Foreword

Mathematicians are happy people, as they spend their lives working on things they love. Professor Elon Lima used to joke that his friend Manfredo do Carmo had become a geometer in order to continue making drawings, which he loved. The authors of "Differential Geometry of Plane Curves", my good friends Hilário Alencar, Walcy Santos, and Gregório Silva Neto, belong to this school. They chose for their work one of the most enchanting subjects in Geometry, both deep and elementary at the same time. It is difficult to think of a more suitable topic to "sharpen the reader's mathematical intuition", as the authors propose to do. Throughout these pages we are introduced to fundamental concepts of Geometry, such as curvature and Frenet frame, and to numerous famous examples. We ponder over "windows" to other areas of Mathematics, such as topology (winding numbers, closed curve theorem) and calculus of variations (isoperimetric inequality). Ideas that seem simple, such as the curve-shortening flow, reveal their richness: they pave the way for constructing sophisticated instruments in contemporary Geometry. In the way of presenting the content, the authors demonstrate how much they have learnt from Manfredo, who was a master of the art of clarifying doubts even before they arise. "Differential Geometry of Plane Curves" is a must-read for anyone who likes math, and even more so for anyone who wants to get into it.

Marcelo Viana

Preface

In this book, we present some geometrical and topological properties of plane curves. Very often, the topological aspects of plane curves have generalizations to higher dimensions. The choice to explore plane curves is due to the fact that in this case, many interesting and inspiring results for differential geometry can be presented in an elementary form. By "elementary" we mean that the prerequisites necessary for understanding this book boil down to a good course in calculus and analytic geometry. In some situations the reader is required to have some knowledge of analysis and differential equations, but nothing that is not intuitive.

The fact that the concepts involved are elementary does not in any way lead to trivial results or simple proofs. In fact, many results, due to the complexity of their proofs, are not proven in undergraduate courses. A typical example is Jordan's theorem for simple closed curves in the plane, which says that such a curve separates the plane into two subsets, exactly one of which is bounded, that share a common boundary which is the image of the curve. This result is perhaps the best example of a statement that we can easily believe in, but whose proof is by no means simple.

The choice of topics covered is based on an attempt to sharpen the reader's mathematical intuition for various geometric concepts and results. The reader will be encouraged to ponder over questions like the relation between the notions of convexity and curvature, the effect of topology on the behavior of the tangent vector of a curve, isoperimetric inequalities, the four-vertex theorem, which studies the curvature function of a closed curve, and some results on the curve-shortening flow.

Of course, there already exist excellent books on differential geometry – for example, the books by Manfredo do Carmo (see [**16**]), Sebastián Montiel and Antonio Ros (see [**51**]), and Wolfgang Kühnel (see [**46**]) – which address the study of curves in Euclidean space. However, our purpose is to restrict ourselves to the plane and study this special case with much more depth.

Background information

This book had a first version that served as the basis for a short course on the Geometry of Plane Curves, presented at the XII School of Differential Geometry, held at the Universidade Federal de Goiás in July 2002. It was expanded and revised to adapt it to a short course on Differential Geometry of Plane Curves given during the 24th Brazilian Mathematics Colloquium. Afterwards, a further revised version was used as teaching material for the course Introduction to Plane Curves at the XV School of Differential Geometry in honor of Manfredo do Carmo's 80th birthday.

The current book has been modified quite a bit from the previous versions: we have added several proofs to provide better clarity for the reader; we have introduced new results – including a chapter on the curve-shortening flow, exercises with corresponding answers, and examples.

This text is indeed an improved and updated English version of our earlier Portuguese book *Geometria Diferencial das Curvas no* $\mathbb{R}^2$, published by the Brazilian Mathematical Society in 2020.

Structure of the book

Chapter 1. We begin by studying curves locally and present the behavior of a differentiable curve in a neighborhood of a point belonging to the image. Here, we explore the concept of curvature of a plane curve, showing that it determines the curve, up to its position in the plane.

Chapter 2. We study continuous plane curves globally. We introduce the notion of the winding number of a curve and present several applications of this concept, such as the fundamental theorem of algebra and some results of complex analysis.

Chapter 3. We study the rotation index of a differentiable curve, defined as the winding number of the curve described by its unit tangent vector. In this context, the turning tangents theorem is the most important result presented.

Chapter 4. We prove Jordan's theorem for regular curves of class $\mathcal{C}^2$.

Chapter 5. We discuss the isoperimetric inequality for closed curves in the plane. This classical result gives us an estimate of the area bounded by a simple closed curve with a fixed perimeter.

Chapter 6. We study convex curves in the plane. In addition to the geometric properties of such curves, we give an introduction to constant-width curves and an isoperimetric inequality for closed convex curves.

Chapter 7. We introduce the conditions necessary to prove one of the most famous classical results of the global geometry of plane curves: the four-vertex theorem.

Chapter 8. We give an introduction to the curve-shortening flow and some recent results regarding it.

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