Preface

This book is an outgrowth of lectures that I gave in the summer of 2020 as part of the Research Experiences for Undergraduates (REU) at the University of Chicago. The REU lectures are not intended to be standard courses but rather tastes of graduate and research level mathematics for advanced undergraduates. The title of the book can be interpreted in two ways. First, this is not a comprehensive survey of an area but rather a "random" sampling of some objects that arise in models in probability and statistical mechanics. The second meaning refers to a prevailing theme in many of these models. Random fields can be studied by exploration, that is, by traveling (perhaps randomly) through the field and observing what one has seen so far and using that to predict the parts that have not been observed.

In order to keep the material accessible to students who have not had graduate material, I have concentrated on discrete models where "measure theoretic" probability is not needed. The formal prerequisites for these notes are advanced calculus, linear algebra, and a calculus-based course in probability. It is also expected that students have sufficient mathematical maturity to understand rigorous arguments. While those are the only formal prerequisites, the intent of these lectures was to give a taste of research level mathematics and I allow myself to venture occasionally a bit beyond these prerequisites.

The first chapter introduces Markov chains and ideas that permeate the book. The focus is on transient chains or recurrent chains with "killing" for which there is a finite Green's function representing the expected number of visits to sites. The Green's function can be seen to be the inverse of an important operator, the Laplacian. Harmonic functions (functions whose Laplacian equal zero) and the determinant of the Laplacian figure prominently in the later chapters. We concentrate mainly on discrete time chains but we also discuss how to get continuous time chains by putting on exponential waiting times. A probabilistic approach dominates our treatment but much can be done purely from a linear algebra perspective. The latter approach allows measures on paths that take negative and complex values. Such path measures come up naturally in a number of models in mathematical physics although they are not emphasized much here.

Chapter 2 introduces an object that has been a regular part of my research. I introduced the loop-erased random walk (LERW) in my doctoral dissertation with the hope of getting a better understanding of a very challenging problem, the self-avoiding random walk (SAW). While the differences between the LERW and SAW have prevented the former from being a tool to solve the latter problem, it has proved to be a very interesting model in itself. One very important application is the relation between LERW and another model, the uniform spanning tree (UST). This relationship is most easily seen in an algorithm due to David Wilson [**20**] to generate such trees.

Analysis of the loop-erasing procedure leads to consideration both of the loops erased and the LERW itself. Chapter 3 gives an introduction to loop measures and soups that arise from this. We view a collection of loops as a random field that is growing with time as loops are added. The distribution of the loops at time 1 corresponds to what is erased from loop-erased random walks. The loop soup at time 1/2 is related to the Gaussian free field (GFF). This chapter introduces the discrete time loop soup which is an interesting mathematical model in itself. This discrete model has characteristics of a number of fields in statistical mechanics. In particular, the distribution of the field does not depend on how one orders the elements, but to investigate the field one can order the sites and then investigate the field one site at a time. For this model, when one visits a site, one sees all the loops that visits that site. This "growing loop" model which depends on the order of the vertices turns out to be equivalent to an "unrooted loop soup" that does not depend on the order.

While we have used the generality of Markov chains for our set-up, one of the most important chains is the simple random walk in the integer lattice. In order to appreciate paths and fields arising from random walk, it is necessary to understand the walk. Chapter 4 discusses the simple random walk on the lattice giving some more classical results that go beyond what one would normally see at an undergraduate level.

We return to the spanning tree in Chapter 5 and consider the infinite spanning tree in the integer lattice as a limit of spanning trees on finite subsets. Whether or not this gives an infinite tree or a forest (a collection of disconnected trees) depends on the dimension. We also give an example of duality on the integer lattice.

Another classical field is the topic of Chapter 6. The multivariate normal distribution is a well known construction and is the model upon which much of classical mathematical statistics, such as linear regression, is based. The (GFF) is an example of such a distribution where some geometry comes into the picture. Here we discuss the GFF coming from a Markov chain. The idea of exploration comes in again as one "samples" or "explores" the field at some sites and uses that to determine distributions at other sites. The global object is independent of the ordering of the vertices but the sampling rule is not. There is a relation between the GFF and the growing loop defined in Chapter 3 discussed in Section 6.6.

In Chapter 7 we introduce some of the continuous models that arise as scaling limits. A proper treatment of this material would require more mathematical background than I am assuming so this should be viewed as an enticement to learn more. The scaling limits we discuss are: Brownian motion, Brownian loop soup, Schramm-Loewner evolution, and the continuous GFF.

In the Appendix, we discuss a couple of topics that arise in the previous chapters but have sufficient independent interest that it seems appropriate to separate them. The first is a basic technique for research probabilists often called the "second moment method". The second, which arises for us primarily in the analysis of the loop models, is an introduction to Lévy processes with an emphasis on the negative binomial and Gamma processes.

There are a number of exercises scattered through the text. It is recommended that the serious reader, that is, those who are considering doing research in this or related areas of mathematics, do as many as possible. I also suggest to be prepared to draw pictures to help understand some of the constructions and the arguments. Of course, the more casual reader can do whatever they please!

I have focused on the mathematics in these lectures and have not discussed the history of the development of these ideas. Clearly, the mathematics in this book is the work of many researchers including many who are active today. I have included a few references for further reading. Many of these works also have extensive bibliographies which can be a good source of original articles.

All of the chapters depend on the material in Chapter 1. Chapter 3 uses Chapter 2 while Chapters 4 and 6 are independent and need only Chapter 1. Chapter 5 uses all then chapters preceding it.

I would like to thank the "random fields" group during the 2020 REU: Nixia Chen, Victor Gardner, Jinwoo Sung, Stephen Yearwood (mentors) and Sam Craig, Nitya Gadhiwala, Jingyi Jia, Ethan Lewis, Mishal Mrinal, Ethan Naegele, Vedant Pathak, Sivakorn Sanguamoo, Rachit Surana, Haozhe Yu, Lingyue Yu, Stanley Zhu (participants), as well as some participants from other groups that interacted with our group: Jake Fiedler, Jessica Metzger, Lucca Prado, Ben Rapport. Among others who commented and sent corrections on early drafts of the notes are Charley Devlin, Vladislav Guskov, Seyhun Ji, Fredrik Viklund, and Zijian Wang.

Research was supported by National Science Foundation grant DMS-1513036.