Preface

In 2013 Park City Math Institute (PCMI) ran a summer program on geometric analysis, and I was invited to give a course at an undergraduate level. I agreed to teach a course introducing students to mathematical general relativity. While the mathematical foundation of general relativity is in differential and (semi-)Riemannian geometry, an active researcher in the field regularly uses techniques from geometric analysis. It is for this reason that the class started with a treatment of Riemannian metrics, tensors and curvature, and ended with a brief introduction into geometric analysis. The overall strategy of the course was to explain the concept of curvature via the Jacobi equation which, through discussion of tidal forces, further helped motivate the Einstein field equations. The book you have in your hands is a modified and expanded version of my PCMI lecture notes. Since 2013 these materials were used extensively by students at Lewis & Clark College.

In the Fall of 2015, on the occasion of general relativity's 100th birthday, I taught a semester-long class at Lewis & Clark College based on the PCMI lecture notes. The class included extra material on general relativity, and omitted the geometric analysis portion of the lecture notes. The prerequisites for the class were sophomore-level Multivariable and Vector Calculus and sophomore-level Linear Algebra. It was recommended, but not required, that students had also taken an upper-level (preferably proof-based) class. The class attracted both physics and mathematics majors. Although a small amount of physics was discussed there were no physics prerequisites. The course was primarily mathematical and did not go into any up-to-date astrophysical information. It ended in addressing the geometry of the Schwarzschild body (black hole). The expansion of the general relativity chapter in this book is based on the material developed for the 2015 course. The geometric analysis chapter of this book is also changed from the original set of PCMI lecture notes. The chapter is now based on analysis techniques I taught those of the Lewis & Clark College students who pursued independent research as undergraduates. Two of their theorems are also included in the chapter.

Courses in differential and/or Riemannian geometry are usually associated with a high upfront cost of prerequisites. A typical undergraduate route departs very little from Gauss's original "Disquisitiones generales circa superficies curvas" ("General Investigations of Curved Surfaces") as it rests on the theory of curves and surfaces in $\mathbb{R}^3$. In contrast, these lecture notes begin with an analysis of one of the central paragraphs of Riemann's Habilitation lecture "Über die Hypothesen, welche der Geometrie zu Grunde liegen" ("On Hypotheses Which Lie at the Bases of Geometry"). As a matter of principle, geometry and curvature in these lecture notes are treated exclusively from the intrinsic standpoint.

Most of us have been taught Riemannian geometry only after we have learned the basics of differential topology. This is a demanding approach because concepts such as tangent and tensor bundles are quite mathematically sophisticated. I have made a deliberate choice to bluntly avoid differential topology altogether. As a result, one will find neither a definition of a manifold nor a definition of tangent vector in these lecture notes. Instead, the narrative relies exclusively on a reader's familiarity with curvilinear coordinatization from Multivariable Calculus. Connections and Christoffel symbols can be explained to anyone fluent in using polar or spherical coordinates in Euclidean space. Results such as asymptotic formulae for volumes of small balls in arbitrary dimension are local in nature and convey much about concepts of curvature without relying on the exact definition of a tangent vector.

I am certain that my choice to avoid differential topology will make some readers see my notes as “lacking mathematical rigor”. I now take a moment to write to such readers.

Over the many years of teaching at a small liberal arts school I came to realize just how non-linear the process of learning mathematics is. The practice of organizing information in a deductive manner, which as mathematicians we are committed to, is not always conducive to learning mathematics and developing intuition. Throughout history people discovered new mathematics without having a complete understanding of subtleties involved in the mathematical foundations underneath. An approach to teaching and learning mathematics which relies on several passes through a subject, each at a higher level of mathematical rigor, is in a sense historically proven to be pedagogically optimal. My personal experience is that of working with students who very rarely start out as “being a part of the math choir”. I dedicate my lecture notes to all those oddball students because it was they who retaught me mathematics.