Preface

This book studies fascinating geometrical, topological and dynamical properties of closed 1-forms on manifolds. Given a closed 1-form ω, we are interested in the number of its zeros, in the geometry of the singular foliation $\omega = 0$, and in the dynamical properties of the gradient-like flows of ω.

A closed 1-form, viewed *locally*, is a smooth function up to an additive constant. All local properties of smooth functions can be translated into the language of closed 1-forms. For example, the notion of a critical point of a function corresponds to the notion of a zero of a closed 1-form. The *global* structure of a closed 1-form ω depends on its de Rham cohomology class $\xi = [\omega] \in H^1(M; \mathbf{R})$. The main subject of this book is to reveal the relations between the global and local features of closed 1-forms.

S.P. Novikov [**N1**], [**N2**] initiated a generalization of Morse theory in which instead of critical points of smooth functions one deals with closed 1-forms and their zeros. He introduced the numbers $b_j(\xi)$ and $q_j(\xi)$ depending on a real cohomology class $\xi \in H^1(M; \mathbf{R})$. We call $b_j(\xi)$ *the Novikov Betti number* and $q_j(\xi)$ *the Novikov torsion number*. In the special case $\xi = 0$ (which corresponds to the classical Morse theory of functions) the number $b_j(\xi)$ equals $b_j(M)$, the Betti number of M, and the number $q_j(\xi)$ coincides with the minimal number of generators of the torsion subgroup of $H_j(M; \mathbf{Z})$. The famous *Novikov inequalities* state that any closed 1-form ω with Morse-type zeros has at least $b_j(\xi) + q_j(\xi) + q_{j-1}(\xi)$ zeros of Morse index j, for any j, where $\xi = [\omega] \in H^1(M; \mathbf{R})$ is the de Rham cohomology class of ω. Nowadays, the Novikov theory is widely known and has numerous applications in geometry, topology, analysis, and dynamics.

This book starts with a detailed introduction into Novikov theory written in textbook style (Chapters 1 and 2). We hope that this material will be useful to readers who wish to apply Novikov theory. The first chapter studies the Novikov numbers $b_j(\xi)$ and $q_j(\xi)$. We describe their main properties and compute them explicitly in some examples. The main issue here is to clarify the character of the dependence of these numbers on the cohomology class ξ. In the second chapter we describe the geometric ideas which led to the discovery of Novikov theory. Here we also give a rigorous proof of the Novikov inequalities.

Subsequent chapters are written in the style of a research monograph. The material described in chapters 3–10 is based mainly on my work; some of these results were obtained jointly with my collaborators, Maxim Braverman, Gabriel Katz, Jerome Levine and Andrew Ranicki, in alphabetical order. The last section of chapter 10 represents a joint work with Thomas Kappeler, Janko Latschev and Eduard Zehnder.

Chapters 3 and 4 describe the universal chain complex. The exposition follows my paper [**Far13**] which develops our joint work with A. Ranicki [**FR**]. These two chapters, playing a central role in this book, give a very general answer to the problem of constructing the *"Novikov complexes"* over different extensions of the group ring of the manifold.

A general well-known intuitive principle of Morse theory says that the topology of the set of critical points of a function *dominates* (in some sense) the topology of the underlying manifold. This principle, when applied to the Morse theory of closed 1-forms, remains true but it requires a different meaning for the word "*dominates*". It turns out that one has to apply to the homotopy type of the manifold a suitable noncommutative localization which appears in the construction of the universal complex.

S.P. Novikov in his work always used a suitable completion to construct the "Novikov complexes". As an alternative it was suggested in my paper [**Far5**] published in 1985 to use a localization instead of the completion. I showed that the localization leads to a smaller ring having many advantages compared to the Novikov completions.

The initial fundamental idea of S.P. Novikov [**N2**] was based on a plan to construct the Novikov complex using dynamics of the gradient flow in the abelian covering associated with the given cohomology class. The dynamics of the gradient flows is used traditionally in Morse theory providing a bridge between the critical set of a function and the global ambient topology. A completely different approach to prove the Morse inequalities was first suggested by E. Witten [**Wi2**]; it is based on the spectral theory of the Laplace operator deformed by the given Morse function.

The construction of the universal complex described in Chapters 3 and 4 uses a new method of *algebraic collapse* suggested originally in our work with A. Ranicki [**FR**]. This technique, combinatorial and algebraic in nature, is quite simple and powerful. It allows one to avoid heavy analytic problems arising when dealing with the two approaches mentioned above (dynamics and spectral theory).

