Preface

The emergence of differential topology and dynamical systems can be traced back to the work of Poincaré on *analysis situs* at the dawn of the 20th century. Poincaré's recognition that the existence and form of solutions of differential equations were intimately connected with the "topology" of the space where the equations found their natural definition led to new ideas in analysis and the development of previously vague notions such as that of "manifold". After these concepts crystalized somewhat, an immediate basic problem was (and still is) to relate the complexity of flows to the topological complexity of the underlying manifold. In this context, a first step was to estimate the number of invariant (or rest) points for the particular case of gradient flows or, equivalently, to estimate the minimal number of critical points of functions on the manifold. Morse's work in the late 20's and early 30's led to such estimates for particular generic functions: those whose critical points were non-degenerate.

Around the same time, L. Lusternik and L. Schnirelmann ([**LS34**]) described a new invariant of a manifold called *category*. Their aim in creating this notion was to provide a lower bound on the number of critical points for *any* smooth function on the manifold. While this aim was analytical in nature, it had far-reaching consequences in geometry as well. As we shall see later (see Theorem 9.12), the general approach of Lusternik and Schnirelmann can be applied to obtain results such as the existence of a closed geodesic. Indeed, Lusternik and Schnirelmann were able to use their new invariant to prove wonderful results such as the existence of three closed geodesics on the sphere. Furthermore, once reformulated by Fox ([**Fox41a**]), category (or *Lusternik-Schnirelmann category* as it became known) found a useful niche in algebraic topology. For example, the category of a space X was used by G. Whitehead to bound from above the nilpotency class of the group of homotopy classes from X to a group-like space ([**Whi54**]). Thus began a long association of category with the notion of nilpotency.

Category continued to be a tool in critical point theory, but it also became a main focus of the "numerical invariant" movement in homotopy theory in the 50's. After foundational results were obtained in the 1960's, the problem list of T. Ganea ([**Gan71**]) served to motivate further study of category by topologists. The development of localization techniques in topology and, particularly, the creation of Sullivan's version of rational homotopy theory spawned new approximating invariants which energized the field and which led to greater understanding in areas as diverse as the study of the homotopy Lie algebra ([**Fél89**]) and the number of fixed points for certain diffeomorphisms on some types of manifolds (see Theorem 8.28). Recently, new approximating invariants for category have been successfully employed to solve an example of the latter problem called the Arnold conjecture for symplectic manifolds ([**Rud99a**], [**RO99**]). Simultaneously, some of the recent, purely homotopical work on LS-category has also been seen to have direct implications in critical point theory ([**Cor98a**]). Thus, Lusternik-Schnirelmann category has come full circle and once again has found a place in the toolkit of researchers in dynamical systems.

It is also important to emphasize that Lusternik-Schnirelmann category is a living, breathing subject which presently (i.e. as of 2002) is undergoing a startling revival. The reasons for this are many, but suffice it to say that recently we have seen problems from Ganea's list solved (see [**Iwa98**] and [**LSV02**]), stable homotopy theory introduced into the subject (see [**Rud99b**] and [**SST01**]), old homotopical tools such as Hopf invariants re-developed in exciting new ways and the general category framework extended to encompass areas such as the theory of foliations (see [**CMV01**] and [**SV02**]).

Much of the recent work on category is included (or mentioned) in the present book, as well as applications to subjects as varied as, for example, 3-manifold topology and the complexity of algorithms. Morse theory has already been at the center of a good number of high quality surveys and treatises, so its appearance here will be only incidental. This book covers the homotopical side of category in a reasonably complete way, but it is not intended as an exhaustive monograph on the more analytical side of the subject. Rather, this book focusses on three recurring themes that give structure and perspective to a vast territory.

- The nilpotencies of various algebraic objects associated to a space are related to the category of the space.
- Homotopically, Hopf invariants provide the most refined tool available for estimating category.
- Homotopy theoretical properties may be translated into critical point properties and vice versa using appropriate notions of stabilization.

