.13, then we find

$$m_\lambda(X_n)m_\mu(X_n) = \sum_\nu a^{\nu}_{\lambda,\mu}(n+j)m_\nu(X_n).$$

Comparing this last line with equation (1.1) and using the fact that the monomial symmetric polynomials are a basis for $\Lambda(X_n)$, we see $a^{\nu}_{\lambda,\mu}(n+j) = a^{\nu}_{\lambda,\mu}(n)$, which is what we wanted to prove. □

As we mentioned above, the moral of Corollary 1.14 is that if we have enough variables (that is, if n is large enough), then the algebraic properties of $\Lambda_k(X_n)$ and our basis of monomial symmetric polynomials do not depend on exactly how many variables we have. However, we also saw that "enough" means different things in different contexts. If we only cared about the product $m_{11}(X_n)m_{21}(X_n)$, then enough would be four. But if we are actually interested in the product $m_{4321}(X_n)m_{77221}(X_n)$, then enough is nine.

Instead of worrying about how much is enough in every new situation, we would like to just assume we have infinitely many variables. To do this, we need to lay some formal groundwork involving "polynomials" which are allowed to have infinitely many terms. We start with a formal definition.

Definition 1.15. Let $\mathbb{N}$ be the set of nonnegative integers and set $\mathbb{N}^\infty = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots$. A *formal power series with coefficients in* $\mathbb{Q}$ is a function $f : \mathbb{N}^\infty \to \mathbb{Q}$ such that if $f(a_1, a_2, \ldots) \neq 0$, then only finitely many of $a_1, a_2, \ldots$ are nonzero.

At first glance a formal power series does not seem to be related to a polynomial which may have infinitely many terms. But we connect these ideas by identifying each tuple $(a_1, a_2, \ldots)$ with the monomial $x_1^{a_1} x_2^{a_2} \cdots$, viewing $f(a_1, a_2, \ldots)$ as the coefficient of $x_1^{a_1} x_2^{a_2} \cdots$, and identifying the function f with the sum

$$\sum_{(a_1, a_2, \ldots) \in \mathbb{N}^\infty} f(a_1, a_2, \ldots) x_1^{a_1} x_2^{a_2} \cdots. \tag{1.2}$$

For example, if f is given by

$$f(a_1, a_2, \ldots) = \begin{cases} a_1 & \text{if } a_2 = a_3 = \cdots = 0, \\ 0 & \text{otherwise,} \end{cases}$$

then we identify f with the series $x_1 + 2x_1^2 + 3x_1^3 + \cdots = \sum_{j=1}^\infty j x_1^j$. For a formal power series f, the sum in (1.2) is "formal" in the sense that we generally do not assign it meaning in the usual sense of addition.

A formal power series can have finitely many terms or infinitely many terms, so every polynomial is also a formal power series. In fact, formal power series also inherit some terminology from polynomials. For instance, the *total degree* (or *degree*, for short) of a term $f(a_1, a_2, \ldots) x_1^{a_1} x_2^{a_2} \cdots$ is $a_1 + a_2 \cdots$, which is finite by the last condition in Definition 1.15. We say a formal power series is *homogeneous of degree* k whenever each of its terms has total degree k, and we note that every formal power series f can be written uniquely as a sum $f = f_0 + f_1 + \cdots$, where f_k is homogeneous of degree k for all $k \geq 0$.

In some contexts it is useful to consider convergence properties of formal power series, since this opens up the possibility of using tools from complex analysis to draw conclusions about the coefficients in a given series. However, we will not be concerned with questions of convergence. Instead, it is the algebra of formal power series which will be of most interest to us. For instance, we can add formal power series and multiply them by scalars, just as we do for polynomials, so the set of formal power series in X is a vector space over $\mathbb{Q}$. In fact, as the next two examples suggest, we can also multiply formal power series.

Example 1.16. Compute the product of the formal power series $f(x) = \sum_{j=0}^{\infty} x^j$ and $g(x) = \sum_{j=0}^{\infty} jx^j$.