The universal complex uses the notion of noncommutative localization in the sense of P. Cohn [**Co**]. We find the algebraic condition on the ring which implies the validity of the Novikov Principle. The universal complex gives many different "Novikov complexes" and many different inequalities comparing the numbers of zeros to the Betti numbers of certain local coefficient systems. It is shown by example that these new inequalities are sometimes stronger than the Novikov inequalities.

In Chapter 5 we present several different generalizations of the Novikov inequalities. First, we remove the Morse nondegeneracy assumption replacing it by nondegeneracy in the sense of Bott. First inequalities of this kind were obtained jointly with Maxim Braverman [**BF1**], [**BF3**]. They relate the Poincaré polynomials of different connected components of the set of zeros to the Novikov counting polynomial of the manifold. A typical inequality (see (5.3)) claims

$$\sum_Z b_{i-\mathrm{ind}(Z)}(Z) \;\geq\; b_i(\xi).$$

Here Z runs over all connected components of the set of zeros of ω, $\mathrm{ind}(Z)$ denotes the index of Z, $b_j(Z)$ stands for the Betti number of Z and ξ denotes the cohomology class of ω. Several theorems of this chapter are new; among them Theorems 5.5, 5.6 and 5.7. For example, Theorem 5.7 gives the inequality

$$\sum_Z b_{i-\mathrm{ind}(Z)}(Z) \;\geq\; b_i(\xi) + q_i(\xi) + q_{i-1}(\xi)$$

(see (5.12)) which obviously generalizes the Novikov inequality. It is obtained under the assumption that the negative normal bundle of the set of zeros is orientable and the integral homology of the set of zeros has no torsion.

We describe in Chapter 6 Novikov-type inequalities where one uses the von Neumann Betti numbers instead of the Novikov numbers; these results were originally obtained in [**Far9**].

Chapter 7 suggests an equivariant version of the critical point theory for closed 1-forms. Although this material originates from a joint work with Maxim Braverman [**BF2**], [**BF4**], the exposition here is quite different and contains some new results. In this chapter we describe relations in an equivariant setting between the topology of the set of zeros of a closed equivariant basic 1-form and suitable equivariant cohomological invariants of the manifold. One defines integers (which are called *equivariant Novikov numbers*) playing a key role in this problem. As an application it is shown how these results (i.e., the equivariant generalization of Novikov theory) help to compute the cohomology of the fixed point set of a symplectic circle action. Finally, we present a formula expressing the signature of a symplectic manifold with a symplectic circle action through the Novikov numbers. This result was originally published in [**Far8**]; it generalizes a theorem of J.D.S. Jones and J.H. Rawnsley [**JR**], who studied the special case of Hamiltonian circle actions.

Next Chapter 8 describes the main theorem of the paper [**Far5**] about the exactness of the Novikov inequalities for manifolds with an infinite cyclic fundamental group. Roughly, it states that in any nonzero cohomology class one may find a closed 1-form for which the Novikov inequalities become equalities. This result is in the spirit of Smale's theorem [**Sm2**] about the existence of minimal Morse functions on simply connected manifolds. It solves a problem raised by S.P. Novikov [**N2**]. The well-known result of W. Browder and J. Levine [**BL**] giving conditions for fibering a manifold over a circle is a consequence of this theorem. This chapter also contains a finiteness theorem for codimension two stable knots: it states that such knots are determined up to a finite ambiguity by their Alexander modules and Milnor form.

E. Calabi [**Ca**] raised the problem of whether it is possible to improve the inequalities for closed 1-forms with Morse-type zeros if one additionally assumes that the 1-form is harmonic with respect to a Riemannian metric. This problem is discussed in Chapter 9, representing the results of a joint work [**FKL**] with Gabriel Katz and Jerome Levine and also the subsequent work of K. Honda [**Ho**]. We prove in this chapter that the harmonicity imposes no further Morse restrictions on the number of zeros. This chapter also contains a detailed study of the geometric properties of singular foliations of closed 1-forms.

Chapter 10 suggests a Lusternik-Schnirelman-type critical point theory for closed 1-forms. The main distinction from Novikov theory is that here one makes no additional requirements about the nature of the zeros of a closed 1-form. Recall that Novikov theory is based on the assumption of nondegeneracy of zeros which plays an important role there. Chapter 10 gives a generalization of the notion of the Lusternik-Schnirelman category. For any pair (X, ξ) consisting of a polyhedron X and a real cohomology class $\xi \in H^1(X; \mathbf{R})$, we define a nonnegative integer $\operatorname{cat}(X, \xi)$, *the category of X with respect to the cohomology class ξ*. The number $\operatorname{cat}(X, \xi)$ depends only on the homotopy type of (X, ξ) and coincides with $\operatorname{cat}(X)$