The brief description above hints at the dual nature of our exposition. In this book, we wish to study category, not simply as a homotopy invariant, but as a useful notion in geometry and dynamical systems. Thus, we will speak to rather different audiences of topologists, geometers and dynamicists.

Here is a chapter by chapter description of the main subjects discussed in the book. Exact references (as well as proofs) for the various results mentioned below are contained in the body of the book and so we omit them here.

Chapter 1 introduces the main definitions and basic properties of category. The category, $\mathrm{cat}(X)$, of a topological space X is defined as the least natural number n such that there is a covering of X by $n+1$ open sets, each of which can be contracted to a point inside X. As we shall see, this simple definition is already quite useful. In particular, it may be used to show that if $\mathrm{cat}(X) \leq n$, then the cup-product of more than n cohomology classes in reduced cohomology necessarily vanishes. Thus, the nilpotency aspect of category shows up quite naturally. Moreover, the covering definition was also the one used by Lusternik and Schnirelmann to show that, under suitable assumptions (satisfied by any CW-complex, for example), the minimal number of critical points of a function $f\colon M \to \mathbb{R}$ has $\mathrm{cat}(M)+1$ as a lower bound. However, this same direct definition is not very practical for homotopical computations. Fortunately, there are two other equivalent definitions which allow computation in important cases. The first alternative definition is due to Whitehead and the second, which will play a key role in the book, is due to Ganea. Ganea's description of category is based on the so-called (Ganea) fibre-cofibre construction which, starting from the path-loop fibration $\Omega X \to PX \to X$, produces a series of fibrations

$$F_n(X) \to G_n(X) \xrightarrow{p_n} X \ .$$

Ganea showed, again under appropriate restrictions, that $\mathrm{cat}(X) \leq n$ precisely when the fibre map p_n has a section. It is important to realize that these fibrations are not as special as they might seem at first sight. Under a different name, they are quite familiar objects. Recall that Milnor introduced a method, now called the Milnor classifying construction, to construct the classifying space for G-fibrations where G is any topological monoid. His construction provides a sequence of fibrations $G \to E_n(G) \to B_n(G)$ as well as coherent inclusions

$$t_n(G)\colon B_nG \longrightarrow BG = B_\infty(G) \ .$$

It was noticed some time after Milnor and Ganea introduced their respective sequences of fibrations that, for an appropriate monoid model G_X of ΩX, the maps p_n and $t_n(G_X)$ may be canonically identified up to homotopy (see Exercise 2.16). Therefore, we see that Ganea's fibrations are, in some sense, universal objects. Furthermore, since $BG_X \simeq X$ and $G_n(X) \simeq B_n(G_X)$, the Ganea spaces $G_n(X)$ are better and better approximations to X. LS-category itself is one measure of the faithfulness of these approximations.

As the reader will see, the computation of category is a very difficult task, so Chapter 2 is devoted to defining and calculating more easily computable invariants which serve as lower bounds for category. Many of these invariants are more algebraic in nature than category itself. A prototype is the Toomer invariant, $e(X)$. Toomer initially introduced his invariant using a certain Milnor-Moore spectral sequence, but the existence of the Ganea fibrations provides a simplified description of $e(X)$ as the least n for which the Ganea projection p_n above is surjective in homology. The various algebraic approximations of category are not only easier to compute, but, moreover, their behavior with respect to simple operations involving spaces — for example products — is much easier to understand than for category itself. For example, $e(X \times Y) = e(X) + e(Y)$, but this compatibility with products is most definitely not the case for category. Although it is rather simple to see that $\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y)$, evaluating the error, $\mathrm{cat}(X) + \mathrm{cat}(Y) - \mathrm{cat}(X \times Y)$, is a different matter altogether. Indeed, the apparently innocuous question — raised by Ganea some thirty years ago — of whether $\mathrm{cat}(X \times S^n)$ equals $\mathrm{cat}(X) + 1$ for all X became known subsequently as *the* Ganea conjecture and has only very recently been *disproved* by N. Iwase. The various lower bounds discussed in this chapter reinforce the link between category and nilpotency. For example, we shall see that the notion of weak category measures the nilpotency of the reduced diagonal. Furthermore, in Chapter 2, we shall see the first appearance of stable invariants such as sigma-category and category weight (which is also developed in Chapter 8).