Solution. To express this product as a formal power series, we need to determine, for each j, the coefficient of x^j in the product

$$(1 + x + x^2 + x^3 + \cdots)(x + 2x^2 + 3x^3 + \cdots).$$

Since every term in the second factor has a factor of x, the constant term in $f(x)g(x)$ will be 0. Before we combine like terms, the terms in $f(x)g(x)$ are exactly the products of one term from $f(x)$ and one term from $g(x)$. The only way such a term can have the form ax is if we choose 1 from $f(x)$ and x from $g(x)$, so the coefficient of x is 1. There are two ways such a term can have the form ax^2: we can choose 1 from $f(x)$ and $2x^2$ from $g(x)$, or we can choose x from $f(x)$ and x from $g(x)$. Therefore the coefficient of x^2 in $f(x)g(x)$ is $2 + 1 = 3$. In general, there are j ways a term in $f(x)g(x)$ can have the form ax^j: for each m with $1 \leq m \leq j$, we can choose x^{m-1} from $f(x)$ and $(j - m + 1)x^{j-m+1}$ from $g(x)$. Therefore the coefficient of x^j in $f(x)g(x)$ is $1 + 2 + \cdots + j = \binom{j+1}{2}$, and we have

$$f(x)g(x) = \sum_{j=1}^{\infty} \binom{j+1}{2} x^j. \qquad \square$$

Example 1.17. Compute the product of the formal power series $f = \sum_{j=1}^{\infty} x_j x_{j+1}$ and $g = \sum_{j=1}^{\infty} x_j$.

Solution. Working formally, we could write fg as

$$fg = \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} x_j x_{j+1} x_m.$$

Although this expresses fg correctly, some terms appear more than once on the right side, so it gives us less insight into fg as a formal power series than we would like. To gain more insight, we need to determine the coefficient of every monomial in $x_1, x_2, \ldots$ in the product

$$(x_1x_2 + x_2x_3 + x_3x_4 + \cdots)(x_1 + x_2 + x_3 + \cdots).$$

Before we combine like terms, the terms in fg are exactly the products of one term from f and one term from g. As a result, the only terms in fg with nonzero coefficients are those of the form $x_m x_j x_{j+1}$, where $m < j-1$ or $m > j+2$, those of the form $x_j^2 x_{j+1}$, those of the form $x_j x_{j+1}^2$, and those of the form $x_j x_{j+1} x_{j+2}$. We can only obtain terms of the first form in one way: choose $x_j x_{j+1}$ from f and x_m from g. Therefore, each of these terms has coefficient 1. Similarly, we can only obtain terms of the second and third forms in one way, so each of these also has coefficient 1. But we can obtain a term of the form $x_j x_{j+1} x_{j+2}$ in two ways: choose $x_j x_{j+1}$ from f and x_{j+2} from g, or choose $x_{j+1} x_{j+2}$ from f and x_j from g. Therefore, each of these terms has coefficient 2. Combining these observations, we can express fg as

$$fg = \sum_{j=1}^{\infty} \sum_{\substack{m=1 \\ m \neq j-1 \\ m \neq j+2}}^{\infty} x_j x_{j+1} x_m + 2 \sum_{j=1}^{\infty} x_j x_{j+1} x_{j+2}. \qquad \square$$

In each of these examples we have a product in which each factor has infinitely many terms. However, there are only finitely many ways to obtain any particular monomial in the result, so it is still possible to express each product as a formal power series. Moreover, this is true in general. That is, if X is any set of variables, then any monomial in the variables in X can be expressed as a product of monomials in only finitely many ways. Therefore, for any formal power series f and g in X, the product fg is a well-defined formal power series.