in the case $\xi = 0$. If $\xi \neq 0$, then $\text{cat}(X, \xi) < \text{cat}(X)$. We show by example that the difference $\text{cat}(X) - \text{cat}(X, \xi)$ may be arbitrarily large. The main theorem of this chapter states that any smooth closed 1-form ω on a smooth closed manifold M must have at least $\text{cat}(M, \xi)$ geometrically distinct zeros, where $\xi = [\omega] \in H^1(M; \mathbf{R})$ denotes the cohomology class of ω, assuming that ω admits a gradient-like vector field with no homoclinic cycles. Viewed differently, the main theorem of Chapter 10 claims that *any gradient-like vector field of a closed 1-form ω has a homoclinic cycle if the number of zeros of ω is less than* $\text{cat}(M, \xi)$.

Let us rephrase this surprising new *"focusing"* phenomenon: when the number of zeros of a closed 1-form ω becomes less than $\text{cat}(M, \xi)$ (which is a homotopy invariant!), any gradient-like vector field for ω has a homoclinic cycle. This result is a manifestation of a deep interaction between homotopy theory and dynamics.

Chapter 10 mainly follows my paper [**Far16**]. A slightly different version of the Lusternik-Schnirelman theory for closed 1-forms was suggested in [**Far17**] and in a more general form in [**FK**]. The results of this chapter correct some of my earlier statements made in [**Far11**] and [**Far12**].

The last section of Chapter 10 describes the notion of a Lyapunov 1-form of a flow and gives necessary and sufficient conditions for the existence of a Lyapunov 1-form in a prescribed cohomology class $\xi \in H^1(M; \mathbf{R})$. Here we use the notion of an asymptotic cycle introduced by S. Schwartzman [**Sch**]. The exposition is based on a joint work with T. Kappeler, J. Latschev and E. Zehnder [**FKLZ**], [**FKLZ1**].

In a series of appendices we give an exposition of Morse-Bott theory for manifolds with corners. This subject belongs to the mathematical folklore and is known to experts although no systematic treatment of these topics seems to exist in the literature.

This book is not designed to be an encyclopedia on the theory of closed 1-forms. It does not cover all results where the topology of closed 1-forms plays a role in mathematics. Unfortunately several important topics were left outside the scope of the book and the interested reader is invited to complete the picture by reading the original journal articles. We will mention briefly some of these topics.

In 1999 M. Hutchings and Y.-J. Lee [**HL1**], [**HL2**] made the fascinating discovery that the Lefschetz ζ-function counting the closed orbits of the gradient flow of a Morse closed 1-form can be computed in terms of the Reidemeister torsion of the Novikov complex. This result was later generalized by several authors; see [**P8**], [**Schu1**], [**Schu2**].

The methods of Novikov theory play an important role in group theory in studying finiteness properties of discrete groups. This research was initiated by J.-Cl. Sikorav in his thesis [**Si1**] written in 1987. J.-Cl. Sikorav proved that the vanishing of the Novikov-Sikorav homology in dimension one with respect to a cohomology class ξ is equivalent to the kernel $\ker \xi$ being finitely generated. Some further results and references can be found in the paper of M. Damian [**Da**] where the relations with the invariants of Bieri-Neumann-Strebel [**BNS**] and Bieri-Renz [**BR**] are explained. Here I would like to mention the related work of A. Ranicki [**Ran**] which proves that the vanishing of the Novikov-Sikorav homology is equivalent to the finite domination of the space of infinite cyclic covering.

A few words on the terminology. The terms "Novikov homology", "Novikov ring" and "Novikov complex" are used too often in the mathematical literature and the meaning of these terms varies in different papers. This may lead to ambiguity and misunderstanding. For example, the term "Novikov ring" denotes both the commutative ring **Nov** of the formal power series (see §1.2) and also the completion $\widehat{\mathbf{Z}\pi_\xi}$ of the group ring $\mathbf{Z}\pi$ determined by a cohomology class $\xi : \pi \to \mathbf{R}$ (see §3.1.5). The latter noncommutative ring was first introduced by J.-Cl. Sikorav [**Si1**]. I suggest to resolve this ambiguity by calling the ring $\widehat{\mathbf{Z}\pi_\xi}$ the Novikov-Sikorav completion and the corresponding homology the Novikov-Sikorav homology.

It is a pleasure for me to thank S.P. Novikov, who initiated the theory described in this book, for his great inspiration, encouragement and support.

I would also like to thank many colleagues for discussions which were helpful in writing this book. Among them I would like to thank my friends Jerome Levine, Thomas Kappeler, Vladimir Turaev and Shmuel Weinberger.

Finally I would like to thank the anonymous referee for generously sharing his ideas and for his highly professional and helpful comments.

Michael Farber