Chapter 3 discusses some upper bounds for category. Geometrically, it is natural to cover the space X with sets which are contractible in themselves, not only within X. By adapting the definition of category to coverings with contractible sets (or, respectively, to coverings with balls if X is a manifold) we obtain some homeomorphism invariants: the geometric category, $\mathrm{gcat}(X)$, and the ball category, $\mathrm{ballcat}(X)$. While these *are* homeomorphism invariants, they *are not* homotopy invariants, so their computation or approximation is even more difficult than that of category. Of course, there is an obvious way to obtain out of $\mathrm{gcat}(-)$ a homotopy invariant: namely, consider the minimum of the values of gcat(X') for all spaces X' having the homotopy type of X. This is the strong category of X, denoted Cat(X). We have $\mathrm{cat}(X) \leq \mathrm{Cat}(X) \leq \mathrm{cat}(X) + 1$ and, therefore, many of the properties of Cat($-$) shed light on those of cat($-$). One such property, which is important in obstruction theory arguments, and which will also be applied in Chapter 7, is that, if $\mathrm{Cat}(X) \leq n$, then the homotopy type of X may be constructed by n cone attachments

$$X \simeq (\ldots(((\Sigma A) \cup C\Sigma A_1) \cup C\Sigma^2 A_2) \ldots) \cup C\Sigma^n A_n.$$

There is a second important method which allows us to get closer to homotopical invariance for invariants like gcat($-$) or ballcat($-$); geometric stabilization. This notion consists of considering the geometric invariants of the product of the initial space X with a sufficiently high-dimensional disk. Under suitable assumptions, for ballcat($-$), we shall see that geometric stabilization gives, for instance,

$$\mathrm{cat}(X) \leq \mathrm{ballcat}(X \times D^k) \leq \mathrm{cat}(X) + 1,$$

for k sufficiently large.

Chapter 4 explores the relationship between category and localization in homotopy theory. Localization of abelian groups and nilpotent spaces is a process that focusses attention on the p-primary information carried by the space (or group) for each prime p separately. For a space X, we denote by $X_{(p)}$ its localization at p. The analogue for an abelian group A is $A_{(p)} = A \otimes Z_{(p)}$, where $Z_{(p)}$ is the ring obtained from the integers by inverting all the primes different from p. Working with each p one at a time simplifies many computations and, ideally, it would be possible to recover cat(X) from knowing cat($X_{(p)}$) for all primes p. However, category shows its "teeth" again here because, as we shall see, under appropriate restrictions, if $m = \max\{\mathrm{cat}(X_{(p)}) : p \text{ prime}\}$, then $\mathrm{cat}(X) \leq 2m$, while, in general, $\mathrm{cat}(X) \neq m$. A general question which then arises is whether two spaces, not of the same homotopy type, but having homotopy equivalent p-localizations for all p, have the same category. Chapter 4 provides some answers in certain cases for this question of the genericity (in the sense of Mislin) of LS-category. There is also a different type of construction — a fibrewise construction with respect to a functor λ — which plays an important role further on in the book. This construction works for any Bousfield localization, $\lambda = L_f$, but we will focus here on a variant of $Q(-) = \Omega^\infty\Sigma^\infty(-)$. This type of construction associates to a fibration such as the Ganea fibration $F_n(X) \to G_n(X) \to X$ a new homotopy fibration $Q(F_n(X)) \to \overline{G_n(X)} \xrightarrow{\overline{p_n}} X$. Further, a section for the original Ganea fibration implies the existence of a section for $\overline{p_n}$, so defining the Q-category of X, Qcat(X), as the minimal n for which $\overline{p_n}$ has a homotopy section, we have $\mathrm{Qcat}(X) \leq \mathrm{cat}(X)$. Homotopically, applying the functor Q to a space is an efficient way to move into the stable homotopy category and, as a consequence, Qcat($-$) is the most efficient homotopical stabilization of category.