Now that we have our usual algebraic operations on formal power series, we would like to extend the action of permutations on polynomials to an action of permutations on formal power series. There are several ways one could do this, but we will use one that is particularly simple: if $\pi \in S_n$ and we have variables $x_1, x_2, \ldots$, then we will define $\pi(x_j) = x_{\pi(j)}$ as usual for $1 \leq j \leq n$, and we will set $\pi(x_j) = x_j$ for $j > n$. Now for any formal power series $f = f(x_1, x_2, \ldots)$ and any permutation $\pi \in S_n$, we define $\pi(f)$ by $\pi(f) := f(\pi(x_1), \pi(x_2), \ldots)$. With this definition, we have the following natural analogue of Proposition 1.2.

Proposition 1.18. *Suppose $X = \{x_j\}_{j=1}^{\infty}$ is a set of variables, f and g are formal power series in X, c is a constant, and $\pi, \sigma \in S_n$ are permutations. Then*

(i) $\pi(cf) = c\pi(f)$;

(ii) $\pi(f+g) = \pi(f) + \pi(g)$;

(iii) $\pi(fg) = \pi(f)\pi(g)$;

(iv) $(\pi\sigma)(f) = \pi(\sigma(f))$.

Proof. This is similar to the proof of Proposition 1.2. □

Now that we know how a permutation acts on a formal power series, we are ready to discuss symmetric functions.

Definition 1.19. Suppose $X = \{x_j\}_{j=1}^{\infty}$ is a set of variables and f is a formal power series in X. We say f is a *symmetric function in* X whenever, for all $n \geq 1$ and all $\pi \in S_n$, we have $\pi(f) = f$. We write $\Lambda(X)$ to denote the set of all symmetric functions in X, and we write $\Lambda_k(X)$ to denote the set of all symmetric functions in X that are homogeneous of degree k. Often X is clear from context, in which case we say f is a *symmetric function*, we write Λ to denote the set of all symmetric functions, and we write Λ_k to denote the set of all symmetric functions which are homogeneous of degree k.

When we had finitely many variables, we constructed the monomial symmetric functions by adding all of the distinct images of a given monomial. We can do the same thing when we have infinitely many variables, but this time we get a formal power series, which is not a polynomial in general.

Definition 1.20. Suppose λ is a partition and $X = \{x_j\}_{j=1}^{\infty}$ is a set of variables. Then the *monomial symmetric function in* X, which we write as $m_\lambda(X)$, is the sum of all monomials Y for which there exists $n \geq l(\lambda)$ and a permutation $\pi \in S_n$ such that $\pi(Y) = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$.

When the set X of variables is clear from context, we often omit it, writing m_λ instead of $m_\lambda(X)$. Similarly, if we write m_λ and no set of variables has been indicated, then we will assume our set of variables is $X = \{x_j\}_{j=1}^{\infty}$.

Example 1.21. Compute m_{21}. Is $m_{21} = m_2 m_1$?

Solution. By definition m_{21} is the sum of the images of $x_1^2 x_2$ under all permutations of $x_1, x_2, \ldots$, so it is the sum of all monomials of the form $x_j^2 x_k$, where j and k are distinct. Formally, we have

$$m_{21} = \sum_{j=1}^{\infty} \sum_{\substack{k=1 \\ k \neq j}}^{\infty} x_j^2 x_k.$$

The product $m_2 m_1 = (x_1^2 + x_2^2 + \cdots)(x_1 + x_2 + \cdots)$ includes terms of the form x_j^3, so it is not equal to m_{21}. However, we do have

$$m_{21} = (x_1^2 + x_2^2 + \cdots)(x_1 + x_2 + \cdots) - (x_1^3 + x_2^3 + \cdots),$$

so $m_{21} = m_2 m_1 - m_3$. □

As we might expect, the monomial symmetric functions form a basis for Λ_k, as their polynomial counterparts did for $\Lambda_k(X_n)$.