Chapter 5 is concerned with results that are specific to yet another type of localization; rationalization. The rationalization of a space is obtained by localizing with respect to zero (that is, inverting all primes). Rational (simply connected or nilpotent) spaces are faithfully modeled by commutative, augmented, differential graded rational algebras. It was discovered in the 70's by Felix and Halperin that the rationalization of the Ganea space $G_n(X)$ (with X a simply connected finite type space) admits an algebraic model $\mathcal{A}$ with the property that the augmentation ideal $\overline{\mathcal{A}}$ satisfies the nilpotency condition $\overline{\mathcal{A}}^{n+1} = 0$. This has led to an efficient description of the category of rational spaces and to a number of remarkable other results. In particular, in the rational world, LS-category is additive with respect to products (so the Ganea conjecture is true in this setting), the strong category of a rational space equals the minimal degree of nilpotency of the algebraic models of X and, for rational Poincaré duality spaces, the strong and usual categories coincide with the rational Toomer invariant.

Chapter 6 centers on the refined computational tool known as the Hopf invariant. In the study of LS-category, this invariant was introduced by Berstein and Hilton. They considered a space $Y = X \cup e^r$ obtained by a cell-attachment from some other space X and used a certain version of the Hopf invariant to compare $\mathrm{cat}(Y)$ to $\mathrm{cat}(X)$. Clearly, from the definition, we have $\mathrm{cat}(Y) \leq \mathrm{cat}(X) + 1$, but Berstein and Hilton proved that if the respective Hopf invariant vanishes, then $\mathrm{cat}(Y) \leq \mathrm{cat}(X)$. In many delicate estimates of category and of its approximations, the key tool turns out to be precisely the Hopf invariant. Moreover, recently, Iwase used Hopf invariant methods to disprove the Ganea conjecture. His approach, which we describe in this chapter together with various other computations, also leads to a better understanding of the relations between LS-category and its homotopical stabilization $\mathrm{Qcat}(-)$. In this chapter, we also consider how counterexamples to Ganea's conjecture may be classified. In particular, we shall see that $\mathrm{Qcat}(X)$ is a tantalizing candidate for an invariant measuring the failure of the Ganea conjecture for a space X. Is the strict inequality $\mathrm{Qcat}(X) < \mathrm{cat}(X)$ equivalent to the failure of Ganea: that is, to the existence of a sphere S^r with $\mathrm{cat}(X \times S^r) = \mathrm{cat}(X)$? In fact, for all computed examples where the space is a manifold M, if $\mathrm{Qcat}(M) \neq \mathrm{cat}(M)$, then $\mathrm{Qcat}(M)+1 = \mathrm{cat}(M)$ and M does *not* verify the Ganea conjecture. Furthermore, the relationship between cat and Qcat goes beyond the homotopical, for we shall see in Chapter 7 that Qcat also figures prominently in critical point estimates.

In Chapter 7, the results and techniques presented earlier in the book come together in the study of the problem of constructing functions with few critical points. Stabilization, this time in a dynamical sense, is again important here and, from this perspective, the key concept is that of *functions quadratic at infinity* on a manifold M. Such a function is defined on the total space of a vector bundle with base space M so that it restricts to a quadratic form along each fibre. Denote by $\widetilde{\mathrm{Crit}}(M)$ the minimal number of critical points of such functions. The existence of the particular cone-decompositions mentioned in the description of Chapter 3 translates into the inequality $\widetilde{\mathrm{Crit}}(M) \leq \mathrm{cat}(M)+2$ (for M simply connected). The stable version of category, $\mathrm{Qcat}(-)$, of Chapter 4, also enters the picture via the inequality, $\mathrm{Qcat}(M)+1 \leq \widetilde{\mathrm{Crit}}(M)$. The convergence of homotopical and geometric stabilizations is emphasized by the fact that these upper and lower bounds for $\widetilde{\mathrm{Crit}}(M)$ are very close in all known examples. Besides being the tool necessary to estimate the difference $\mathrm{cat}(M) - \mathrm{Qcat}(M)$, the Hopf invariants presented in Chapter 6 also directly intervene in the unstable version of the problem. This unstable version focusses on trying to reduce the number of critical points of a fixed function $f\colon M \to \mathbb{R}$. We will see that a certain type of Hopf invariant (due to Ganea and presented earlier in Section 6.7) provides the natural obstruction to fusing together the critical points of f (when f is generic).

Chapter 8 focusses on the role of category in symplectic topology. One of the key homotopical ingredients here comes from Chapter 2 and it is a local variant of the Toomer invariant, called *category weight*. Category weight is associated to a cohomology class $u \in H^*(X; A)$ and we shall see that, when M is a symplectic manifold whose symplectic form ω satisfies $\int_{S^2} \omega = 0$ for all smooth maps $S^2 \to M$, then the category weight of the cohomology class $[\omega]$ is 2 and this then implies that $\mathrm{Qcat}(M) = \mathrm{cat}(M) = \dim(M)$. Furthermore, because, for all manifolds, $\mathrm{Crit}(M) \leq \dim(M) + 1$, we have that the minimal number of critical points for all smooth functions on M, $\mathrm{Crit}(M)$, is equal to $\dim(M) + 1$. Somewhat miraculously, deep results in symplectic topology show that, for exactly this type of symplectic manifold, the number of periodic orbits of a Hamiltonian flow on M is in bijection with the number of rest points of a certain gradient-like flow defined on a compact space X_∞ which maps into M by a map injective in cohomology. The algebraic properties of category weight come in handy here because they allow us to deduce from this cohomological condition that the number of rest points is at least equal to $\dim(M)+1$. In this way, the last step in the proof of a form of the celebrated Arnold conjecture is achieved through the use of category methods. In a slightly different direction, the role of $\widetilde{\mathrm{Crit}}(-)$ in symplectic topology had been recognized for a long time in Lagrangian intersection problems and so, the lower bound provided by $\mathrm{Qcat}(-)$ also enters the picture.

Throughout the book, the reader will find many explicit computations, as well as exercises and open problems. Chapter 9 is a repository of other extended examples which are, even if not in the mainstream of our presentation, of sufficient interest so as to be described in detail. In particular, we present Smale's use of category ideas in complexity theory and, following Singhof, we calculate category for certain Lie groups. We also present a somewhat simplified approach to the calculation of the category of 3-manifolds from the fundamental group alone. Other applications are included as well.

Because this book is intended for rather different audiences — topologists and geometers as well as analysts — we have included two appendices. The first appendix concerns topology and analysis. It includes topological definitions (such as that of an ANR) as well as a very brief recollection of basic Morse theory. The second appendix is much more detailed and presents various technical results and constructions in homotopy theory. In particular, it contains facts about homotopy pullbacks, pushouts and limits which are used in many places in the book, and which may prove enlightening as well.

The standard prerequisites for reading this book are an understanding of basic topological concepts such as homotopy, cohomology, fibration and cofibration, (as found, for example, in a first year-long course in algebraic topology), as well as fundamental notions of critical point theory (as found, for example, in a first course in Morse theory). More complicated homotopical constructions may be found in Appendix B. With the exception of these prerequisites, we have tried to make this book as self contained as possible. Where this has not proved to be possible, we have provided guides to the appropriate references.

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Let's now begin.