Proposition 1.22. *For all $k \geq 0$, the set $\{m_\lambda \mid \lambda \vdash k\}$ of monomial symmetric functions is a basis for Λ_k. In particular,* $\dim \Lambda_k = p(k)$*, the number of partitions of k.*

Proof. To show $\{m_\lambda \mid \lambda \vdash k\}$ is linearly independent, first note that if we set $x_j = 0$ in m_λ for all $j > n$, then we obtain the monomial symmetric polynomial $m_\lambda(X_n)$. Now suppose we have constants a_λ for which $\sum_{\lambda \vdash k} a_\lambda m_\lambda = 0$. If we set $x_j = 0$ for all $j > k$, then we find $\sum_{\lambda \vdash k} a_\lambda m_\lambda(X_k) = 0$. But $\{m_\lambda(X_k) \mid \lambda \vdash k\}$ is linearly independent by Proposition 1.12, so $a_\lambda = 0$ for all λ, as desired.

To show $\{m_\lambda \mid \lambda \vdash k\}$ spans Λ_k, suppose $f \in \Lambda_k$. We argue by induction on the number of terms in f of the form $a x_1^{a_1} x_2^{a_2} \cdots$, where a is a nonzero constant and $a_1 \geq a_2 \geq \cdots$. Note that for any such term the sequence $a_1, a_2, \ldots$ is a partition of k, so there are at most $p(k)$ such terms. If f has exactly one term of the given form, then all of its images under any permutation are also terms in f, and we have $f = a m_\lambda$, where $\lambda = a_1, a_2, \ldots$. Now suppose f has more than one term of the given form. For any such term $a x_1^{a_1} x_2^{a_2} \cdots$, all of the images of this term must also be terms in f. Therefore, $f - a m_\mu \in \Lambda_k$, where $\mu = a_1, a_2, \ldots$. Moreover, $f - a m_\mu$ has fewer terms of the given form than f, so by induction $f - a m_\mu$ is a linear combination of the elements of $\{m_\lambda \mid \lambda \vdash k\}$. Now the result follows. □

1.4. Problems

1.1. Find all permutations $\pi \in S_4$ for which $\pi(f) = f$, where $f(X_4) = x_1 x_2^2 x_4^4 + x_2^2 x_3^4 x_4 + x_1^4 x_2^2 x_3$.

1.2. Find all permutations $\pi \in S_4$ for which $\pi(f) = f$, where $f(X_4) = x_1 x_2 x_3^2 x_4^2 + x_1 x_2^2 x_3 x_4^2 + x_1^2 x_2 x_3^2 x_4 + x_1^2 x_2^2 x_3 x_4$.

1.3. Find a polynomial f in x_1, x_2, x_3 which has $\pi(f) = \text{sgn}(\pi) f$ for all $\pi \in S_3$, and which has $x_1^7 x_3^2$ as one of its terms.

1.4. For any set P of polynomials in $x_1, \ldots, x_n$, let $S_n(P)$ be the set of permutations $\pi \in S_n$ such that $\pi(f) = f$ for all $f \in P$. Prove that for every P, the following hold.
   (a) $S_n(P)$ is nonempty.
   (b) $S_n(P)$ is closed under multiplication of permutations. That is, if $\pi, \sigma \in S_n(P)$, then $\pi\sigma \in S_n(P)$.
   (c) $S_n(P)$ is closed under taking inverses. That is, if $\pi \in S_n(P)$, then $\pi^{-1} \in S_n(P)$.

1.5. Prove or disprove: if P is a set of polynomials in $x_1, \ldots, x_n$, then for any permutation $\pi \in S_n$ and any $\sigma \in S_n(P)$, we have $\pi\sigma\pi^{-1} \in S_n(P)$.

1.6. For any set $T \subseteq S_n$, let $\text{Fix}(T)$ be the set of polynomials $f(X_n)$ such that $\tau(f) = f$ for all $\tau \in T$. Prove that for every T, the following hold.
   (a) $\text{Fix}(T)$ is a subspace of the vector space of polynomials in $x_1, \ldots, x_n$ with coefficients in $\mathbb{Q}$.
   (b) $\text{Fix}(T)$ is closed under multiplication. That is, if $f \in \text{Fix}(T)$ and $g \in \text{Fix}(T)$, then $fg \in \text{Fix}(T)$.

1.7. Prove or disprove: if $T \subseteq S_n$, then for any polynomial $f \in \text{Fix}(T)$ and any polynomial $g(X_n)$, we have $fg \in \text{Fix}(T)$.

1.8. Prove, or disprove and salvage: for any $n \geq 1$ and any set $T \subseteq S_n$, we have $T = S_n(\text{Fix}(T))$.

1.9. Prove, or disprove and salvage: for any $n \geq 1$ and any set P of polynomials in $x_1, \ldots, x_n$, we have $P = \text{Fix}(S_n(P))$.

1.10. For any $n \geq 1$, any $k \geq 0$, and any $T \subseteq S_n$, let $\text{Fix}_k(T)$ be the set of polynomials in $\text{Fix}(T)$ which are homogeneous of degree

k. Show $\mathrm{Fix}_k(T)$ is a subspace of the space of all homogeneous polynomials of degree k in $x_1, \ldots, x_n$ with coefficients in $\mathbb{Q}$.

1.11. For $n \geq 1$, let C_n be the set containing just the permutation $(1, 2, \ldots, n)$. Find and prove a formula for $\dim \mathrm{Fix}_2(C_n)$.

1.12. Show that the space $\Lambda(X_n)$ is infinite dimensional for all $n \geq 1$.

1.13. Show that for all $n, k \geq 0$, the space $\Lambda_k(X_n)$ is a subspace of $\Lambda(X_n)$.

1.14. We showed in Proposition 1.12 that if $n \geq k$, then $\Lambda_k(X_n)$ is finite-dimensional. Show $\Lambda_k(X_n)$ is also finite dimensional when $0 \leq k < n$. More specifically, show $\dim \Lambda_k(X_n)$ is the number of partitions of k with at most n parts by finding a basis whose elements are indexed by these partitions.

1.15. Show that for all $n, k_1, k_2 \geq 0$, if $f \in \Lambda_{k_1}(X_n)$ and $g \in \Lambda_{k_2}(X_n)$, then $fg \in \Lambda_{k_1+k_2}(X_n)$.

1.16. Suppose $n \geq 1$, $k_1 \geq k_2 \geq 0$, and $k_1 + k_2 = n$. Write the product $m_{k_1} m_{k_2}$ as a linear combination of $\{m_\lambda \mid \lambda \vdash n\}$.

1.17. Write the product $m_4 m_3 m_2 m_1$ as a linear combination of $\{m_\lambda \mid \lambda \vdash 10\}$.

1.18. If we write the product $m_6 m_5 m_4 m_3 m_2 m_1$ as a linear combination of $\{m_\lambda \mid \lambda \vdash 21\}$, what is the coefficient of $m_{(12,9)}$?

1.19. Suppose $n \geq 1$ and $k_1 + k_2 = n$. Write the product $m_{1^{k_1}} m_{k_2}$ as a linear combination of $\{m_\lambda \mid \lambda \vdash n\}$.

1.20. If we write the product $m_{1^{k_1+1}} m_{1^{k_2+1}}$ as a linear combination of monomial symmetric functions, what is the coefficient of $m_{2,1^{k_1+k_2}}$?

1.21. Write $\prod_{j=1}^{\infty}(1 + x_j)$ and $\prod_{j=1}^{\infty}(1 - x_j)$ in terms of the monomial symmetric functions.

1.22. Write the sum

$$\sum_{\lambda} m_\lambda$$

as an infinite product in as simple a form as possible. Here the sum is over all partitions.

1.5. Notes

The background on formal power series we have developed here will be enough for our work with symmetric functions. However, more information is available in a variety of sources, including [**Loe17**, Ch. 7], [**Niv69**], and [**Wil05**, Ch. 2].